SOME COMMON MISCONSTRUCTIONS AND MISINTERPRETATIONS IN BASIC ALGEBRA: A CASE OF STUDENTS OF UNIVERSITY FOR DEVELOPMENT STUDIES AND NAVRONGO SENIOR HIGH SCHOOL IN GHANA

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ABSTRACT: *This study investigated some common misconstructions and misinterpretations in basic algebra among students of University for Development Studies (UDS) and Navrongo Senior High School (NAVASCO) in Ghana, with a view to exposing the nature and origin of these errors and making suggestions for classroom teaching. The study employed both quantitative and qualitative approach to data collection process, involving the use of pencil-andpaper tests and interviews. The quantitative data involved a pre-tested test for its validity and reliability given to 50 students. Furthermore, interviews were later organised directly for ten students purposefully selected to identify their misconstructions, misinterpretations and reasoning processes. Data analyses were largely done through descriptive statistics and incorporated elements of inferential statistics such as independent t-test. The main conclusions drawn from this study were attributed to lack of conceptual knowledge and basic understanding of algebra.*

KEYWORDS: Variables, Expressions, Algebra, Misconstruction, Misinterpretation.

INTRODUCTION

As a branch of mathematics, algebra emerged at the end of 16th century in Europe, with the work of François Viète. Algebra can essentially be considered as doing computations similar to those of arithmetic but with non-numerical mathematical objects. However, until the 19th century, algebra consisted essentially of the theory of equations. For example, the fundamental theorem of algebra belongs to the theory of equations and is not, nowadays, considered as belonging to algebra. Algebra is one of the most abstract strands in mathematics. According to Greens and Rubenstein (2008), until relatively recently, the study of algebra was reserved for college-bound students. After a widespread push by The National Council of Teachers of Mathematics (NCTM) and teachers nationwide, algebra is now a required part of most curricula in many countries including in the US and Canada and Ghana as well. However, many attempts to better prepare students for algebra have not resulted in greater achievement in first-year algebra. Students with high intelligent ability are still struggling with algebraic concepts and

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skills (Greens and Rubenstein, 2008). Many are discontinuing their study of higher-level mathematics because of their lack of success in algebra in many schools and colleges in Ghana.

This work was motivated by the fact that, available literature within the scope of the authors showed no evidence of study of this type conducted in Ghana. It is based on this account that this study was carried out among students of tertiary and senior high schools in Kassena Nankana Municipality to ascertain the misconstructions and misinterpretations in algebra with a view to exposing the nature and origin of these errors and misunderstandings. To achieve this goal, the study focused on the nature of students' learning basic concepts by analyzing their errors in solving well-designed problems used to assess those concepts. In exploring such issues, the four algebra concepts, **variables**, **expressions**, **equations** and **word problems** were chosen to analyze students' misconstructions. Through focusing on these four fundamental algebraic concepts could lead to the exposure of some more general difficulties of understanding and learning principles.

To achieve the objectives of this paper, a thorough assessment was done on misconstructions and misunderstandings within students, differences among students when solving algebraic expressions involving variables, equations and word problems and students' problem solving processes and reasoning.

However, in specific terms, the study also seeks to identify the sources of errors in algebra, present the differences in understanding among students when solving algebraic problems as well as assessing the ways of presentations of students' in solving algebraic problem.

LITERATURE REVIEW

Although there are many causes of students' difficulties in learning mathematics, the lack of enough support from research fields for teaching and learning is an important one. If research could characterize students' learning difficulties, it would be possible to design effective instructions to help students learning. As Booth (1988) pointed out (see also Welder, R.M), "one way of trying to find out what makes algebra difficult is to identify the kind of errors students commonly make in algebra and then investigate the reasons for these errors". The research on students' errors and misconstructions is a way to providing such support for both teachers and students. In this paper, we review literature on three aspects: The nature of algebra, Students' problem solving mental strategies and Errors and possible misconstructions in algebra.

Nature of Algebra

There are many conceptions about algebra in literature. It is correct to say many historically developed concepts about algebraic expressions are present in the current secondary and high school algebra curricula throughout the world. This means that the inclusion of algebra in secondary and high schools shows how important this branch of mathematics is and how it is related to other branches of mathematics.

These conceptions are important when selecting algebraic concepts in a test for any secondary and high school students. There are four conceptions in algebra and these are; algebra as generalized arithmetic, algebra as a study of procedures for solving certain kinds of problems,

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algebra considered to be the study of relationship among quantities and finally algebra as a study of structures (Usiskin,1988).

Students' problem solving strategies and mental strategies

Students' construction of knowledge in mathematical problems solving is influenced in their use of strategies as they attempt to master a problem situation. Various stages of the solving process will bring different sets of challenges to them. According to Polya (1957), a problem solving strategy follows a four-phase heuristic process. The stages under this model include: understanding the problem, devising a plan, carrying out the plan and looking back. Furthermore, a student has to strictly follow the outlined stages in order to fully understand the problem and get a correct solution. Polya (1957) advocates a linear type of approach to problem solving strategy.

