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THE STOCHASTIC INTEGRAL AS A PEDAGOGIC TOOL IN FINANCE

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Abstract: *The stochastic integral is expressly designed as a tool for financial modeling and it is now the backbone of a large body of academic teaching and research on asset pricing, corporate finance and investment behavior. The note offers an outline of the nature of the subject along with a brief exposition of why it is so. This lecture note will serve as the basis for asset pricing at the graduate class. It should be accessible even to advanced undergraduates. The note also at the end contains several exercises without solution for the practitioners. This note is written with a certain mathematical rigor because of the complexity of the material.*

Keywords: Rieman - Stieltjes Integral, Brownian motion , Martingale, Stochastic Integral Black- Scholes Option Price Theory

1.0 The Stochastic Integral as a Model in Finance

In standard calculus and ordinary differential equations, a central object of study is the derivative df / dt of a function $f(t)$. If f is a function and X is continuous, then the Riemann-Stieltjes integral $\int_0^t X\, df$ is well defined. The Lebesgue - Stieltjes integral can then be generalized by measurable integrands. However, this process is much less well behaved. For example, with probability one, the sample paths of standard Brownian motion are nowhere differentiable. Furthermore, they have infinite variation over bounded time intervals. Consequently, if ξ is such a process, then the integral $\int_0^t \zeta\, dX$ can not defined using the standard methods[1].The typical stochastic calculus-based financial model describes the random variation of the market price, say X, at time t , of some financial asset. For proper formulations, one fixes a probability space ($\Omega, \Re, P$) (a measure space with $p(\Omega) = 1$) as well as a filtration $\{ \Re_t : t \in [0, \propto] \}$ of sub-- σ – algebras of $\Re$ that determines the timing Of the revelation of information. The 'usual condition' for a filtration is laid out by Protter (2004). One may loosely view $\Re_t$ as the set of events (elements of $\Re$) whose outcomes are certain to be revealed to investigate as true or false by or at time t. For any event A the probability assigned to A by ors is P (A). The price process

[1] In finance often a more direct approach is taken. The idea is that we simply define the stochastic integral such that the required elementary properties are satisfied. That is, it should agree with the explicit expressions for certain simple integrands, and should satisfy the bounded and dominated convergence theorems. Much of the theory of stochastic calculus follows directly from these properties, and detailed constructions of the integral are not required for many practical applications. Note that, whereas the value of a standard Lebesgue integral is just a real number, stochastic integrals take values in the space of random variables. It is therefore possible to weaken some of the properties required of such integrals. First, any identity is only required to be satisfied almost surely. That is, on a set of probability one. Second, the notion of convergence of a sequence of real numbers can be replaced by the much weaker idea of convergence in probability.

X={ X_t : $t \in [0, \infty]$} is adapted[2] to the filtration, meaning that $X_t : \Omega \rightarrow \Xi$ is a random number whose outcome is revealed to investigate at or before time t[3].

Occasionally the market efficiency is meant to imply that the price process *X* must meet a martingale[4], meaning essentially that the current price X_t is a conditionally unbiased predictor of the price X_u at any future time *u*. This is a misconception. Investors would generally not take the risk of owning an asset unless they are compensated by an expected returns. Beyond a compensation for risk, even a risk-free asset must offer a certain return that compensates the investor for tying up capital. Allowing for nonzero expected changes, it is, therefore, natural to treat the price process *X* as, loosely speaking, a 'martingale plus something' or, to pick a precise and a natural definition, a semi martingale.[5] The most classical example of a semi martingale used in financial modeling is a geometric Brownian motion as given by Paul Samuelson (1965).

The stochastic integral should satisfy bounded convergence in probability. That is, if ξ^n is a sequence of predictable processes converging to a limit ξ, and is uniformly bounded $|\xi^n \le k$ or some constant $K > 0$, then the integrals converge. These properties are enough to define stochastic integration for bounded and predictable integrands

$$\int_0^t \xi^n \, dX \rightarrow \int_0^t \xi \, dX$$

2.0 A trading strategy

A trading strategy θ determines the quantity $\theta_t(\omega)$ of the asset held in each state $\omega \in \Omega$ and at each time *t*, for a model of a well-functioning market, it is crucial to rule out trading strategies that are based on unlimited profits at no risk. The natural corresponding measurability restriction is that θ is a predictable process. Given price process

[2] A process *X* is adapted if for all t, X_t is $\Re_t$ - measurable.

