International Journal of Mathematics and Statistics Studies, 13 (2), 53-74, 2025
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)
Website: https://www.eajournals.org/
Publication of the European Centre for Research Training and Development -UK

# A Two-Point Off-Grid Colocation Point for the Direct Solution of Fourth Order Ordinary Differential Equations

**M. O.Alabi**
Department of Physical Sciences, Chrisland University, Abeokuta, Ogun State.

**M. T. Raji**
Department of Mathematics, Federal University of Agriculture, Abeokuta, Ogun State

**M. A.Kehinde**
Department of Mathematics, Federal College of Education (Special), Oyo, Oyo State.

doi: https://doi.org/10.37745/ijmss.13/vol13n25374 Published June 25, 2025

**Citation**: Alabi M.O., Raji M.T., Kehinde M.A (2025) A Two-Point Off-Grid Colocation Point for the Direct Solution of Fourth Order Ordinary Differential Equations, *International Journal of Mathematics and Statistics Studies*, 13 (2), 53-74

**Abstract:** *The desire to find solutions to differential equations cannot be over emphasized based on the importance of such equations. Many people have developed different initial value solvers to handle various differential equations based on the order of the differential equations. In solving differential equations of order greater than one, it is often the practiced to resolve such a differential equation into system of first order ordinary differential equations and then an appropriate method is applied. Also, in some cases the analytical solutions to some of the differential equations are intractable, hence there is need to circumvent this hurdle, and this is done by the introduction of approximate solution otherwise referred to as Numerical solution. This presentation focuses on derivation and implementation of a direct method to solve directly the fourth order ordinary differential equations by interpolating at some selected grid points and collocating at both grid and off grid points. Also, the derived method shall be applied to solve some fourth order ordinary differential equations to compare the level of accuracy of the method with the analytical solution.*
**Key Words:** grid point, collocation, interpolation, error constant, convergency, consistency

## INTRODUCTION

The analytical solution to some differential equations may be highly intractable, then an approximate solution is hereby sought for. Many of the approximate methods are either one step method such as Runge – Kutta method, Eulers Method, Heuns Method, Picard iterative method among others. Lack of accuracy of single step methods gave birth to Multistep methods which are in the form of Predictor – Corrector mode. Some of the methods are Simpson's methods, Adams Method (Adams Moulton and Admas Bashford method) to mention just a few. [8, 13]

In terms of the level of accuracy of the results generated using predictor corrector methods, there comes the introduction of Block method. One of the major setbacks of the predictor corrector method is that the method depends on another information for starting values thereby altering the accuracy of the results. The new method proposed named Block method has the advantage of self-starting thereby improving upon the output of the result. [8,13]

This paper presents the derivation and application of Block Linear Multistep method for the direct solution of fourth order initial value problems of the form

$$y''''(x) = f(x, y(x), y'(x), y''(x), y'''(x))$$

(1)

together with

$$y(x_0) = y_0,\ y'(x_0) = y_1, y''(x_0) = y_2, y'''(x_0) = y_3 \tag{2}$$

The method is based on interpolating the polynomial equation at some selected grid points while collocation was done at some selected grid points and two off grid points. Many authors have worked extensively on Linear Multistep Method to solve differential equations in which most of them are Predictor – Corrector in nature which is an improvement over and above the known single step methods [1, 2, 5, 6, 14]. These methods have some level of better accurate results over and above any known single step method, but they equally suffer some setback in terms of accuracy because they depend on single step methods in determining their starting value, also it is often laborious to develop an appropriate corrector method.

To circumvent this hurdle of developing an appropriate predictor – corrector method, researchers came up with an alternate method that has a better level of accuracy, and this method is given the name Block Linear Multi- step method. This method has an advantage of self-starting which does not depend on additional information in getting its starting value as against the Predictor – Corrector method. Among researchers that have worked extensively on this method include [3, 7, 10, 11, 12] to mention but a few.

## METHODOLOGY

In the course of developing initial value solvers for Ordinary Differential Equations, many authors make use of power series method of the form

$$y(x) = \sum_{r=0}^{\infty} a_r x^r \quad (3)$$

where $a_r$ is an arbitrary constant, while some used either perturbation method, Canonical polynomials, and some make use of Legendry polynomial or Hermite Polynomial as basic functions. [1, 14].

In this presentation, Chebyshev polynomial is hereby used as basis function to derive the proposed Linear Multistep method. The choice of Chebyshev polynomial is because it is the most accurate monomial among all other monomials within the interval $[-1, 1]$ in which some authors have used this polynomial as basic functions in deriving initial value solvers for ordinary differential equations. [4, 9].

Here, the linear multistep method

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h^n \sum_{r=0}^{k} \beta_r f_{n+r} + \beta_j f_{n+j} + \beta_v f_{n+v} \quad (4)$$

is hereby proposed where $n$ is the order of the differential equation, both $\alpha$ $and$ $\beta$ are arbitrary constants and not necessary $\alpha_r = 0$, $h$ is the step length while $j$ $and$ $v$ are non-integer collocation points. It is important to note that the smaller the value of $h$, the better the accuracy of the method [13].

## DERIVATION OF THE METHOD

Consider the polynomial equation

$$y(x) = \sum_{n=1}^{k} a_n T_n(x) \quad (5)$$

in which $a_n$ is an arbitrary constant and $T_n(x)$ is the Chebyshev polynomials. Note that the Chebyshev polynomial can be generated recessively using the relation

$$T_{n+1} = 2x\,T_n(x) - T_{n-1}(x), \quad n = 0(1)k \quad (6)$$

where $T_0\,(x) = 1 \; and \; T_1(x) = x$ (Fox and Parker).