On the other hand Schoenfeld (1983), devised a model for analysing problem solving that was derived from Polya's model. This model describes a mathematical problem solving in five levels and these include: reading, analysis, exploration, planning/implementation and verification. In applying this framework, Schoenfeld discovered that expert mathematicians returned several times to different heuristics episodes. For instance, in one case a problem solver engaged in the following sequence of heuristics: read, analyse, plan/ implement, verify, analyse, explore, plan/implement and verify. Therefore, Schoenfeld established that the solving model is rather cyclic than linear.

Errors and Misconceptions in Algebra

Research has shown that systematic errors have documented that students hold mini theories about scientific and mathematical ideas. Research has also shown that students have many naïve theories, preconceptions or misconceptions about mathematics that interfere with their learning processes (Posamentier, 1998). Research also shows that because students have actively perceived their misconstructions from their experiences, they are very much attached to them. Similarly, students' errors and misconceptions contribute to the process of learning. Errors and misconceptions do not originate in a consistent conceptual framework based on earlier acquired knowledge but rather are usually outgrow of an already acquired system of concepts and beliefs wrongly applied to an extended domain (Nesher, 1987).

It is true that quite often, intuitive background knowledge hinders the formal interpretation or use of algorithmic procedures as seen in students' misinterpretations of $(a+b)^2$ as a^2+b^2 and this can be categorized as evolving from the application of the distributive law intuitively. In some instances, solving schema is applied inadequately because of superficial similarities in disregard of formal similarities.

MATERIALS AND METHODS

Study Site and Subjects

The study was conducted at Navrongo Senior High School and University for Development

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Studies (Navrongo Campus), all in the Upper East Region of Ghana, from January to May 2016. A convenient sampling method was employed with a sample size of 50 students, made up of 25 students each from the institutions considered above. The study excluded first year students of the Senior High School.

Research Design

A preliminary test was carried out to gather information about the research and to expose deficiencies of the procedure to be followed in the research. A total of 20 problems under the four conceptual areas namely; Variables, Algebraic Expressions, Equations and Word Problems, were given to a total of 30 students from the two study sites to solve.

Overall structure of the test, suitability and appropriateness of the problems were assessed and 16 out of the 20 problems were selected for the main study.

Administration of the Main Test

After the preliminary test, the main study was conducted with a test of the 16 problems/items. The test comprised of all the four categories already discussed. The students were given instructions to use algebraic methods when solving the problems. After that, the papers were marked and students' answers from the test were carefully analysed and grouped into various error types. The same errors that appeared in different questions were assembled into one category with their percentages. For example, the number of students who made the same error was divided by the total number of students who attempted the question. These percentages were used to calculate the mean number of errors for each conceptual area.

Student Interviews

In this study, interviews were used to explore students' thinking. The students selected for the interviews were those who had shown some misconstructions, misinterpretations or had shown a peculiar way of answering some questions in the test given.

During the interview process, the interviewees were encouraged to explain what they were doing as they attempted to solve the problem. However, short intervening questions were asked during the process in order to probe their thinking thoroughly. Each interview lasted between twenty and thirty minutes. All of the ten students who were marked for the interview turned up.

Data Analysis

All statistical analyses were done using Microsoft Excel, 2013 and Statistical Product and Service Solution (SPSS) Version 16.0.

The study compared errors committed in algebra between male and females and also between Senior High School (SHS) and tertiary students using independent sample t-test.

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RESULTS

Errors Students Make When Solving Problems involving Variables

There were six problems in the test conducted that asked for Students' understanding of variables. The rubric for errors and possible misconstructions for variables was constructed. The type of error(s) or possible misconstruction(s) identified under variables included: assigning labels, values or verbs for variables, assigning labels for constant, misinterpreting the product of two variables, wrong simplification, reversal error and incorrect quantitative comparisons. The results are presented in Table 1**.**

Table 1: ***Errors Students Make When Solving Problems Related to Variables***

Problem	**Expected Answer (correct response)**	**Type of Error or Possible Misconception**	**Students incorrect response(s)**	**Frequency of Incorrect Responses n=50**	**Percentage of Incorrect Responses (%)**
1. Solomon sells *m* mangoes. Martin sells three times as many mangoes as Solomon. A mango costs GH¢ 25 **i.** Name a variable in this problem. **ii.** Name something in the problem that is not a variable.	Variable = *m* and something which is not a variable is GH¢ 25.00	Assigning Labels for variables	3 times or 3	2	4
	Variable = *m* and something which is not a variable is GH¢ 25.00	Assigning values for variables	GH¢ 22.00	1	2
	Variable = *m* and something which is not a variable is GH¢ 25.00	Assigning verbs for variables	Sells	1	2
	Variable = *m* and something which is not a variable is GH¢ 25.00	Assigning constants for variables	Solomon or Martin	2	4

