European Journal of Statistics and Probability

Vol.3, No.4, pp.10-19, December 2015

Published by European Centre for Research Training and Development UK (www.eajournals.org)

STEPS PROBLEM: THE LINK BETWEEN COMBINATORIC AND K-BONACCI SEQUENCES.

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ABSTRACT: *In this paper we generalized the combinatorial formula for k-bonacci numbers where k is a positive integer by using the well-known step problem. We solved this problem using the combinatorial analysis. We established the relationship between the combinatorial analysis and the famous Fibonacci sequences. We extended our analysis to the case of taking 1 or 2 or 3 or 4 up to k steps at a time, which allowed us to derive a combinatorial equivalent of k-bonacci numbers*

KEYWORDS: Sequences, K-bonacci, Combinatoric.

INTRODUCTION

A well-known combinatorial formula for the Fibonacci numbers F_n, defined by $F_0 = 0,\ F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$, is $F_{n+1} = \sum_{i=0}^{\frac{n}{2}} \binom{n-i}{i}$ for $n \geq 0$. Anand (2000) . In this paper we generalized this combinatorial formula for k-bonacci numbers where k is a positive integer by using the well-known step problem.

Basic Definitions

Fibonacci sequence: is the sequence in which any term of the Sequence is the sum of the two preceding terms. e.g 1,1,2,3,5,8…..

K-bonacci sequence: is the generalization of the Fibonacci sequence in which any term of the sequence is the sum of the k preceding terms.

Example:	$k - bonacci\ sequence$	
	$2 - bonacci$	$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$ ….
	$3 - bonacci$	$1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504$ ….
	$4 - bonacci$	$1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773$ ….
	$5 - bonacci$	$1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912$ ….
	$6 - bonacci$	$1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976$ ….

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

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$Doubling$ $1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024$ ....

$k = infinite$

Fibonacci number. A number x is a Fibonacci number if and only if one or both of $5x^2 + 4$ or $5x^2 - 4$ is a perfect square

Combination: is the number of ways of selecting r objects from n different objects without order

RELATED WORKS

Zeynep Akyuz (2013) in the paper, On some combinatorial identities involving the terms of generalized fibonacci and lucas sequences, considered the Horadam sequence and some summation formulas involving the terms of the Horadam sequence. They derived combinatorial identities by using the trace, the determinant, and the nth power of a special matrix

Arthur T. Benjamin, Alex K. Eustis, and Sean S. Plott (2008). In "The 99th Fibonacci Identity"

provided elementary combinatorial proofs of several Fibonacci and Lucas number identities left open in the book Proofs That Really Count , and generalized these to Gibonacci sequences G_n that satisfy the Fibonacci recurrence, but with arbitrary real initial conditions. They offered several new identities as well.

The Pascal matrix and the Stirling matrices of the first kind and the second kind obtained from

the Fibonacci matrix were studied, respectively by Gwang-Yeon Leea(2003). Also, they obtained combinatorial identities from the matrix representation of the Pascal matrix, the Stirling matrices of the first kind and the second kind and the Fibonacci matrix.

The step problem

There is a flight of n steps. A Person standing at the bottom climbs either 1 or 2 steps at a time. In how many ways can he get to the top.. Shmuel (1989). We solved this problem using the combinatorial analysis. We established the relationship between the combinatorial analysis and the famous Fibonacci sequences. We extended our analysis to the case of taking 1 or 2 or 3 or 4 up to k steps at a time, which allowed us to derive a combinatorial equivalent of k-bonacci numbers

Solution of the step Problem.