Here, a four step Block Linear Multistep method is hereby proposed with two off grid collocation points. To achieve this, interpolation was done at four grid points of regular interval $x = x_{n+r}, r = 0(1)3,$ and collocation was done at five grid points and two off grid points $x_{n+r}$, $r = 0(1)4\,,\; x_{n+\frac{1}{2}}, \; and \; x_{n+\frac{3}{2}}$

In order to achieve this, equations (5) and (6) were made use of by letting $k = 10$ which leads to the polynomial equation

$$y(x) = a_{0+}\, a_1 x + a_2(x^2 - 1) + a_3(4x_3 - 3x) + a_4(8x^4 - 8x^2 + 1) +$$
$$a_5(16x^5 - 20x^3 + 5x) + a_6(32x^6 - 48x^4 + 18x^2 - 1) +$$
$$a_7(64x^7 - 112x^5 + 56x^3 - 7x) +$$
$$a_8(128x^8 - 256x^6 + 160x^4 - 32x^1 + 1)$$

(7)

$$a_9(256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x) +$$
$$a_{10}(512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1)$$

Using the shifted Chebyshev polynomial in which

$x = \frac{2x-b-a}{b-a}, x_n \le x \le x_{n+4}, there\,by\; x = \frac{x-kh-2h}{2h}$, hence equation (7) becomes

$$y(x) = a_0 + a_1\left(\frac{x-kh-2h}{2h}\right) + a_2\left\{\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right\}$$

$$a_3\left\{4\left(\frac{x-kh-2h}{2h}\right)^3 - 3\left(\frac{x-kh-2h}{2h}\right)\right\} + a_4\left\{8\left(\frac{x-kh-2h^4}{2h}\right) - 8\left(\frac{x-kh-2h^4}{2h}\right) + 1\right\} +$$

$$a_5\left\{16\left(\frac{x-kh-2h}{2h}\right)^4 - 20\left(\frac{x-kh-2h}{2h}\right)^2 + \left(\frac{x-kh-2h}{2h}\right)\right\} +$$

$$a_6\left\{32\left(\frac{x-kh-2h}{2h}\right)^6 - 48\left(\frac{x-kh-2h}{2h}\right)^4 + 18\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right\} +$$

$$a_7\left\{64\left(\frac{x-kh-2h}{2h}\right)^7 - 112\left(\frac{x-kh-2h}{2h}\right)^5 + 56\left(\frac{x-kh-2h}{2h}\right)^3 - 7\left(\frac{x-kh-2h}{2h}\right)\right\} +$$

$$a_8\left\{128\left(\frac{x-kh-2h}{2h}\right)^8 - 256\left(\frac{x-kh-2h}{2h}\right)^6 + 160\left(\frac{x-kh-2h}{2h}\right)^4 - 32\left(\frac{x-kh-2h}{2h}\right) + 1\right\}$$

$$a_9\left\{256\left(\frac{x-kh-2h}{2h}\right)^9 - 576\left(\frac{x-kh-2h}{2h}\right)^7 + 432\left(\frac{x-kh-2h}{2h}\right)^5 - 120\left(\frac{x-kh-2h}{2h}\right)^3 + 9\left(\frac{x-kh-2h}{2h}\right)\right\} +$$

$$a_{10}\left\{512\left(\frac{x-kh-2h}{2h}\right)^{10} - 1280\left(\frac{x-kh-2h}{2h}\right)^8 + 1120\left(\frac{x-kh-2h}{2h}\right)^6 - 400\left(\frac{x-kh-2h}{2h}\right)^4 + 50\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right\}$$

The fourth derivative of $y(x)$ yields

$$y^{iv}(x) = \frac{12a_4}{h^4} + a_5\left\{\frac{120}{h^4}\left(\frac{x-kh-2h}{2h}\right)\right\} + \frac{72a_6}{h^4}\left\{10\left(\frac{x-kh-2h}{2h}\right)^2 - \right\} +$$

$$\frac{840a_7}{h^4}\left\{4\left(\frac{x-kh-2h}{2h}\right)^3 - \frac{x-kh-2h}{2h}\right\} + \frac{240a_8}{h^4}\left\{56\left(\frac{x-kh-2h}{2h}\right)^4 - 24\left(\frac{x-kh-2h^2}{2h}\right) + 1\right\}$$

$$+ \frac{216a_9}{h^4}\left\{224\left(\frac{x-kh-2h}{2h}\right)^5 - 140\left(\frac{x-kh-2h}{2h}\right)^3 + 15\left(\frac{x-kh-2h}{2h}\right)\right\} +$$

$$\frac{120a_{10}}{h^4}\left\{1344\left(\frac{x-kh-2h}{2h}\right)^6 - 1120\left(\frac{x-kh-2h}{2h}\right)^4 + 210\left(\frac{x-kh-2h}{2h}\right)^2 - 5\right\}$$

Interpolating $y(x)$ at $x = x_{n+k}$ , $k = 0(1)3$ and collocating equation $y^{iv}(x)$ at

$x = x_{n+k}, k = 0\ (1)4\ and\ at\ x = x_{n+\frac{1}{2}}, x_{n+\frac{3}{2}}$ yields the linear equation in matrix form $Ax = b$, in which

$$x = (a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9 \quad a_{10})^T$$

$$b = \left(y_n \quad 2y_{n+1} \quad y_{n+2} \quad 2y_{n+3} \quad h^4 f_n \quad 8h^4 f_{n+\frac{1}{2}} \quad h^4 f_{n+1} \quad 8h^4 f_{n+\frac{3}{2}} \quad h^4 f_{n+2} \quad h^4 f_{n+3} \quad h^4 f_{n+4}\right)^T$$

and

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 12 & -120 & 648 & -2520 & 7920 & -21384 & 51480 \\ 0 & 0 & 0 & 0 & 96 & -720 & 2664 & -6300 & 10020 & -9234 & -1965 \\ 0 & 0 & 0 & 0 & 12 & -60 & 108 & 0 & -360 & 648 & -180 \\ 0 & 0 & 0 & 0 & 96 & -240 & -216 & 2520 & & -1080 & -3078 & 3915 \end{bmatrix}$$

$$\begin{matrix} 0 & 0 & 0 & 0 & 12 & 0 & -72 & 0 & 240 & 0 & -600 \\ 0 & 0 & 0 & 0 & 12 & 60 & 108 & 0 & -360 & -648 & -180 \end{matrix}$$