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Problem	Expected Answer (correct response)	Type of Error or Possible Misconception	Students incorrect response(s)	Frequency of Incorrect Responses n=50	Percentage of Incorrect Responses (%)
2. What does *xy* mean? Write your answer in words.	*xy* means *x* multiplied by *y*	Misinterpreting the product of two variables	*xy* means a variable; *xy* means the multiple of *x* and *y*	25	50
3. Add 3 to 5***y***	3 + 5*y*	Wrong simplification	8*y*	12	24
4. Subtract 3**a** from 7	7 – 3*a*	Wrong simplification	4 or –4*a*	10	20
	7 – 3*a*	Reversal error	3a – 7	6	12
5. Multiply $a+2$ by 5	5*a* + 10	Wrong simplification	10*a* or 2*a* × 5	7	14
	5*a* + 10	Incomplete simplification	5(*a* + 2)	4	8
	5*a* + 10	Invalid simplification	*a* + 10, 10*a* or 2*a* × 5	3	6
6. The letter *n* represents a natural number. Which one is greater than the other? $\frac{1}{n}$ or $\frac{1}{n+1}$? Give reason(s).	$\frac{1}{n} > \frac{1}{n+1}$ Reason: As the denominator of the fraction increases, the values of the fraction decreases.	Incorrect quantitative comparison	$\frac{1}{n+1} > \frac{1}{n}$ Because $n+1 > n$	13	26

Source: Field Survey 2016

Assigning Labels, Arbitrary Values or Verbs for Variables and Constants

From Table 1, it can be seen that some students misinterpreted a variable as a 'label' (4%) or even as a verb such as 'sells' (2%). They did not perceive the correct interpretation of the variable as the 'number of a thing'. It was difficult for students to distinguish between a variable and a non-variable in terms of the varying and non-varying quantities in problem 1. The students were confused with viewing variables as constants or vice-versa. This error was

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noticed when students were asked to name something in the problem that was not a variable, the answer such as 'Solomon', 'mangoes', 'cedis' were given. In a general sense, these answers may be considered as correct. Sometimes, the words 'mangoes' and 'cedis' could be considered as symbols representing variables in some contexts. However, these answers were considered as incorrect in the context of the given problem since there was a variable or a number attached to these words.

Misinterpreting the Product of Two Variables

In problem 2, as high as 50% of students had difficulties to perceive the product of two variables as two separate variables combined together by a sign. They viewed the product as one variable.

Wrong Simplification

24%, 20% and 14% of students had difficulties regarding the simplification of the problems 3, 4 and 5 respectively. They could conjoin, connect or put together the terms without even considering the operations that are to be carried out on these terms. Addition, subtraction, division, and multiplication signs were left out to form a single bundle of strings.

Reversal Error

Incorrect word order matching led to a reversal error when forming algebraic expressions from a word sentence. When the subtrahend was a number (or constant) and the minuend (subtracted) was an algebraic term in a word sentence (problem 4), students carried out the operation in the reverse order by exactly matching the letters in the given word order.

Incorrect Quantitative Comparisons

26% of the answers to problem 6 were attributed to an incorrect quantitative comparison to compare two algebraic fractions. These students substituted numbers to the algebraic expressions in order to compare them. After the substitution, they only compared the magnitudes of the denominators instead of comparing the whole fractions thereby arriving at wrong conclusions. They did not realize that the reciprocal of a number is smaller than the number itself.

Incomplete Simplification

An answer was categorized as incomplete when some students terminated the simplification of the answers of problem 5 somewhere in the middle of the process without reaching the final answer. In the students' point of view, these answers are final but they are incomplete when compared to standard algebraic procedures. Another possibility is that these students probably may not know how to proceed further. Some of them wrote the problem again in another form as the answer or they terminated the procedure abruptly without completion.

This category of errors was also observed in solving problems related to algebraic expressions.

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Errors or Misconstructions Students Make When Solving Problems Related to Algebraic Expressions

In this study, algebraic expressions had the highest number of student's errors. These include: incorrect simplification, incomplete simplification, invalid multiplication, formation of false equations from answers and invalid distribution, which were summarized and presented in Table 2.

Table 2: ***Errors and Misconceptions Students Make When Solving Problems Related to Algebraic Expressions***

Problem	Expected answer (correct response)	Type of error or possible misconception	Students incorrect response (s)	Frequency (incorrect responses) n = 50	Percentage of incorrect responses (%)
7. Simplify the following i. $(-x)^2 y$ ii. $\sqrt{-x}.\sqrt{-y}$	i. $x^2 y$ ii. $\sqrt{xy}$	Invalid multiplication	xy, $-x^2 y$, $-\sqrt{xy}$ Math error, $-x^2 y$	30	60
	ii. $\sqrt{xy}$	Incomplete simplification	$\sqrt{-x} \times -\sqrt{y}$	4	8
8. Write in another form? $\frac{1}{2} - \frac{3}{2x}$	$\frac{x-3}{2x}$, $\frac{x}{2x} - \frac{3}{2x}$, $x-3 \div 2x$	Incomplete simplification	$\frac{1-3}{2}$	22	44
	$\frac{x-3}{2x}$	Formation of false equation	$\frac{x-3}{2x} \times 2x = x-3$	5	10
9. Expand	$m^2 + 2mn + n^2$	Incomplete simplification	$(m + n)(m + n)$	7	14
	$m^2 + 2mn + n^2$	Invalid distribution	$m^2 + n^2$ or $m^2 - n^2$	18	36