Let n_1, n_2 be numbers of times to take one step or two steps respectively then we have the following table:

For 1 step,

	n_1	n_2	$n_1 + n_2$	$\frac{(n_1 + n_2)!}{n_1!\,n_2!}$
	1	0	1	1
			Total nos of ways F_1^2	1

For 2 steps

	n_1	n_2	$n_1 + n_2$	$\frac{(n_1 + n_2)!}{n_1!\,n_2!}$
	2	0	2	1
	0	1	1	1
			Total nos of ways F_2^2	2

For 3 steps

	n_1	n_2	$n_1 + n_2$	$\frac{(n_1 + n_2)!}{n_1!\,n_2!}$
	3	0	3	1
	1	1	2	2
			Total nos of ways F_3^2	3

For 4 steps

	n_1	n_2	$n_1 + n_2$	$\frac{(n_1 + n_2)!}{n_1!\,n_2!}$
	4	0	4	1
	2	1	3	3
	0	2	2	1
			Total nos of ways F_4^2	5

For 5 steps

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

	n_1	n_2	$n_1 + n_2$	$\frac{(n_1+n_2)!}{n_1!\,n_2!}$
	5	0	5	1
	3	1	4	4
	1	2	3	3
			Total nos of ways F_5^2	8

For 6 steps

	n_1	n_2	$n_1 + n_2$	$\frac{(n_1+n_2)!}{n_1!\,n_2!}$
	6	0	6	1
	4	1	5	5
	2	2	4	6
	0	3	3	1
			Total nos of ways F_6^2	13

If we continue this way it gives a sequence of the total number of ways to be:

$$1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \ldots ....$$

This sequence is the famous Fibonacci sequence $F_n^2 = F_{n-1}^2 + F_{n-2}^2$ for $n > 2$. with the close form solution $F_n^2 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$ (see Carl (2003)) Thus we can write a Fibonacci number as the summation of $\frac{(n_1+n_2)!}{n_1!n_2!}$.

Theorem 1

$$\frac{(n_1+n_2)!}{n_1!n_2!} = \binom{n_1+n_2}{n_1} = \binom{n-i}{i}, \quad i = 0,1,2 \ldots$$

Proof

We prove for even n only, the proof for odd n is similar.

For n stairs taking 1 or 2 steps at a time, where n_1 and n_2 are numbers of one and two steps respectively we have

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

Table 1

n_1	n_2	n_1+n_2	$\frac{(n_1+n_2)!}{n_1!\,n_2!}$	$\binom{n-i}{i}$
n	0	n	1	$\binom{n-0}{0}$
$n-2$	1	$n-1$	$\frac{(n-1)}{1}$	$\binom{n-1}{1}$
$n-4$	2	$n-2$	$\frac{(n-2)(n-3)}{2!}$	$\binom{n-2}{2}$
$n-6$	3	$n-3$	$\frac{(n-3)(n-4)(n-5)}{3!}$	$\binom{n-3}{3}$
$n-8$	4	$n-4$	$\frac{(n-4)(n-5)(n-6)(n-7)}{4!}$	$\binom{n-4}{4}$
$n-10$	5	$n-5$	$\frac{(n-5)(n-6)(n-7)(n-8)(n-9)}{5!}$	$\binom{n-5}{5}$
$n-12$	6	$n-6$	$\frac{(n-6)(n-7)(n-8)(n-9)(n-10)(n-1}{6!}$	$\binom{n-6}{6}$
. . .	. . .	. . .	. . .	. . .

The summation of the last Column of table 1 gives,

$$T(n)=\binom{n-0}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\binom{n-3}{3}+\binom{n-4}{4}+\binom{n-5}{5}+\binom{n-6}{6}+\cdots$$

From table 1 , it is clear that for even numbers of steps $0\le i\le \frac{n}{2}$ i.e,

$T(n)=\sum_r^{\frac{n}{2}}\binom{n-i}{i}$, for even n and $T(n)=\sum_r^{\frac{n-1}{2}}\binom{n-i}{i}$ for odd n

Extension to k-bonacci sequences

With a little modification of the step problem where one or two or three steps can be taken, the tribonacci sequence can be easily derived using the same analysis as that of one or two steps

Letting n_1, n_2, n_3 be the number of one, two, and three steps respectively, we have the following tables

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n_1	n_2	n_3	$n_1+n_2+n_3$	$\frac{(n_1+n_2+n_3)!}{n_1!n_2!}$
1	0	0	1	1
			Total nos of ways F_1^3	1

For 2 steps

n_1	n_2	n_3	$n_1+n_2+n_3$	$\frac{(n_1+n_2+n_3)!}{n_1!n_2!}$
2	0	0	2	1
0	1	0	1	1
			Total nos of ways F_2^3	2