0 0 0 0 12 -120 648 -2520 7920 -21384 51480

Solving the matrix equation leads to

$$a_0 = \frac{1}{6048000}\Big\{6048000y_{n+1} - 6048000y_{n+2} + 6048000y_{n+3} + 10571h^4 f_n - 76160h^4 f_{n+\frac{1}{2}} + 302172h^4 f_{n+1} - 106624h^4 f_{n+\frac{3}{2}} + 722378h^4 f_{n+2} + 155564h^4 f_{n+3} + 99h^4 f_{n+4}\Big\}$$

$$a_1 = \frac{1}{84672000}\Big\{-56448000y_n + 84672000y_{n+1} - 169344000y_{n+2} + 141120000y_{n+3} + 284333h^4 f_n + 8817956h^4 f_{n+1} - 1782400h^4 f_{n+\frac{1}{2}} - 2452352h^4 f_{n+\frac{3}{2}} + 19281094h^4 f_{n+2} + 4075092h^4 f_{n+3} + 277h^4 f_{n+4}\Big\}$$

$$a_2 = \frac{1}{1512000}\Big\{1512000y_{n+1} - 3024000y_{n+2} + 1512000y_{n+3} + 3803h^4 f_n - 26880h^4 f_{n+\frac{1}{2}} + 111796h^4 f_{n+1} - 37632h^4 f_{n+\frac{3}{2}} + 266754h^4 f_{n+2} + 60052h^4 f_{n+3} + 107h^4 f_{n+4}\Big\}$$

$$a_3 = \frac{1}{50803200}\Big\{-16934400y_n + 50803200y_{n+1} - 50803200y_{n+2} + 16934400y_{n+3} + 80311h^4 f_n - 534400h^4 f_{n+\frac{1}{2}} + 2442412h^4 f_{n+1} - 733824f_{n+\frac{3}{2}} + 5783218f_{n+2} + 1424444h^4 f_{n+3} + 5039h^4 f_{n+4}\Big\}$$

$$a_4 = \frac{1}{168000}\Big\{109h^4 f_n - 640h^4 f_{n+\frac{1}{2}} + 3988h^4 f_{n+1} - 896h^4 f_{n+\frac{3}{2}} + 8662h^4 f_{n+2} + 2756h^4 f_{n+3} + 21h^4 f_{n+4}\Big\}$$

$$a_5 = \frac{1}{4704000}\left\{-511h^4 f_n - 3200h^4 f_{n+\frac{1}{2}} - 25452h^4 f_{n+1} - 18816h^4 f_{n+\frac{3}{2}} + 11102h^4 f_{n+2} + 36036h^4 f_{n+3} + 841h^4 f_{n+4}\right\}$$

$$a_6 = \frac{1}{6048000}\left\{-1071h^4 f_n + 12160h^4 f_{n+\frac{1}{2}} - 9772h^4 f_{n+1} + 17024h^4 f_{n+\frac{3}{2}} - 32578h^4 f_{n+2} + 13626h^4 f_{n+3} + 601h^4 f_{n+4}\right\}$$

$$a_7 = \frac{1}{5644800}\left\{-609h^4 f_n + 3200h^4 f_{n+\frac{1}{2}} - 11508h^4 f_{n+1} + 18816h^4 f_{n+\frac{3}{2}} - 11102h^4 f_{n+2} + 924h^4 f_{n+3} + 279h^4 f_{n+4}\right\}$$

$$a_8 = \frac{1}{1209600}\left\{-49h^4 f_n + 640h^4 f_{n+\frac{1}{2}} - 1428h^4 f_{n+1} + 896h^4 f_{n+\frac{3}{2}} + 98h^4 f_{n+2} - 196h^4 f_{n+3} + 39h^4 f_{n+4}\right\}$$

$$a_9 = \frac{1}{50803200}\left\{-511h^4 f_n - 3200h^4 f_{n+\frac{1}{2}} + 13748h^4 f_{n+1} - 18816h^4 f_{n+\frac{3}{2}} + 11102h^4 f_{n+2} - 3164h^4 f_{n+3} + 841h^4 f_{n+4}\right\}$$

$$a_{10} = \frac{1}{6048000}\left\{109h^4 f_n - 64h^4 f_{n+\frac{1}{2}} + 119h^4 f_{n+1} - 896h^4 f_{n+\frac{3}{2}} + 262h^4 f_{n+2} - 44h^4 f_{n+3} + 21h^4 f_{n+4}\right\}$$

Substituting the value of $a's$ into $y(x)$ to generate the continuous scheme thus

$$y(x) = \frac{1}{254016000}\left\{\begin{array}{c}
(-338688000x^3 + 84672000x)y_n + \\
(1016064000x^3 + 508032000x^2 - 508032000x)y_{n+1} + \\
(-1016064000x^3 - 1016064000x^2 + 254016000x + 254016000)y_{n+2} + \\
(338688000x^3 + 508032000x^2 + 169344000x)y_{n+3} + \\
\begin{pmatrix} 2343936x^{10} - 654080x^9 - 7176960x^8 + 3225600x^7 + 6322176x^6 - \\ 2407104x^5 + 3999380x^3 - 292152x^2 - 699867x \end{pmatrix} h^4 f_n + \\
\begin{pmatrix} -1376256x^{10} - 4096000x^9 + 20643840x^8 + 18432000x^7 - \\ 53760000x^6 - 26496000x^5 - 9676800x^4 + 2752000x^3 + \\ 3467520x^2 + 652800x - 24192 \end{pmatrix} h^4 f_{n+\frac{1}{2}} + \\
\begin{pmatrix} 2558976x^{10} + 17597440x^9 - 44782080x^8 - 72737280x^7 + \\ 69233472x^6 + 109686528x^5 + 17954200x^4 - 15888880x^3 \\ -8216964x^2 + 933408x + 44898 \end{pmatrix} h^4 f_{n+1} + \\
\begin{pmatrix} -19267584x^{10} - 24084480x^9 + 72253440x^8 + 59607360x^7 \\ -67436544x^6 - 151732224x^5 + 64350720x^3 + \\ 3161088x^2 - 8203776x \end{pmatrix} h^4 f_{n+\frac{3}{2}} + \\
\begin{pmatrix} 5634048x^{10} + 14210560x^9 - 11450880x^8 - 63947520x^7 - \\ 36728832x^6 + 89526528x^5 + 169344000x^4 + 69035960x^3 - \\ 39883536x^2 - 21910728x \end{pmatrix} h^4 f_{n+2} + \\
\begin{pmatrix} -946176x^{10} - 4049920x^9 - 2903040x^8 + 11773440x^7 + \\ 26780544x^6 + 19643904x^5 + 20160x^4 - 6203120x^3 - \\ -1633128x^2 + 154896x + 420 \end{pmatrix} h^4 f_{n+3} + \\
\begin{pmatrix} 451584x^{10} + 1076480x^9 - 80640x^8 - 1627200x^7 - \\ 301056x^6 + 1137024x^5 - 609020x^3 + 18312x^2 + 102276x \end{pmatrix} h^4 f_{n+4}
\end{array}\right\}$$