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Problem	Expected answer (correct response)	Type of error or possible misconception	Students incorrect response (s)	Frequency (incorrect responses) n = 50	Percentage of incorrect responses (%)
10. Simplify where it is possible: $2x - m + n - n - + m + 3n$	$2x + 3n$	Incorrect simplification	$2x - m + n$, $2x - 2m + 3n$, error	42	84
11. Factorize The expression $x^2 + 4x + 3 + mx + 3m$	$(x + 3)(x + 1 + m)$	Incorrect factorization	$x(x + 4 + m) + 3(1 + m)$	24	48

Source: Field Survey 2016

Incorrect Simplification

As high as 84% of the students could not simplify problem 10 correctly. They connected or put together the terms without even considering the operations that are to be carried out on these terms. Students had difficulties in performing arithmetic operations such as $- m + m$ for any real number m.

Invalid Multiplication

Errors in this category were the second highest (60%) among errors observed under algebraic expressions. It could be observed that students had difficulty with operations involving the square and square root of expressions, most especially when negative variables were involved.

Formation of False Equations From Answers

In problem 8, 10% of the students formed invalid equations from the answers in the form of algebraic expressions. These students proceeded further to solve these equations. When simplifying algebraic expressions, students connected the variables in the problem in a meaningless way to form an equation.

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Invalid Distribution

Invalid distribution is a kind of misuse of the distributive property in algebra. The left distributive property states that $a(b+c)=ab+ac$ for real numbers a b, and c . This implies that we can either do the addition first, and then multiply, or multiply first and then add. It makes no difference. However, when unlike terms are inside the brackets, it is impossible to add them. Students had to multiply the brackets by the letter outside of the parenthesis. Actually, the distributive property helps us to simplify algebraic quantities by allowing us to replace terms containing parenthesis with equivalent terms without the parenthesis anymore.

Invalid distribution occurred when raising a binomial to a power. 36% of students mistakenly distributed exponentiation over addition as $(m+n)^2=m^2+n^2$.

Errors and Misconstructions Students Make When Solving Problems Related to Equations

These types of errors or possible misconstructions were presented in Table 3. They included misinterpreting numbers as labels, misinterpreting the elimination method in solving simultaneous equations, wrong solutions to simultaneous equations and wrong simplification.

Table 3: ***Errors and Misconceptions Students Make When Solving Problems Related to Equations***

Problem	Expected answer (correct response)	Type of error or possible misconception	Students incorrect response(s)	Frequency (incorrect responses) n= 50	Percentage of incorrect responses (%)
12. Solve for *y* in the equation below. 6*y* + 13 = 75	*y* = 10.33	Number as labels	*y = 2*	6	12
	y = 10.33	Wrong simplification	$6y=\frac{75}{13}$, $13y=\frac{75}{6}$	7	14

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Problem	**Expected answer (correct response)**	**Type of error or possible misconception**	**Students incorrect response(s)**	**Frequency (incorrect responses) n= 50**	**Percentage of incorrect responses (%)**
13. Consider the system of linear equations: $a + b = 5$ $a - b = 7$ **i.** To eliminate *a*, do you add or subtract the two equations?	13(i) subtract 13(ii) add	Misinterpreting the elimination method when solving simultaneous equations	Subtract when the equation has to be added or vice- versa	38	76
ii. To eliminate *b*, do you add or subtract the two equations? **iii.** Solve the system of equations above,	$a = 6$, $b = -1$	Wrong solution to simultaneous equations	$a = 6$, $b = 1$; $a = -6$, $b = 1$	16	32
14. Solve the system of linear equations: $\frac{x}{2} + \frac{2y}{3} = \frac{7}{3}$ $\frac{3x}{2} + 2y = 5$	There is no unique solution for the system of equations	Misinterpreting the elimination method when solving simultaneous equations	Error	5	10
	There is no unique solution for the system of equations	Wrong solution to simultaneous equations	$x = 4$, $y = 3$	6	12
	There is no unique solution for the system of equations	incomplete	$3x + 4y = 14...(1)$ $3x + 4y = 10…(2)$	25	50

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Numbers as Labels

In Table 3, 12% of the students made this error in problem 12 and it was a different form of the same error discussed under 'variables as labels'. These students used a number as a label to replace or substitute a variable. Solving for y in $6y + 13 = 75$, these students wrote $y = 2$ by pasting the number 2 into the position of y to get 62. These students had understood the property of equivalence as they pasted the correct number to make the equivalence work, although they did not follow the normal equation solving procedures. This error may had occurred due to students' previous knowledge of number equations where students had to insert a number to satisfy a numeric equation. Similarly, these students might have used the number as a label for a letter to satisfy the equation numerically.