For 3 steps

n_1	n_2	n_3	$n_1+n_2+n_3$	$\frac{(n_1+n_2+n_3)!}{n_1!n_2!}$
3	0	0	3	1
1	1	0	2	2
0	0	1	1	1
			Total nos of ways F_3^3	4

For 4 steps

n_1	n_2	n_3	$n_1+n_2+n_3$	$\frac{(n_1+n_2+n_3)!}{n_1!n_2!}$
4	0	0	4	1
2	1	0	3	3
1	0	1	2	2
0	2	0	2	1
			Total nos of ways F_4^3	7

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

For 5 steps

n_1	n_2	n_3	$n_1+n_2+n_3$	$\frac{(n_1+n_2+n_3)!}{n_1!\,n_2!}$
5	0	0	5	1
3	1	0	4	4
2	0	1	3	3
1	2	0	3	3
0	1	1	2	2
			Total nos of ways F_5^3	13

From the arrangement of the total sums $\frac{(n_1+n_2+n_3)!}{n_1!n_2!}$, we have the sequence, $1,\ 2,\ 4,\ 7,\ 13, \ldots \ldots$

This sequence is called the tribonacci sequence (Jay (2002)), with the property that any term of the sequence is the sum of the 3 preceding terms.. Another way round this is explained below.

Since the first move is either 1, 2, or 3 steps, leaving $n-1, n-2, or\ n-3$ steps, hence:

$$F_n^3 = S_n = S_{(n-1)} + S_{(n-2)} + S_{(n-3)} \quad n > 3 \quad \text{(Tribonacci Sequence)}$$

So, after generating S_1, S_2, and S_3 as in the table above, one can use the recurrence S_n to generate the remaining values to any required point.

Example

$S_1 = 1,\ S_2 = 2,\ S_3 = 4$, then

$$S_4 = S_3 + S_2 + S_1 = 4 + 2 + 1 = 7$$

$$S_5 = S_4 + S_3 + S_2 = 7 + 4 + 2 = 13$$

From the example above, the following theorem can be established

Theorem 2

n steps can be climbed in 1 or 2 or 3 or 4 up to k ways in k-bonacci number of ways

Proof

The first move is either $1, 2, 3, 4, \ldots k$ step, leaving $n-1, n-2, n-3, n-4, \ldots\ n-k$ steps, hence

$$S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + \cdots + S_{n-k} \quad n > k$$. This is k-bonacci number, then,

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$F_n^k = S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + \cdots + S_{n-k} \quad n > k$ ■

Theorem 3 (Trinomial coefficient of order n)

If $n, n_1, n_2, n_3 \in N$ and $n_1 + n_2 + n_3 \leq n$, then $\binom{n}{n_1, n_2, n_3} = \frac{n!}{n_1!, n_2! n_3!}$

Proof

$$\binom{n}{n_1, n_2, n_k} = \binom{n}{n - n_1}\binom{n - n_1}{n_2} = \frac{n!}{n_1!, n_2!\, n_3!} \quad ■$$

Using theorem 3, we have $\frac{(n_1 +, n_2, + n_k)!}{n_1!, n_2! n_3!} = \binom{n}{, n_1, n_2, n_3}$ where (n_1, n_2, n_3) is a weak composition of (n)

Thus a tribonacci number can be written uniquely as $\sum_{\substack{n_1 +, n_2, + n_k \leq n \\ n_i \in n}} \binom{n}{, n_1, n_2, n_3}$.

$$F_n^3 = \sum_{\substack{n_1 +, n_2, + n_k \leq n \\ n_i \in n}} \binom{n}{, n_1, n_2, n_3}$$

Theorem 4 (k-nomial coefficients of order n)

For all $n \in \mathbb{N}$, $all\ k \in +\mathbb{N}$, and for every weak composition $(n_1, n_2, \ldots \ldots, n_k)$ of n with k parts, the k-nomial coefficients of order n, $\binom{n}{n_1, n_2, \ldots \ldots, n_k} = \frac{n!}{n_1!, n_2!, \ldots\ldots, n_k!}$ Carl (2003)