Further simplification yields

$\alpha_0(t) = \frac{1}{3}(t - 4t^3)$, $\alpha_1(t) = 4t^3 + 2t^2 - 2t$, $\alpha_2(t) = 3 + 3t - 4t^2 - 4t^3$ and

$\alpha_3(t) = \frac{1}{3}(2t + 6t^2 + 4t^3)$,

$$\beta_0(t) = \frac{1}{10080}\begin{pmatrix} 93t^{10} - 26t^9 - 285t^8 + 128t^7 + 215t^6 - 96t^5 - 89t^4 + 159t^3 - \\ 12t^2 - 28t \end{pmatrix}$$

$$\beta_{\frac{1}{2}}(t) = \frac{1}{10080}\begin{pmatrix} -55t^{10} - 163t^9 + 819t^8 + 731t^7 - 2133t^6 - 1051t^5 - 384t^4 \\ 109t^3 + 1380t^2 + 108t \end{pmatrix}$$

$$\beta_1(t) = \frac{1}{10080}\begin{pmatrix} 102t^{10} + 697t^9 - 1777t^8 - 2886t^7 + 2747t^6 + 4353t^5 + \\ 712t^4 - 631t^3 - 433t^2 + 37t + 2 \end{pmatrix}$$

$$\beta_{\frac{3}{2}}(t) = \frac{1}{10080}\begin{pmatrix} -765t^{10} - 956t^9 + 2867t^8 + 2365t^7 - 2676t^6 - 6021t^5 \\ + 1937t^4 + 2554t^3 + 125t^2 - 326t \end{pmatrix}$$

$$\beta_2(t) = \frac{1}{10080}\begin{pmatrix} 224t^{10} + 564t^9 - 454t^8 - 2538t^7 - 1458t^6 + 3553t^5 \\ + 6720t^4 + 2740t^3 - 1584t^2 - 869t \end{pmatrix}$$

$$\beta_3(t) = \frac{1}{10080}\begin{pmatrix} -38t^{10} - 161t^9 - 151t^8 + 467t^7 + 1063t^6 + 780t^5 + \\ t^4 - 246t^3 - 65t^2 + 6t \end{pmatrix}$$

$$\beta_4(t) = \frac{1}{10080}(18t^{10} + 43t^9 - 3t^8 - 65t^7 - 12t^6 + 45t^5 - 24t^3 + t^2 + 4t)$$

Let $t = \frac{x-kh-2h}{kh}$, and evaluating $x\ at\ x_{n+4}$, this makes $t = 1$, then this leads to the discrete scheme of the form

$$y_{n+4} - 4y_{n+3} + 6y_{n+2} - 4y_{n+1} + y_n = \frac{h^4}{10080}\Big(7f_{n+4} + 1692f_{n+3} + f_{n+2} - 896 f_{n+\frac{3}{2}} + 2924f_{n+1} - 640f_{n+\frac{1}{2}} + 95f_n\Big) \quad (8)$$

The first derivative of the continuous scheme yields

$$\alpha'_0(t) = \frac{1}{3h}\left(-6t^2 + \frac{1}{2}\right)$$

$$\alpha'_1(t) = \frac{1}{h}(6t^2 + 2t - 1)$$

$$\alpha'_2(t) = \frac{1}{2h}(-12t^2 - 8t + 3)$$

$$\alpha'_3(t) = \frac{1}{3h}(6t^2 + 6t + 1)$$

$$\beta'_0(t) = \frac{1}{10080h}(465t^9 - 117t^8 - 1140t^7 + 448t^6 + 753t^5 - 240t^4 - 178t^3 + 238.5t^2 - 12t - 14)$$

$$\beta'_{\frac{1}{2}}(t) = \frac{1}{10080h}(-275t^9 - 733.5t^8 + 3276t^7 + 2558.5t^6 - 6399t^5 - 2627.5t^4 - 768t^3 + 163.5t^2 + 1380t + 54)$$

$$\beta'_1(t) = \frac{1}{10080h}(510t^9 + 3141t^8 - 7108t^7 - 10101t^6 + 8241t^5 + 10882.5t^4 + 1424t^3 - 946.5t^2 - 433t + 18.5)$$

$$\beta'_{\frac{3}{2}}(t) = \frac{1}{10080h}(-3825t^9 - 4302t^8 + 11468t^7 + 8277.5t^6 - 8028t^5 - 15052.5t^4 + 3874t^3 + 3831t^2 + 125t - 163)$$

$$\beta'_2(t) = \frac{1}{10080h}(1120t^9 + 2538t^8 - 1816t^7 - 8883t^6 - 4374t^5 + 8882.5t^4 + 13440t^3 + 4110t^2 - 1584t - 434.5)$$

$$\beta'_3(t) = \frac{1}{10080h}(-190t^9 - 724.5t^8 - 460t^7 + 1634.5t^6 + 3189t^5 + 1950t^4 + 2t^3 - 369t^2 - 65t + 3)$$

$$\beta'_4(t) = \frac{1}{10080h}(90t^9 + 193.5t^8 - 12t^7 - 227.5t^6 - 36t^5 + 112.5t^4 - 36t^2 + t + 2)$$