Misinterpreting the Elimination Method When Solving Equations Simultaneously

When eliminating a variable from a system of linear equations in problem 13, 76% of students misjudged the operations to be performed. Some of them chose the reverse operation, for example, adding when it had to be subtracted or vice versa. Probably, this misunderstanding came from their fragile understanding of simplifying integers and manipulating signs. Their difficulties were aggravated when the variables in the two equations had opposite signs ($-b$, $+b$).

Wrong Solution to Simultaneous Equations

In this study, students used two methods to solve a linear system: the substitution method and the elimination method. In the substitution method, students had to isolate a variable from one equation and substitute its value into the second equation. 32% of students applied the substitution method wrongly which led to errors in problem 13. For these students, one variable was made the subject of one equation but had problems of carrying out proper substitution and expansion in the other equation. Where correct substitution and expansions were done, misuse of the "change- side, change-sign" rule led to errors. This error was observed in the last steps of the equation solving process. Some students carried over the terms to the other side of the equation without properly changing the signs or without executing proper operations (for instance $a + a - 5 = 7, \Rightarrow 2a = 7 - 5$). Also 76% of the students misinterpreted the elimination method when solving simultaneous equations in problem 13.

Problem 14 could not be solved simultaneously. 50% of the students could not indicate this statement as their answers due to lack of understanding whiles 12% of the students used the coefficients of x and y as the solution of x and y respectively.

Errors or Misconstructions Students Make When Solving Word Problems

Many empirical studies have indicated in the past that students face difficulties in translating algebra word problems that state relationships between two or more variables into a symbolic form. In our study, there were four word problems which consisted of mainly word sentences. Students had to read the problems, convert them into algebraic forms, and solve them. In some questions, students had to provide reasons for their answers. Among others, there were two main processes involved in solving a word problem. One was the translation process, which was to read and translate the words of the problem into an algebraic representation. The solution process was to apply rules of algebra in order to arrive at a solution. Several types of errors

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were seen from the careful analysis of answers. One observation was that a considerable number of students used arithmetic methods rather than algebraic methods to solve word problems. These errors were summarized and presented in Table 4.

Table 4: ***Errors and Misconceptions Students Make When Solving Word Problems***

Problem	**Expected answer(correct response)**	**Type of error or possible misconception**	**Students incorrect response(s)**	**Frequency (incorrect responses) n= 50**	**Percentage of incorrect responses (%)**
15. Shirts cost GH¢ *x* each and pants cost GH¢ *y* a pair. If one buys 3 shirts and 2 pairs of pants, explain what $3x + 2y$ represents?	It represents the total cost of 3 shirts and 2 pairs of pants	Assigning labels for variables	It represents buying 3 shirts and 2 pants	6	12
	It represents the total cost of 3 shirts and 2 pairs of pants	Assigning labels for variables	Sum of 3 shirts plus 2 pairs of pants	14	28
	It means the total cost of 3 shirts and 2 pairs of pants.	Lack of understanding of the unitary concept when dealing with variables	$3x$ = 3 shirts $2y$ = 2 pants	17	34
16. Muniru gave his stamp collection to his three children: Solo, Ado and Leti. Leti received 5 times the number of stamps than Solo did, and 4 less than those received by Ado. The whole quantity received by Solo and Ado is 22 stamps. How many stamps did Muniru give to each child?	Muniru gave to Solo, Ado and Leti, 3, 19, 15 stamps respectively	Lack of understanding of problem	$x = 11$	25	50
	Muniru gave to Solo, Ado and Leti, 3, 19, 15 stamps respectively.	Incomplete		4	8

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Another possible errors made by some students occurred at their difficulties in understanding the unitary concept when multiplying a variable with a constant in problem 15. Given the price of a shirt as GH¢ x and when they were asked what $3x$ means, some students failed to understand that $3x$ is the cost of 3 shirts. This is a basic arithmetic concept. The only difference in this question is that the price was given as a variable. They interpreted the term '$3x$' as '3 shirts for cedis'. Here again, it is evident that, in addition to the incorrect calculation, 40% considered x as the label for 'shirts', rather than the unit price of a shirt and at the same time considered x as the item price whiles 34% of the students lacked understanding of the unitary concept when dealing with variables.

50% of the students had $x = 11$ in problem 16 which showed their lack of understanding of the problem whiles 8% never completed the solution to the problem.

Differences Among Students When Solving Problems in Variables, Algebraic Expressions, Equations and Word Problems

Table 5 presents the mean and standard deviation according to the four conceptual areas: variables, expressions, equations and word problem. In Table 5, it can be seen that averagely, students committed more errors when solving algebraic expressions than problems related to identifying variables and solving equations. The data on the mean number of errors committed in the four conceptual areas was presented as a bar chart in Figure 1.