[3] Stochastic integration with respect to standard Brownian motion was developed by Kiyoshi Ito. This required restricting the class of possible integrands to be adapted processes, and the integral can then be constructed using the Ito isometric; see Ito (1944. Ito laid the foundation for stochastic calculus with his model of a stochastic process *X* that solves a stochastic differential equation of the form: $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s$ where *B* is a standard Brownian motion and μ and σ are functions from Ξ to Ξ satisfying some technical conditions. (A Lipchitz condition suffices). As a generalization of it, Ito has generalized a class of processes of this form $X_t = X_0 + \int_0^t H_s\,ds + \int_0^t V_s\,dB_s$ for adopted process *H* and *V* satisfying suitable technical conditions. Such a process is now called the 'Ito Process', a special case of what later became known as a semi martingale.

[4] A martingale is an integrable adapted process *M* whose conditional change expected change $E(M_u - M_t \mid \Re)$ is zero whenever $u \ge t$, see Harrison and Pliska (1981), Harrison and Kreps (1979), Das (2011)

[5] A semi-martingale is defined as the sum $M + A$ of local martingale M, a slight relaxation of a martingale and adopted process A whose sample paths have finite variation on each bounded time integral.

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X and a trading strategy θ satisfying some technical conditions, the total financial gain $\int \theta_u \, dX_u$ between any two times s and $t \geq s$ as defined as stochastic integral[6].

***Definition* 1**

Let X be a process. The stochastic integral up to time $t > 0$ with respect to X, if it exists, is a map

$$Bp \rightarrow L^0 \quad , \qquad \xi \leftrightarrow \int_0^t \xi \, dX$$

Which agrees with this explicit expression for bounded elementary integrands ξ, satisfies bounded convergence in probability.

Proving the existence of the stochastic integral for an arbitrary integrator X is, in general, quite a difficult problem. However, uniqueness is a simple consequence of the monotone class theorem. Also, note that the requirement that the integral is a linear function of the integrand was not mentioned in the definition above. However, this property is again a simple consequence of the monotone class theorem.

***Lemma* 1**

Let X be a stochastic process. If the stochastic integral up to time $t > 0$ as given by Definition 1 exists, then it is uniquely defined. Furthermore, linearity in the integrand is satisfied.

Definition 2

A semi martingale $\boldsymbol{X}$ is a cad lag adapted process[7] such that, for each $t > 0$, the stochastic integral given by *Definition* 1 exists.

An elemental type of trading strategy is a 'buy and hold' strategy θ which indicates a position immediately after some stopping time T and close it at some later time stopping type U. For a position size $\overline{\theta}$ that is $\Re_T$ -measurable, the trading strategy θ is defined by $\theta_t = 1_{\{T \angle t \angle U}\,\overline{\theta}$. The total gain from trade for this 'buy and hold' strategy is naturally $\int_0^U \theta_t dX_t = \overline{\theta}\,(X_U - X_T)$ the position size multiplied by the interim price change. The gain from trade for a general stochastic trading strategy can be defined as the total gain of an approximating portfolio of 'buy and hold' strategies, in a particular limiting sense[8]. For the cases most commonly encountered in financial applications, based on Brownian motion, Shreve (2004) gives a clear explanation of this limit.

A typical financial model allows for n different securities, with price processes $X \ldots. \quad X_n$ An investor can choose an associated n –dimensional trading strategy $\theta = (\theta_1, .. \quad \theta_n)$ from some allowable set ψ determining the total gain from trade process

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[6] For general settings, minimal restrictions on the trading strategy θ and the price process X for $\int \theta_t \, dx_t$ to be a well-defined stochastic integral in Protter (2004)

[7] A cadlag function is a function, defined on $\mathbb{R}$ or a subset of $\mathbb{R}$, that is right continuous and has a left limit. The acronym *Cadlag* comes from the French "continue à droite limit à gauche," which translates to the English "right-continuous with left limits" (sometimes abbreviated "RCLL"). All continuous functions are "cadlag" See Davidson (1994).

[8] See Protter (2004) for the case of semi-martingale

$$\int \theta_t \, dX_t = \sum_{i-1}^{n} \int \theta_{it} \, dX_{it}$$

In addition to incorporating technical restrictions under which these stochastic integrals are well defined, the allowable set Ψ can enforce budget limits, credit constraints, short-sales limitations or various other natural investment restrictions. Significant strands of research literature address the following two classes of problems:

- Given some utility functional U on the space of potential gains from trade, solve the optimization problem $sup_{\theta \in \psi} \, U \, (\int_0^t \theta_s \, dX_s)$. The utility functional U among other properties can encode preferences regarding risk, inter temporal substitution and the timing of information about trading gains.