The second derivative of the continuous scheme yields

$$\alpha''_0(t) = \frac{-2t}{h^2},\ \alpha''_1(t) = \frac{1}{h^2}(6t + 1),\ \alpha''_2(t) = \frac{1}{h^2}(-6t - 2),\ \alpha''_3(t) = \frac{1}{h^2}(2t + 3)$$

$$\beta''_0(t) = \frac{1}{10080h^2}(2092.5t^8 - 468t^7 - 3990t^6 + 1344t^5 + 1882.5t^4 - 480t^3 - 267t^2 + 238.5t - 6)$$

$$\beta''_{\frac{1}{2}}(t) = \frac{1}{10080h^2}(-1237.5t^8 - 2934t^7 + 11466t^6 + 7675.5t^5 - 15997.5t^4 - 5255t^3 - 1152t^2 + 163.5t + 690)$$

$\beta''_1(t) = \frac{1}{10080h^2}(2295t^8 + 12564t^7 - 24878t^6 - 30303t^5 + 20602.5t^4 + 21765t^3 + 2136t^2 - 946.5t - 216.5)$

$\beta''_{\frac{3}{2}}(t) = \frac{1}{10080h^2}(-17212.5t^8 - 17208t^7 + 40138t^6 + 24832.5t^5 - 30105t^4 - 30105t^3 + 5811t^2 + 3831t + 62.5)$

$\beta''_2(t) = \frac{1}{10080h^2}(5040t^8 + 10152t^7 - 6356t^6 - 26649t^5 - 10935t^4 + 17765t^3 + 20160t^2 + 4110t - 792)$

$\beta''_3(t) = \frac{1}{10080h^2}(-855t^8 - 2898t^7 - 1610t^6 + 4903.5t^5 + 7972.5t^4 + 3900t^3 + 3t^2 - 369t - 32.5)$

$\beta''_4(t) = \frac{1}{10080h^2}(405t^8 + 774t^7 - 42t^6 - 682.5t^5 - 90t^4 + 225t^3 - 36t + 0.5)$

The third derivative of the continuous scheme leads to

$\alpha'''_0(t) = \frac{-1}{h^3},\ \alpha'''_1(t) = \frac{3}{h^3},\ \alpha'''_2(t) = \frac{-3}{h^3},\ \alpha'''_3(t) = \frac{1}{h^3}$

$\beta'''_0(t) = \frac{1}{10080h^3}(8370t^7 - 1638t^6 - 11970t^5 + 3360t^4 + 3765t^3 - 720t^2 - 267t + 119.25)$

$\beta'''_{\frac{1}{2}}(t) = \frac{1}{10080h^3}(-4950t^7 - 10269t^6 + 34398t^5 + 19188.75t^4 - 31995t^3 - 7882.5t^2 - 1152t + 81.75)$

$\beta'''_1(t) = \frac{1}{10080h^3}(9180t^7 + 43974t^6 - 74634t^5 - 75757.5t^4 + 41205t^3 + 32647.5t^2 + 2136t - 473.25)$

$\beta'''_{\frac{3}{2}}(t) = \frac{1}{10080h^3}(-68850t^7 - 60228t^6 + 120414t^5 + 62081.25t^4 - 40140t^3 - 45157.5t^2 + 5811t + 1915.5)$

$$\beta'''_2(t) = \frac{1}{10080h^3}(20160t^7 + 35532t^6 - 19068t^5 - 66622.5t^4 - 21870t^3 + 26647.5t^2 + 20160t + 2055)$$

$$\beta'''_3(t) = \frac{1}{10080h^3}(-3420t^7 - 10143t^6 - 4830t^5 + 12258.75t^4 + 15945t^3 + 5850t^2 + 3t - 184.5)$$

$$\beta'''_4(t) = \frac{1}{10080h^3}(1620t^7 + 2709t^6 - 126t^5 - 1706.25t^4 - 180t^3 + 337.5t^2 - 18)$$

Evaluating the first derivative of the continues scheme at $x_{n+k}, k = 0(1)4$ leads to the following discrete schemes.

$$hy'_n + \frac{11}{6}y_n - 3y_{n+1} + \frac{1}{2}y_{n+2} - \frac{1}{3}y_{n+3} = \frac{h^3}{10080}\left(427.5f_n + 2201f_{n+\frac{1}{2}} + 360.5f_{n+1} - 11032f_{n+\frac{3}{2}} - 573f_{n+2} + 18f_{n+3} + 1.5f_{n+4}\right)$$

$$hy'_{n+1} + \frac{1}{3}y_n + \frac{1}{2}y_{n+1} - 2y_{n+2} + \frac{1}{6}y_{n+3} = \frac{h^3}{10080}\left(50f_n - 451f_{n+\frac{1}{2}} + 152f_{n+1} - 411f_{n+\frac{3}{2}} + 280f_{n+2} - 15f_{n+3} - 2f_{n+4}\right)$$

$$hy'_{n+2} - \frac{1}{6}y_n + y_{n+1} - \frac{3}{2}y_{n+2} - \frac{1}{3}y_{n+3} = \frac{h^3}{10080}\left(-14f_n + 54f_{n+\frac{1}{2}} + 18.5f_{n+1} - 163f_{n+\frac{3}{2}} + 434.5f_{n+2} + 3f_{n+3} + 2f_{n+4}\right)$$

$$hy'_{n+3} + \frac{1}{3}y_n - \frac{3}{2}y_{n+1} + 2y_{n+2} - \frac{11}{6}y_{n+3} = \frac{h^3}{10080}\left(24f_n + 387f_{n+\frac{1}{2}} + 481f_{n+1} + 344f_{n+\frac{3}{2}} + 1759f_{n+2} + 119f_{n+3} - 3f_{n+4}\right)$$

$$hy'_{n+4} + \frac{11}{6}y_n - 7y_{n+1} + \frac{17}{2}y_{n+2} - \frac{13}{3}y_{n+3} = \frac{h^3}{10080}\left(203.5f_n + 3371f_{n+\frac{1}{2}} + 5628f_{n+1} - 3795f_{n+\frac{3}{2}} + 12999f_{n+2} + 4970f_{n+3} + 87.5f_{n+4}\right)$$