Table 5: ***Descriptive Statistics: Mean Number of Errors in Algebra***

Areas	*n*	Minimum	Maximum	Sum	Mean	Std. Deviation
Variables	50	1	4	86	1.72	0.834
Expression	50	2	6	156	3.12	1.118
Equations	50	0	2	64	1.28	0.536
Word problems	50	1	3	76	1.52	0.677

Source: Field survey 2016

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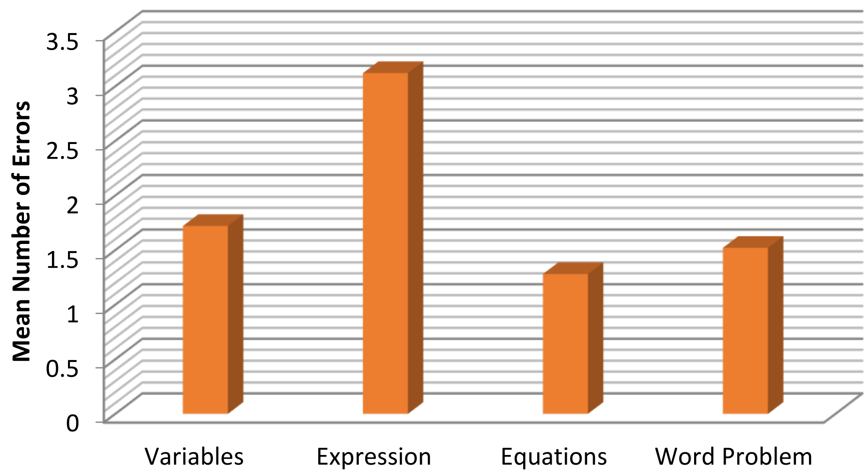

Figure 1: Mean Number of Eroors Commited by the Students

The summary measures in Table 6 shows the mean errors committed in algebra by sex (male and female). However, in Table 7, further analysis was performed to determine the mean difference in the errors committed using the independent sample *t*-test.

Table 6: *Group Statistics* of Errors Committed in Algebra by Males and Females

	Variable	*n*	Mean	Std. Deviation	Std. Error
Total Errors	Male	25	7.58	1.447	0.284
	Female	25	7.71	1.706	0.348

Source: Field Survey 2016

Table 7: *Independent Sample Test*

		Levene's Test for Equality of Variances		*t* -test for Equality of Means					
		F	Sig.	*t*	Mean Diff.	Sig.	Mean Diff.	95% Confidence Interval of the Diff. Lower	Upper
Total Errors	Equal Variance Assumed	1.116	0.296	-0.294	48	0.770	-0.131	-1.029	0.766
	Equal Variance not Assumed			-0.293	45.306	0.771	-0.131	-1.036	0.773

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Further, it was observed in Table 7 that in testing for the equality of variance of the two groups, an F-value of 1.11 was recorded with a significant value of 0.296. *p value* > 0.05, indicated that the variance of the two variables were not different, hence we use equal variance assumed under the column Levene's test for equality of variances. Since *p value* > 0.05 under *t*-test for equality of means, there is no significant difference between errors committed by males and females.

Figure 2 shows the performance of males and females in the four conceptual areas: variables, expressions, equations and word problem. It can be observed that more errors were committed in algebraic expressions than in variables, equations and word problem by both males and females. This simply implies that the Students found expressions to be more difficult than variables, equations and word problem.