- Apply the laws of supply and demand to characterize the price processes of the available financial securities. A minimal restriction on the behavior of prices is the absence of arbitrages; demand and supply could never be matched in the presence of arbitrage.[9]

The field of finance is replete with many other applications of stochastic calculus such as the financial policies of corporations, the design of securities and risk management, which usually involves control of characteristics of the left tail of the probability distribution of the gain from trade $\int_0^t \theta_s \, dX_s$

3.0 Asset Pricing: Black - Scholes Theory

In its simplest form, the Black- Scholes (- Merton) model (1973) involves only two underlying assets, a riskless asset, cash bond and a risky asset, stock. The asset cash bond appropriates at the short rate or riskless rate of return, *r* which (at least for now) is assumed to be non random, although possibly time-varying. Thus the price of the cash B bond at time *t* is assumed to satisfy the differential equation.

$$\frac{d\,B_t}{dt} = r_t \, B_t$$

Where unique solution for the value $B_0 = 1$ is

$$B_t = exp \int_0^t r_s \, d\,s$$

The share price S_t of the risky asset stock at time *t* is assumed to follow a stochastic differential equation of the form

$$dS_t = \mu_t \, S_t \, dt + \sigma S_t \, d\,\omega_t \text{ where } \omega_t \}_t \geq 0$$

where μ_t is a non random (but not necessarily constant) function of t and $\sigma > 0$ a constant called the volatility of the stock[10].

[9] Karatzas and Shreve (2005) Shreve's work (2005) focuses on arbitrage free asset pricing perhaps because most of the readers for whom the book is intended are aiming for business careers in finance. The theory of optimal investment, while a significant subject area in academia, has achieved much less traction in business practice. Shreve and others provide a detailed treatment of mathematical models of optimal investment.

[10] The Black-Scholes model can be derived in all detail from a special portfolio called the delta hedge, Black-Scholes

Proposition 1

If the drift coefficient function μ_t is bounded[11], then the stochastic differential equation has a unique solution with initial condition S_0 and is given by

$$S_t = S_0 \; exp \; (\sigma \omega_t - \sigma^2 (t/2) + \int_0^t \mu_s \, d s)$$

Moreover, under the risk-neutral measure, it must be the case that

$$r_t = \mu_t$$

Corollary 1

Under the risk neutral measure the log of the discounted stock price at time *t* is normally distributed with mean $\log S_0 - \sigma^2 \, t/2$ and variance $\sigma^2 T$

Exercises

Exercise 1

Assume that the value of an index *X* follows geometric Brownian motion with drift σ An asset *V* promises that when X reaches *Q*, the bearer will be paid R and the asset will be retired The economy is risk neutral and the risk-free discount rate is *r* .

a. What is the value of the asset? [Hint: $V = A X^{\gamma}$ where *A* and γ are constants. Also the boundary condition requires $V(Q) = R$

b. What are the sufficient conditions for $\gamma > 0$?

Exercise 2

Refer to page 3, footnote 3.

a. Establish that for any smooth function $f : \Xi \rightarrow \Xi$

$$f (X_t) = f (X_0) + \int_0^t (f'(X_s) H_s + \frac{1}{2} f''(X_s) V_s^2) d S + \int_0^t f'(X_s) V_s \, dB_s$$

This is actually the Ito's formula[12]

(1973). Black and Scholes produced two separate proofs one from the delta hedge and the other via CAPM. The CAPM assumes a different risk-reduction strategy.

[11] If exact valuations, as in exotic options are usually difficult, it is always useful to narrow the searching of the prices by placing boundary restrictions. Exotic options are very powerful and widely used tools of risk management. However, because of the complexity of those contracts, the basic characteristics of many exotic options are not clearly understood, or are even misunderstood. See Ye (2009), Fackler (2000)

[12] Jarrow and Protter (2004) offer a colorful and ex excellent but complete history of these of this type of development up to the time of modern financial theory.

Exercise 3

Assume X follows geometric Brownian motion, with drift α and volatility σ. Assume Y follows geometric Brownian motion with drift β and volatility λ. The correlation between the Wiener[13] components of the two processes is ρ; $d z_x dz_y = \rho dt$

a Let $V = XY$ What process does V follow? Define your process 9 i.e., define α_V, σ_V and dz_V so that it can be written as $dV/V = \alpha_V dt + \sigma_V dZ_V$

4.0 Remarks

I operated within the conceptual apparatus of the modern theory of asset pricing. This note is based largely on my own previous published contributions and which can be seen as a synthesis of work accomplished over the last 15 years. In spite of a technical presentation, this work may be difficult, however, and it might require a certain intellectual investment from students...

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[13] Norbert Wiener (1923) first showed the existence of such a process in 1923

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