The Straight Line Theorem

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ABSTRACT: Proposition: A straight line measures the shortest distance between two points in a plane.

KEY WORDS: circle, compass, line of radius, straightedge.

INTRODUCTION

To introduce the topic, recall how the straightedge and compass form the two main instruments of plane geometry. A straightedge may be a ruler, yardstick, bar of iridium, or piece of string that is drawn tightly between two points in a plane, against which a pencil may draw a straight line.

A compass forms a virtual triangle. It has two legs of nearly equal length that join together at its top, with an adjustable setting for the angle between its legs. A compass is held above a plane with its legs touching the surface, letting the tips or ends of its legs serve as the endpoints for a virtual line that measures the distance between its legs as the third side of a virtual triangle.

In other words, the legs of a compass form a virtual line on a plane, identified by its endpoints. The virtual line of a compass forms the third side of a virtual triangle, and its two legs form the other two sides.

One leg of the compass has a sharp point at its end, which lets the compass fix its location as one endpoint of the virtual line. The other leg has a pencil at its tip, which marks the other endpoint.

Since a compass is able to maintain the angle between its legs, it can rotate the leg with a pencil around its leg with a sharp point, which maintains its fixed position so that the leg with a pencil draws an arc that maintains a fixed distance from the fixed endpoint, which is equal to the length of its virtual line. In other words, a compass is able to rotate its virtual line around its fixed endpoint.

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If a compass rotates its virtual line around its fixed endpoint by three hundred and sixty degrees, the arc it draws intersects with itself to form the circumference of a circle. The fixed endpoint of the virtual line defines the center of a circle, while the length of its virtual line defines the radius of the circle.

Moreover, the pencil of the compass is assumed to be ideal, meaning that it keeps its sharpness and does not break, so that the compass draws the circumference as an arc that is smooth and continuous, without any missing points or segments.

From another perspective, the circumference of the circle forms a closed figure with an interior and exterior, whose boundary distinguishes between whether an element of the plane such as a point lies inside or outside the figure.

For example, the circumference or boundary of a circle distinguishes between whether a point lies inside or outside the circle. A point inside the circle lies a distance away from its center that is less than its radius, while a point outside the circle lies a distance away from its center that is greater than its radius.

The circumference of a circle describes its location in a plane. The location of a circle may also be described by the location of its center and length of its radius, which are used to draw its circumference.

Since every point on the circumference of a circle lies the same distance away from its center, a circle may be described as the boundary of a closed figure where every point on its boundary lies equidistant from a single point inside of it.

A compass may duplicate a circle by rotating the same virtual line around a different point in the plane to construct a circle with the same radius and circumference, but in a different location.

With this in mind, using the compass and straightedge, two circles of any size may be placed along a horizontal line like the x axis of a Cartesian coordinate system to where they intersect with each other at a single point.

This construction of two circles that intersect at a single point may be used to suggest a second proposition, that the distance between the centers of two circles that intersect with each other at single point is equal to the sum of their radii.

Second proposition: the distance between the centers of two circles that intersect with each other at a single point is equal to the sum of their radii.

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To start the proof of this second proposition, it may be observed that since the center of each circle defines a distinct point in a plane, a line may be drawn between the centers of the two circles by using a straightedge and pencil, where the straightedge is assumed to be long enough to cover the distance between the centers of the circles, which serve as endpoints for the line.

The pencil is assumed to be ideal, meaning that it keeps its sharpness and does not break, so that as it moves along the straightedge, it draws a line that is smooth and continuous, without any missing points or segments.

Moreover, the line between the centers of the circles is assumed to remain fixed in length since its endpoints are assumed to remain fixed in location. Plane geometry assumes that a plane or flat surface remains fixed in size and shape.

In contrast to a line that is bounded by two endpoints, a line of infinite length is unbounded. A line of infinite length is often indicated by a ray, or arrow placed at the tip of one of its endpoints to indicate it continues beyond the endpoint.

A line of infinite length is often assigned a point of reference or origin such as the point of origin of a Cartesian coordinate system, which is commonly used to count intervals or measure distance along a line.

A line possesses two directions by which an observer may see or move along the line. For a line that is bounded by two endpoints, which may be called A and B, an observer at point A sees the line move toward B and away from A, while an observer at point B sees the line move toward A and away from B.