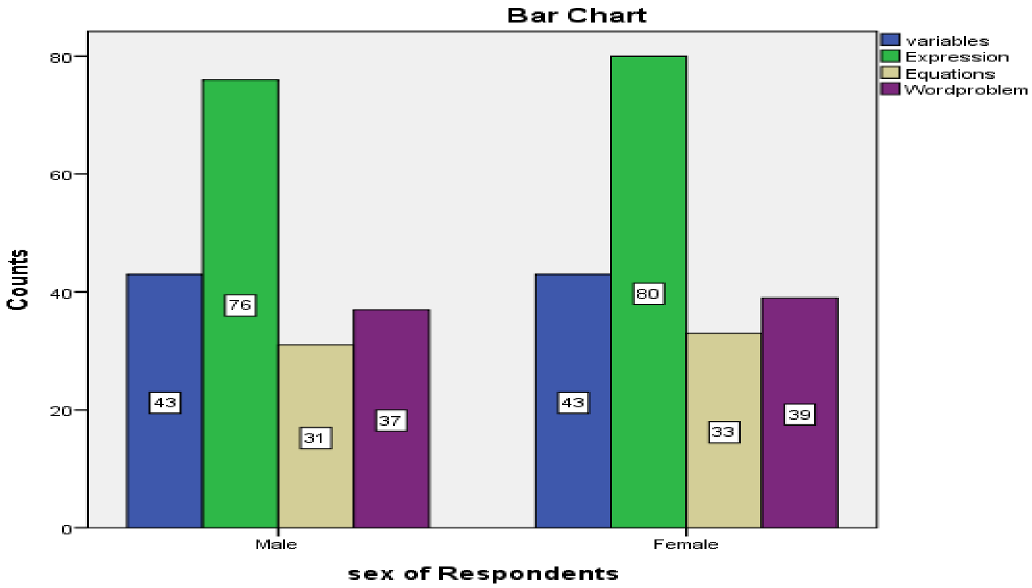

Figure 2: Errors Committed by Students by Gender

Table 8 showed the mean error committed by students of UDS and that of the Navrongo Senior High School. Clearly, there were differences in the means and this was statistically proven by the independent sample *t*-test in Table 9.

From Table 9, it was observed that in testing for the equality of variance of the two groups, the *F*-value is 0.000 and the significant value is 0.995. Since *p value* > 0.05, we have no evidence against the null hypothesis, thus it implies the variance of the two variables are not different, hence we use equal variance assumed under the column Levene's Test for equality of variances. Since *p value* < 0.05 under *t*-test for equality of means, we reject the null hypothesis that there is significant difference between errors committed by SHS and Tertiary students.

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Table 8: ***Group Statistics of Errors Committed in Algebra by Tertiary Students and SHS Students***

	Level	*n*	Mean	Std. Deviation	Std. Error
ERRORS	Tertiary (UDS)	25	5.720	3.021	0.604
	SHS	25	9.560	3.150	0.630

Source: Field Survey 2016

Table 9: ***Independent Sample Test***

		Levene's Test for Equality of Variances		*t*-test for equality of Means					
		F	Sig.	*t*	DF	Sig. (2-tailed)	Mean Diff.	95% Confidence Interval of the Diff. Lower	Upper
Errors	Equal Variances Assumed	0.000	0.995	-4.399	48	0.000	-3.840	-5.595	-2.085
	Equal Variances not Assumed			-4.399	47.916	0.000	-3.840	-5.595	-2.085

Source: Field Survey 2016

DISCUSSION

Diverse forms of misconstructions result in errors observed in dealing with problems in algebra. To achieve the objectives of this study, the following were addressed: categories of errors and misconstructions in solving problems related to variables, algebraic expressions, equations, word problems; whether or not existing theoretical explanations account for the errors or misconstructions observed in this study and what can be learned from students' problem solving processes and reasoning in algebra.

Misconstructions/Errors in Solving Problems Related to Variables

The categories of students' errors related to variables included assigning labels for variables, misinterpreting the product of two variables, wrong simplification, incomplete simplification and incorrect quantitative comparisons.

Some students misinterpreted a variable for a 'label', or as a verb such as 'sell' rather than as the "number of things". This was common for problem 15 for example, when the price of the

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shirt was x Ghana cedis, the students gave a response that $3x$ stood for a label for “3 shirts”. This was a clear misinterpretation of the algebraic term. The result was consistent with Philip (1999) that explained a similar use of letters as labels as used in $3f = 1y$ to denote 3 feet equals 1 yard. In this interpretation, f and y stand for ‘feet’ and ‘yard’ respectively. The letter in this case was used to denote the name of the unit. Further, the use of the letter as a label was found when students solved $6y + 13 = 75$, by pasting the number 2 as a label for y but not by substituting it. This is similar to finding the number to satisfy a number equation in arithmetic. It was also discovered that, 12% of the students found it difficult to differentiate between variables and nonvariables. This was evident in problem 1, when students provided names of persons, things and letters for non-variables. Some of these solutions were correct, but unacceptable under algebraic interpretations. Misinterpretation of letters as labels is an error that may lead to many other errors in Algebra. For instance in the famous Student-Professor Problem in Clement et al. (1981) and Kaput (1985), college students made similar interpretations of variables. In the Student-Professor Problem, students used p to represent professors rather than the number of professors and similarly s to represent students rather than the number of students. The result for this was that a reversal error (writing the equation as $6s = p$ instead of $6p = s$) had occurred.

A number of students viewed the product of two variables as one variable. For instance, they perceived the product of xy in problem 2 as a single variable. In this case the Students did not take note of the multiplication sign between the letters and simply thought of xy as a number similar to 15. In (Macgregor and Stacey, 1997), the misinterpretation is considered to come from other numerical system such as the Roman numeral system in which the number *IV* is understood as five less one. This was consistent with the findings from this study. Students who said that there was one variable might have seen xy as a conjoined answer.

A good number of students had the reversal error committed problem 4 (12%), where they formed the expression in the reverse order. The problem read ‘subtract $3a$ from 7’. 12% of the students wrote $3a – 7$ instead of the expected answer ‘$7 – 3a$’. In this problem, the students had to read the word sentence and translate it into an algebraic form. In reality, the reverse order of the answers showed that students literally matched the word order given in the problem into algebraic form rather than the actual understanding of the correct relationship among the given variables.