In like manner, the evaluation of the second derivative of the continuous scheme at $x = x_{n+k}, k = 0(1)4$ yields

$$h^2y''_n - 2y_n + 5y_{n+1} - 4y_{n+2} + y_{n+3} = \frac{h^2}{10080}\Big(-922.5f_n - 5881f_{n+\frac{1}{2}} - 3140.5f_{n+1} + 27378.5f_{n+\frac{3}{2}} + 1739f_{n+2} - 58.5f_{n+3} - 7f_{n+4}\Big)$$

$$h^2y''_{n+1} - y_n + 2y_{n+1} - y_{n+2} = \frac{h^2}{10080}\Big(-107f_n + 98f_{n+\frac{1}{2}} + 21f_{n+1} + 2027f_{n+\frac{3}{2}} - 37f_{n+2} + 4f_{n+3} + f_{n+4}\Big)$$

$$h^2y''_{n+2} - y_{n+1} + 2y_{n+2} - y_{n+3} = \frac{h^2}{10080}\Big(-6f_n + 690f_{n+\frac{1}{2}} + 216.5f_{n+1} + 62.5f_{n+\frac{3}{2}} + 792f_{n+2} - 32.5f_{n+3} + 0.5f_{n+4}\Big)$$

$$h^2y''_{n+3} + y_n - 4y_{n+1} + 5y_{n+2} - 2\,y_{n+3} = \frac{h^2}{10080}\Big(96f_n - 782f_{n+\frac{1}{2}} + 2624f_{n+1} - 385f_{n+\frac{3}{2}} + 7007f_{n+2} + 872f_{n+3} - 9f_{n+4}\Big)$$

$$h^2y''_{n+4} + 2y_n - 7y_{n+1} + 8y_{n+2} - 3\,y_{n+3} = \frac{h^2}{10080}\Big(346.5f_n - 6581f_{n+\frac{1}{2}} + 3018.5f_{n+1} - 9920.5f_{n+\frac{3}{2}} + 12495f_{n+2} + 11014.5f_{n+3} + 554f_{n+4}\Big)$$

So also, the evaluation of the third derivative of the continuous scheme at $x = x_{n+k}, k = 0(1)4$ result to the following schemes

$$h^3y'''_n + y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3} = \frac{h}{10080}\Big(1223f_n + 4818f_{n+\frac{1}{2}} + 22504f_{n+1} - 58624f_{n+\frac{3}{2}} - 39906f_{n+2} + 83f_{n+3} + 8f_{n+4}\Big)$$

$$h^3y'''_{n+1} + y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3} = \frac{h}{10080}\Big(95f_n + 2689f_{n+\frac{1}{2}} - 317f_{n+1} - 7548f_{n+\frac{3}{2}} - 1800f_{n+2} + 69f_{n+3} + 16f_{n+4}\Big)$$

$$h^3y'''_{n+2} + y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3} = \frac{h}{10080}\Big(119.25f_n + 81.75f_{n+\frac{1}{2}} - 473.25f_{n+1} + 1915.5f_{n+\frac{3}{2}} + 2055f_{n+2} - 184.5f_{n+3} - 18f_{n+4}\Big)$$

$$h^3y'''_{n+3} + y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3} = \frac{h}{10080}\Big(152f_n - 4389f_{n+\frac{1}{2}} + 7599f_{n+1} - 5322f_{n+\frac{3}{2}} + 12016f_{n+2} + 3703f_{n+3} - 12f_{n+4}\Big)$$

$$h^3y'''_{n+4} + y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3} = \frac{h}{10080}\Big(1019.25f_n - 2580f_{n+\frac{1}{2}} - 21722.25f_{n+1} - 24153.75f_{n+\frac{3}{2}} - 3006f_{n+2} + 15479.25f_{n+3} + 2636.25f_{n+4}\Big)$$

## ANALYSIS OF THE METHOD

At this juncture, some basic analysis of the method shall be carried out such as order, consistency and error constant of the derived scheme as well as to test for zero stability of the derived method.

**Theorem 1:** The necessary and sufficient condition for a Linear Multistep method to converge is that it must be both consistent and zero stable. [13]

It is important to state that consistency controls the magnitude of the local truncation error while zero stability concerns the manner of error committed at each stage of computation. So also, for a Linear Multistep method to be consistent, it must have order $p \geq 1$ while a Linear Multistep method is said to be zero stable if no root of the first characteristics polynomial $\rho(r)$ has modulus greater than one and if every root with modulus one is simple.[13]

The Linear Multistep method is said to be of order $p$ if $C_0 = C_1 = C_2 \ . \ . \ . C_{p+3} = 0$ while $C_{p+4} \neq 0$ and $C_{p+4}$ is referred to as error constant. To determine the order and error constant of the derived method, the method hereunder is applied.

$$C_0 = \sum_{j=0}^{k} \alpha_j, \quad C_1 = \sum_{j=1}^{k} j\alpha_j, \quad C_2 = \sum_{j=1}^{k} j^2\alpha_j, \quad C_3 = \sum_{j=1}^{k} j^3\alpha_j$$

$$C_4 = \frac{1}{4!}\sum_{j=1}^{k} j^4\alpha_j - \sum_{j=0}^{k} \beta_j - \beta_{\frac{1}{2}} - \beta_{\frac{3}{2}} \text{ and}$$

$$C_q = \frac{1}{q!}\sum_{j=1}^{k} j^q\alpha_j - \frac{1}{(q-4)!}\sum_{j=1}^{k} j^{q-4}\beta_j + \left(\frac{1}{2}\right)^{q-4}\beta_{\frac{1}{2}} + \left(\frac{3}{2}\right)^{q-4}\beta_{\frac{3}{2}}, \quad q = 5(1)k$$

Applying the above method, it was discovered that the derived method is of order 8 in which $C_{12} = -2.09983E10^{-3}$

Also to determine the zero stability of the method, the roots of the first characteristics polynomial of the derived method were sought for, that is

$$r^4 - 4r^3 + 6r^2 - 4r + 1 = 0 \qquad (9)$$

Solving equation (9) yields $r = 1$ four times and this shows that the method is both consistent and zero stable, hence the method satisfies the condition for convergence. At this juncture, some numerical experiments shall be carried out using the derived method to solve some fourth order ordinary differential equations and the output shall be compared with the analytical solution to see the level of convergency of the method derived.