For a line that is unbounded, its directions are typically given with respect to a point of origin that lies along the line. For example, for a horizontal line, a point of origin divides the line into the directions of right and left. For a vertical line, a point of origin divides the line into the directions of up and down.

A line possesses the property that another line drawn between a pair of points that lie along the line will lie within its original endpoints or boundary so that a line is reflexive in its identity.

In other words, a subdivision or subset of a line that is identified by a pair of endpoints that lie within its boundary or two endpoints creates a line segment that is shorter than the original line, but possesses its properties of being smooth and continuous, and has the same slope.

On the other hand, a line may have its boundary increased by placing a straightedge between a pair of points that lie along it, and drawing the line along the straightedge past one of its two endpoints to increase its length. This creates an extension that intersects the line at one of its endpoints in a smooth, manner, and possesses the same slope.

In other words, since a line is reflexive in its identity, an extension of the line that is constructed by using two points that lie along it retains its properties of being smooth and continuous, and possesses the same slope. A line may also have its length increased by using a longer straightedge to redraw the line beyond its original endpoints.

Since a pencil draws a line by moving along a straightedge in a smooth, continuous manner, every point and segment of the line lies along the straightedge, which gives it a uniform slope.

The uniform slope of a line enables an observer to view or see along the line without having his view obstructed by a hill or valley. A line of sight lets an observer see past an endpoint of a line by following an actual or hypothetical extension of the line, which possesses the same slope.

In terms of a Cartesian coordinate system, a pencil moving along a straightedge draws a line to where segment of the line possesses the same slope and intercept just as a compass draws the circumference of a circle to where every point on the circumference lies the same distance away from its center.

Since a line may be drawn between the centers of two circles, from any point on the line an observer may view or see the center of each circle by looking in its direction along the line, which follows a line of radius for each circle.

A line of radius may be defined as a line whose endpoints are the center of a circle and a point on its circumference so that its length equal to the radius of the circle. Any line that starts from the center of a circle follows a line of radius for that circle since the line starts from the center of the circle and moves toward its circumference, which encompasses every direction going around the center of the circle.

A line of radius possesses two directions where an observer on the line will either look inward to the center of the circle or outward to its circumference.

From another perspective, every point inside a circle lies along a line of radius since, as a compass rotates its leg with a pencil that draws the circumference of the circle, its virtual line sweeps out the interior of the circle.

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With this in mind, a line that is drawn between the centers of two circles follows a line of radius for each circle since it starts from the center of each circle. Moreover, each line of radius that the line follows possesses the same slope as the other since the entire line possesses a fixed or uniform slope.

In other words, a line that connects the centers of two circles may be viewed as incorporating a line of radius from each circle that has the same slope since the entire line possesses a fixed or uniform slope.

If the distance between the centers of the two circles is greater than the sum of their radii, the line that connects their centers covers the intervening distance by using an extension of a line of radius from one of the circles. If the distance between the centers of the circles is less than the sum of their radii, their lines of radius overlap.

If two circles intersect with each other at a single point, or there is some intervening distance between them, the line that connects their centers traverses or crosses the full length of a line of radius from each circle in order to reach its circumference.

With this in mind, if two circles intersect with each other at a single point, or there is some intervening distance between them, there exists a point of crossing lying on the circumference of each circle that identifies the specific point through which the line that connects their centers crosses the circumference of the circle.

If two circles intersect with each other at a single point, the line that connects their centers goes through their point of intersection where it lies on the circumference of each circle. In other words, their point of intersection is able to serve as the point of crossing for each circle since it alone satisfies the criterion for continuity as a point that forms a smooth, continuous bridge between the circumferences of the circles.

In other words, the single point of intersection between the two circles provides the only pair of crossing points between the two circles that forms a smooth, continuous junction between the circles, letting their crossing points to sit along a smooth, continuous line that connects their centers.

Any other pair of crossing points would introduce a discontinuity into the line that connects the centers of the circles since it would use a line of radius from each circle that do not join together at their endpoints like the lines of radius represented at the single point of intersection between the two circles.

Since a line that connects the centers of two circles that intersect with each other at a single point is smooth and continuous, as the line starts from the center of one circle, it travels a distance that is equal to its radius in order to reach its first point of crossing, which lies on the circumference of the circle.