Proof

$$\binom{n}{n_1, n_2, \ldots \ldots, n_k} = \binom{n}{n - n_1}\binom{n - n_1}{n_2}\binom{n - n_1 - n_2}{n_3} \ldots \binom{n - n_1 - n_2 - \cdots - n_{k-2}}{n_{k-1}}$$
$$= \frac{n!}{n_1!, n_2!, \ldots \ldots, n_k!} \quad ■$$

Theorem 5

$$F_n^k = \sum_{\substack{n_1 +, n_2, + \cdots + n_k \leq n \\ n_i \in n}} \frac{n!}{n_1!, n_2!, \ldots\ldots, n_k!} = \sum_{\substack{n_1 +, n_2, + \cdots + n_k \leq n \\ n_i \in n}} \binom{n}{, n_1, n_2, \ldots n_k}$$

Proof

Combining theorem 2 and 4, the proof of theorem 5 is obvious. Or by induction, for n number of stair cases taking 1 or 2 steps we have $\sum_{i=0}^{\frac{n+1}{2}} \binom{n - i}{i} = F_n$. Fibonacci number

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

For n number of stair cases taking 1 or 2 or 3 steps at a time we have $F_n^3 = \sum_{\substack{n_1+,n_2,+n_3\leq n \\ n_i \in n}} \binom{n}{n_1, n_2, n_3} = \sum_{\substack{n_1+,n_2,+n_3\leq n \\ n_i \in n}} \frac{n!}{n_1!,n_2!n_3!}$ tribonacci number

For n number of stair cases taking 1 or 2 or 3 or 4 steps at a time we have $F_n^4 = \sum_{\substack{n_1+,n_2,+n_3+n_4\leq n \\ n_i \in n}} \binom{n}{n_1, n_2, n_3, n_4} = \sum_{\substack{n_1+,n_2,+n_3+n_4\leq n \\ n_i \in n}} \frac{n!}{n_1!,n_2!n_3!,n_4!}$ Tetrabonacci number

For n number of stair cases taking 1 or 2 or 3 or 4 or 5 steps at a time we have $F_n^5 = \sum_{\substack{n_1+n_2+,n_3,+n_4+n_5\leq n \\ n_i \in n}} \binom{n}{n_1, n_2, n_3, n_4, n_5} = \sum_{\substack{n_1+n_2+,n_3,+n_4+n_5\leq n \\ n_i \in n}} \frac{n!}{n_1!,n_2!n_3!,n_4!,n_5!}$ 5-bonacci number

Therefore for n number of stair cases taking 1 or 2 or 3 or 4 or 5 up to k steps at a time we have $F_n^k = \sum_{\substack{n_1+,n_2,+\cdots+n_k\leq n \\ n_i \in n}} \frac{n!}{n_1!,n_2!,\ldots\ldots,n_k!} = \sum_{\substack{n_1+,n_2,+\cdots+n_k\leq n \\ n_i \in n}} \binom{n}{,n_1, n_2, \ldots n_k}$ k-bonacci number by using theorem 4 $F_n^k = \sum_{\substack{n_1+,n_2,+\cdots+n_k\leq n \\ n_i \in n}} \frac{n!}{n_1!,n_2!,\ldots\ldots,n_k!} = \sum_{\substack{n_1+,n_2,+\cdots+n_k\leq n \\ n_i \in n}} \binom{n}{,n_1, n_2, \ldots n_k}$ ∎

CONCLUSION

The Fibonacci sequence is merely the beginning of a rich set of relationships associated with the golden mean. K-Bonacci sequences have been shown to represent the result of a family of constrained step problems.

The k-bonacci sequences always appear in many disciplines where you less expect them, such as architecture, Planning, Nature, Computer science, Mathematics, Physics, Building, Engineering, Music, Agriculture, Biology, Botany and so on.

RECOMMENDATION

Many applications and relationship of the Fibonacci numbers to many aspects of nature, disciplines, and formulas are still not discovered. The study of these remarkable numbers must be taken seriously.

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