## SOME NUMERICAL COMPUTATIONS

Illustration I. Determine the solution to the differential equation

$$y^{iv} - 5y'' + 4y = 0: \ y(0) = 3, y'(0) = 5, y''(0) = -2, y'''(0) = 4$$

The analytical solution is

$$y(x) = 9.5e^{x} - 2.5e^{-x} - 2.5e^{2x} - 1.5e^{-2x}$$

The numerical computation to Illustration I is as shown in Table 1 below

Table 1: Numerical Computation to Illustration I

| x | y- Exact | y- Computed | Absolute Error |
|---|---|---|---|
| 0 | 3.0000000000 | 3.0000000000 | 0.0000000000 |
| 0.1 | 3.9554271520 | 3.9554267810 | 3.70611345E – 7 |
| 0.2 | 4.8214575070 | 4.8214579123 | 4.05629981E – 7 |
| 0.3 | 5.5930986650 | 5.5930986817 | 1.67000000E – 8 |
| 0.4 | 6.2586887450 | 6.2586894524 | 7.07400000E – 7 |
| 0.5 | 6.7990016890 | 6.7990019135 | 2.24035153E – 7 |
| 0.6 | 7.1860158886 | 7.1860158992 | 1.04349130E – 8 |
| 0.7 | 7.3812920978 | 7.3812929607 | 8.62233099E – 7 |
| 0.8 | 7.3338905724 | 7.3338991064 | 8.533994393E – 6 |
| 0.9 | 6.9777384132 | 6.9777319206 | 6.492674771E – 6 |
| 1.0 | 6.2283355947 | 6.2283359043 | 3.09049219E – 7 |

Illustration II: Solve completely the differential equation

$$y^{iv}(x) - 4y'''(x) + 5y''(x) - 4y'(x) + 4y(x) = 0$$

Subject to

$$y(0) = 4,\ y'(0) = 6,\ y''(0) = 8 \ and \ y'''(0) = -6$$

The theoretical solution is

$$y(x) = 6.24e^{2x} - 4.8xe^{2x} - 2.24\cos x - 1.68\sin x$$

Table 2 below shows the numerical output to Illustration II

International Journal of Mathematics and Statistics Studies, 13 (2), 53-74, 2025
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)
Website: https://www.eajournals.org/
Publication of the European Centre for Research Training and Development -UK

Table 2: The Output to Numerical Illustration II

| x | y- Exact | y- Computed | Absolute Error |
|---|---|---|---|
| 0 | 4.0000000000 | 3.9971824673 | 2.8175327E – 3 |
| 0.1 | 4.6387504168 | 4.6387569032 | 6.486386876E – 6 |
| 0.2 | 5.3477207926 | 5.3477814017 | 6.060825418E – 5 |
| 0.3 | 6.1097425587 | 6.1097429702 | 4.11157979E – 7 |
| 0.4 | 6.8969373685 | 6.8969301864 | 7.183042454E – 6 |
| 0.5 | 7.6669823776 | 7.6669829011 | 5.23306771E – 7 |
| 0.6 | 8.3582417278 | 8.3582499013 | 8.173606549E – 6 |
| 0.7 | 8.8834436851 | 8.8834409334 | 2.75703608E – 6 |
| 0.8 | 9.1214965568 | 9.1214969062 | 3.49400000E – 7 |
| 0.9 | 8.9069275947 | 8.9069279904 | 3.95827600E – 7 |
| 1.0 | 8.0162923625 | 8.0162928705 | 5.07661766E – 7 |

Numerical Experimentation III

Determine the complementary solution to the differential equation

$$y^{iv} + 3y'' + 2y = 0 \; subject\ to\ y(0) = 4, y'(0) = 3, \; y''(0) = 5, \; y'''(0) = 9$$

The analytical solution is

$$y(x) = 7\cos x + 7\sin x - 3\cos 2x - 2\sin 2x$$

The numerical experimentation to Illustration III is as shown in Table 3 below

Table 3: Output to Numerical Experimentation of Illustration III

| x | y- Exact | y- Computed | Absolute Error |
|---|---|---|---|
| 0 | 4.0000000000 | 3.9999856721 | 1.43279E – 5 |
| 0.1 | 4.3263246776 | 4.3263299503 | 5.2719398685E – 6 |
| 0.2 | 4.7091316937 | 4.7091376027 | 5.908871837E – 6 |
| 0.3 | 5.1507050786 | 5.1507009762 | 4.102789517E – 6 |
| 0.4 | 5.6485230437 | 5.6485299112 | 6.8668597878E – 6 |
| 0.5 | 6.1952078158 | 6.1952009227 | 6.893541819E – 6 |
| 0.6 | 6.7786951827 | 6.7786990172 | 3.834431474E – 6 |
| 0.7 | 7.3826182329 | 7.3826190584 | 8.25422386E – 7 |
| 0.8 | 7.9868909626 | 7.9868919675 | 1.004952328E – 6 |
| 0.9 | 8.5684691677 | 8.5684690832 | 8.44099000E – 8 |
| 1.0 | 9.1022586906 | 9.1022580943 | 5.9642232E – 7 |