As the line reaches the first point of crossing, it reaches the second point of crossing that lies on the circumference of the other circle since the points of crossing overlap, and from that point the line travels an additional distance that is equal to the radius of the other circle in order to reach its center.

With this in mind, the line travels a total distance that is equal to the radius of the first circle, added to the radius of the other or second circle, or the sum of their radii. In other words, a line that connects the centers of two circles that intersect with each other at a single point has a length that is equal to the sum of their radii.

Moreover, this distance of the sum of the radii of two circles that intersect with each other at a single point represents the shortest distance between the centers of the circles since the radius of a circle represents the shortest distance between its center and circumference.

In other words, as measured along a line that starts from the center of a circle, any distance that is shorter than the radius of the circle falls within its interior since the radius of a circle measures the distance between its center and circumference.

With this in mind, the sum of the radii of two circles that intersect with each other at a single point represents the shortest distances between the centers of the circles since their point of intersection lies on the circumference of each circle, where it represents the endpoint of a line of radius from that circle, and the shortest distance between its circumference and center.

From another perspective, the merger of a line of radius with another line of radius at their endpoints where both lines possess the same slope since they lie along the same line, measures the shortest distance between their combined endpoints, or the centers of the two circles.

Since a line that connects the centers of two circles that intersect with each other at a single point has a length that is equal to the sum of their radii, and this distance represents the shortest distance between their centers, a straight line measures the shortest distance between two points in a plane as a proof by substitution.

In other words, if two circles intersect with each other at a single point, the endpoints of the line that connects their centers may be viewed as two distinct points in a plane so that a line that is drawn between the centers of the circles represent the shortest distance between the centers as two points in a plane.

In other words, by substituting for two points in a plane the centers of two circles, which have their radii chosen to where they intersect with each other at a single point, it may be shown that a line that connects the two points has a length that is equal to the sum of the radii of the circles, and this distance represents the shortest distance between the centers of the circles as two points in a plane.

In summary, the compass shows how a straightedge draws a line with the shortest distance between two points in a plane. Since a straight line is often assumed to measure the shortest distance between two points in a plane, this theorem may be treated as a fundamental theorem of geometry.

SECOND PERSPECTIVE

(If two circles intersect with each other at a single point, their point of intersection may be used as a common endpoint to draw two lines independent of each other, and possess the same slope while they reach the center of each circle as their other endpoint so that they form a single line with a length equal to the sum of the radii of the circles.)

If two circles intersect with each other at a single point, their point of intersection may be used as a common endpoint to construct two lines independently of each other.

From the point of intersection, one line may have its other endpoint chosen to be the center of one of the two circles, making it a line of radius for that circle, while the other line may have its other endpoint chosen to be the center of the other circle, making it a line of radius for the other circle.

In other words, if two circles intersect with each other at a single point, from their point of intersection that lies on the circumference of each circle, two independent lines may be drawn that reach to the center of each circle as its other endpoint, making each line a line of radius for each circle.

Moreover, each line of radius possesses the same slope since they possess a common endpoint at the point of intersection between the circumferences of the circles. While from their point of intersection, the lines may be seen as pointing in opposite directions toward the center of each circle, a line moves freely between its endpoints.

From another perspective, since each point on the circumference of a circle may be associated with a specific line of radius that has its own slope, if two circles intersect with each other at a single point, their point of intersection identifies a line of radius from each circle that possesses the same slope.

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With this in mind, if two circles intersect with each other a single point, each line of radius that is represented at their point of intersection has the same slope. As a result, by using their point of intersection as a common endpoint to draw a line that reaches to the center of each circle constructs a single line between the centers of the circles that is smooth and continuous, and possesses a uniform slope.

The two lines merge at their common endpoint to form a single line that is smooth and continuous, and possesses a uniform slope with a length that is equal to the sum of the radii of the circles.

Using this construction, the proposition that the distance between the centers of two circles that intersect with each other at a single point is equal to the sum of their radii may be considered proven.

From another perspective, if two circles intersect with each other at a single point, their point of intersection aligns a line of radius from each circle that possesses the same slope to where they intersect at their endpoints, forming a natural bridge or straight line that goes from the center of one circle to the center of the other circle.

Since this construction joins together a line of radius from each circle that has the same slope, it forms a single line that is smooth and continuous and possesses a uniform slope, just as the line that connects the centers of two circles may be drawn in a smooth, continuous manner along a straightedge. The two lines are identical.

Any other pair of crossing points than the point of intersection between the two circles requires the introduction of a line segment between the circumferences of the circles, so that the line becomes disjointed, and having a length that is longer than the sum of the radii of the circles.

THIRD PERSPECTIVE

(If two circles intersect with each other at a single point, their point of intersection may be used as an endpoint to extend a line of radius from one circle into the center of other circle with a length that is equal to the radius of the other circle.)

(From this extended endpoint, the compass may be used to construct a new circle that replicates the other circle in both size and location to show that the distance between the centers of the two circles is equal to the sum of their radii.)

If two circles intersect with each other at a single point, from their point of intersection a line of radius from one circle may be extended into the other circle where the extension is given a length that is equal to the radius of the other circle by using an extension of the line of radius from the first circle so the extended line maintains a uniform slope.

From the endpoint of this extension, which has a length that is equal to the radius of the other circle, a new circle may be constructed by rotating the compass around the endpoint of the extended line, where the length of the virtual line that the compass rotates may be set equal to the radius of the other circle so that the compass draws a new circle that replicates the other circle in size and location.

This new circle replicates the other circle in size and location since its radius is equal in length to the radius of the other circle, and its circumference intersects the circumference of the first circle at the original point of intersection between the circles.

In other words, this new circle replicates the other circle in location since the original point of intersection between the two circles is its only point of intersection with the first circle, so that it satisfies the sole condition of uniqueness the other circle had with the first circle in their single point of intersection.

Since this new circle was constructed by extending a line of radius from the first circle with a length that is equal to the radius of the other circle, the total length of the extended line of radius is equal to the radius of the first circle added to the radius of the second circle, or the sum of their radii.

With this in mind, the second proposition that the distance between the centers of two circles that intersect with each other at a single point is equal to the sum of their radii may be considered proven, and used to show that a straight line measures the shortest distance between two points in a plane.

DISCUSSION

(This proof may be extended to other types of surfaces that are smooth and continuous, and fixed in size and shape if the virtual line of a compass is able to follow, or conform to the shape of the surface.)

Since the rotation of a line of fixed length around a point in plane, which defines the center of a circle, gives a way to find the shortest distance from that point to another point in the plane, since the rotation encompasses every direction going around that point, the proof may be extended to other types of surfaces that resemble a plane by being smooth and continuous, and fixed in size and shape.

This proof may be extended as long as the compass may be rotated from a fixed point on the surface, its virtual line is able to conform to the shape of the surface, and the rotation intersects with itself to form a closed figure, which encompasses every direction going around its point of rotation.

For example, a line of fixed length may be rotated around a point on a gently curved surface like a large sphere to find the shortest distance from that point to another point on the surface. This principle is illustrated by the use of Great Circle or Polar Routes by ships or aircraft that travel long distances over the Earth's surface, which is nearly spherical in shape.

In other words, the compass may be used on surfaces that are gently curved, such as a globe, which gives a spherical representation of the Earth's surface, to find the shortest distance between two points on it as long as the virtual line of the compass is able to conform to the shape of the surface.

From the viewpoint of a cartographer, who systematically projects the curved surface of the Earth onto a flat piece of paper by using some type of scale to compress its distances, Great Circle routes tend to curve slightly from a straight line that is drawn between two points on a map since they reflect the gentle curvature of the Earth's surface.

Cartographers or map makers have devised various projections of the Earth's curved surface onto a two dimensional map that project accuracy of shape or area, or balance the two, or project an accurate shape but seem to grow in size as the map approaches the Earth's poles like the Mercator projection.

From another perspective, using a straightedge to draw a line between two points is like using a piece of string to measure the distance between two points on a surface that may be curved, but is otherwise smooth and continuous and fixed in size and shape.

The string is analogous to a straight line as long as it is drawn tightly enough between the two points so that it conforms to the shape of the surface, it does not fold or break so that it keeps its smoothness and continuity, and its length remains fixed.

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