Illustration IV: Solve completely the differential equation

$$y^{iv} - 7y'' - 18y = 0, \text{ subject to}$$

$$y(0) = 1,\ y'(0) = 3,\ y''(0) = -4,\ y'''(0) = 2$$

The theoretical solution is

$$y(x) = 0.7\cos\sqrt{2}\,x + 1.7\sin\sqrt{2}x + 0.3\cos 3x + 0.2\sin 3x$$

The output to Illustration IV is as shown in Table 4 below

Table 4: The Computational Result to Illustration IV

| x | y- Exact | y- Computed | Absolute Error |
|---|---|---|---|
| 0 | 1.0000000000 | 0.9998976435 | 1.023565E – 4 |
| 0.1 | 0.41444878516 | 0.4144489764 | 1.911914776E – 7 |
| 0.2 | 0.18544967057 | 0.1854489027 | 7.67870000E – 7 |
| 0.3 | -0.02361051641 | -0.02361097833 | 4.619182019E – 7 |
| 0.4 | -0.19621158857 | -0.19621170912 | 1.205287269E – 7 |
| 0.5 | -0.33366275639 | -0.33366272305 | 3.33723102E – 8 |
| 0.6 | -0.44284711285 | -0.44284719021 | 7.73066337E – 8 |
| 0.7 | -0.53039610736 | -0.53039609542 | 1.19508374E – 8 |
| 0.8 | -0.60052532177 | -0.60052535801 | 3.62437427E – 8 |
| 0.9 | -0.65417021927 | -0.65417007853 | 1.407662188E – 7 |
| 1.0 | -0.68883259955 | -0.68883250974 | 8.9909047E – 8 |

**DISCUSSION OF RESULTS**

At this juncture, a critical analysis of the results generated from the numerical illustrations as presented in Tables 1 to 4 shall be analyzed.

First and foremost, the derived method satisfied the conditions for which a Multistep Method will converged as earlier enumerated. In this presentation, four critical illustrations were carried out to compare the level of accuracy of results generated using the derived method with the analytical method. From the table of results, that is from Tables 1 to 4, it was observed that the column of y-computed which shows the results generated using the derived method compares favorably well with the results of analytical solutions to the differential equations under consideration, as shown in the column of y - exact. The absolute error is very minimal in all illustrations used.

In like manner, as the Grid Block Multistep Methods can be used to solve ordinary differential equations, it is hereby clearly shown that Off – Grid Block Linear Multistep Method can equally be used to solve differential equations as shown in the tables of results as presented in Tables 1 to 4.

It is not an overemphasized thing to say that this method is noble because the method is self-starting whereby it requires no additional information to get its starting values thereby reducing the error in the computation unlike the Predictor - Corrector methods. Also, the method affords the users to make use of the method in solving a fourth order ordinary differential equations directly without necessarily resolving such a differential equation to system of first order ordinary differential equations before solving which is often laborious.

At this juncture, this method is hereby unequivocally recommended to be used for finding numerical solutions to Fourth Order Ordinary Differential Equations without any need of resolving such differential equations to system of first order ordinary differential equations.

## REFERENCES

1 Aboiyar, T., Luga, T., Iyorter, B. V. (2015) "Derivation of Continuous Linear Multistep Methods Using Hermite Polynomials as Basis Functions" American Journal of Applied Mathematics and Statistics Vol.3 No. 6, pp 220 – 225

2 Adeniyi, R. B. and Alabi, M. O. (2006) "Derivation of Continuous Multistep Methods Using Chebyshev Polynomial Basis Function". Abacus Vol. 33. Pp. 351 – 361.

3 Adesanya, A. O., Momoh, A. A., Alkali, M. A., and Tahir, A. (2012) "Five Steps Block Method for the Solution of Fourth Order ODEs". International Journal of Engineering Research and Application, Vol. 2(5) pp 991 – 998.

4 Alabi, M. O. (2008) "A Continuous Formulation of Initial Value Solvers with Chebyshev Basis Function in a Multistep Collocation Technique" Ph. D. Thesis, University of Ilorin, Ilorin Kwara State.

5 Alabi, M.O., Oladipo, A.T, Adesanya A.O, Okedoye, M.A, Babatunde,O.Z, (2007), "Formulation of Some Linear Multistep Schemes for Solving First Order Initial Value Problems Using Canonical Polynomials as Basis Functions." Journal of Modern Mathematics and Statistics. 1(1) Pp 3 - 7.

6 Alabi, M.O., Oladipo, A.T., and Adesanya, A.O. (2008), "Initial Value Solvers for Second Order Ordinary Differential Equations using Chebyshev Polynomial as Basis Functions". Journal of Modern Mathematics and Statistics 2(1) Pp.18 – 27.

7 Alabi, M. O., Raji, M. T., and Alabi, I. Y. (2025) "Off Grid Initial Value Solver for First Order Ordinary Differential Equations in Block Form Using Chebyshev Polynomial as Basis Function" International Journal of Mathematics and Statistics Studies Vol. 13 (1), pp. 13 – 35.

8 Butcher, J. C. (2003) "Numerical Methods for Ordinary Differential Equations". John Waley and Sons, London.
9 Fox, L. and Parker, I. B., (1968) "Chebyshev Polynomials in Numerical Analysis" Oxford University Press, London.
10 James, A. A., Adesanya, A. O., and Sunday, J. (2013) "Continuous Block Method for the Solution of Second Order Initial Value Problems of ODEs." International Journals of Pure and Applied Mathematics. Vol 83(3) pp 405 - 416
11 James, A. A., Odesanya, A. O., Sunday, J. and Yakubu, D. G. (2013) "Half- Step Continuous Block Method for the Solution of Modeled Problems of ODEs. "American Journal of Computational Mathematics, Vol.3, pp 261 – 269.
12 Jator, S. N. (2007) "A sixth Order Linear Multistep Method for the Direct Solution of $y'' = (x, y, y')$" International Journal of Pure and Applied Mathematics 40, pp. 457 – 472.
13 Lambert, J. D. (1938) "Computational Methods in Ordinary Differential Equations. JohnWaley and Sons, London.
14 Omolehin, J.O., Ibiejugba, M. A, Alabi, M.O. and D.J. Evans. (2003), "A new Class of Adams Bashforth Schemes for ODES" Intern. J. Computer Math. Vol. 80(5), Pp 629 – 638.