Misconstructions/Errors in Solving Problems Related to Algebraic Expressions

This category includes the following: incorrect simplification, incomplete simplification, invalid multiplication, formation of false equations from answers and invalid distribution.

An answer was categorized incomplete if students terminated the simplification of an algebraic expression before the accepted answer. In this situation, a student would start the problem and proceed with one or two steps and abruptly terminates the process before reaching the final answer. Through the written test and the interviews conducted with the students it was observed that probably there were two reasons for this. Firstly, it could be seen as insufficient knowledge by students on how to proceed with the problem. Secondly, students could have wrongly thought they had reached the final answer. For the incomplete answers, further simplification was possible to reach the desirable solution. According to Booth (1984) such errors were because of students’ lack of knowledge or lack of confidence in the problem solving process. The cause of incomplete simplification could have been due to lack of knowledge.

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The misconception of 'invalid distribution' had a variety of forms. One common example of this was the failure of the students to retrieve the correct expansion of a binomial such as $(m+n)^2$. According to literature, this error is as a result of deficiency in the mastery of prerequisite facts and concepts. Another explanation is that students overgeneralized a correct rule to misapply it in another situation as a result of explicit, declarative knowledge gained from the curriculum (Macgregor and Stacey, 1997; Matz, 1982). Finally, the students misused the distributive law because these errors have their roots in arithmetic misconstructions. Lack or incomplete understanding of arithmetical concepts or failure to transfer arithmetic understandings to algebraic context are the leading factors (Macgregor and Stacey, 1997; Norton and Irvin, 2007; Piaget, 1970). However, from the findings of this study, it was found that no single reason is responsible for students to have invalid or incomplete distribution but a combination of deficiency in the mastery of prerequisite facts and concepts and overgeneralizing a correct rule to misapply it in another situation.

Misconstructions or Errors in Solving Problems Related to Equations

There were four categories of errors under the category of equations solving. These included numbers as labels, misinterpreting the elimination method, wrong use of the substitution method and misuse of the "change-side and change-sign" rule.

The students had to solve the question which stated: "Solve for $6y + 13 = 75$." The students simply pasted the number 2 into the position of *y* to have a complete equation as "62 + 13 = 75". This seemed to have made sense as it showed that the students understood the equivalence property as he pasted the correct number to make the equivalence work, although he did not follow the normal procedure. The only reason for this error could be that they might have used the previous knowledge of arithmetic of number equations to insert a number to satisfy the numerical equation.

Students misconstrued the elimination method when solving the linear systems of simultaneous equations. Students often misjudged the operation to be performed and chose a reverse operation that is they added when they needed to subtract or vice versa (76%). Other misunderstandings of simultaneous equations were that students concentrated on one equation of the system. Both misconceptions showed that students' had incomplete understanding of the elimination method of solving linear systems of simultaneous equations.

This was further evidenced from the reluctance of students to solve the equations using the elimination method as most of them used the substitution method to solve problem 13. One main problem faced was lack of understanding when starting to solve the problem. Another observation was that students had difficulty in arriving at the conclusions intuitively. It was difficult for students to deduce whether the solution was the correct one.

The misuse of the "change-side, change-sign" rule was a common error in solving equations as evidenced in this study. This happened because they attempted to separate the letter and constants in an algebraic term. The main reason for this is the lack of understanding of the basic features of algebra.

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Misconstructions or Errors in Solving Word Problems

Many of the difficulties that students face in solving word problems mainly emanated from their failure to translate the word problems into algebraic language, varying relationships between variables as well as guessing without reasoning. Few students used guesses that were not educated guesses. An important characteristic of an educated guess is that the guess will improve every time based on previous guesses.

The independent sample *t*-test was used to test if there was significant difference in errors committed between males and females and also between tertiary students and SHS students. The results revealed that there was no significant difference in errors committed between males and females. This could be attributed to the fact that both males and females received equal treatment during mathematics instruction. Further, this could mean that the teachers of mathematics are not gender biased when teaching and they look at males and females as equal partners in mathematics achievement. This was not consistent with findings from the previous researchers who found out that boys performed better than girls in Algebra (Usiskin, 1982). On the other hand, the test also showed that there are significant difference in errors committed between tertiary and SHS students. This could also be attributed to the fact that higher education imparts more experience in students when solving algebra thereby committing less error.

Lessons from Students' Problem Solving Processes and Reasoning in Algebra

The findings from this study showed that misconstructions are robust; this simply meant that they could not easily be dislodged. It is evident from the interview conducted that in many instances students appear to overcome a misconstruction only to have the same misconstruction later. According to Piaget (1970), when students constructed knowledge, they became attached to what they had constructed (radical constructivism). Therefore, one major requirement in trying to eliminate these misconceptions is to make sure that students actively participate in the process of overcoming their misconstructions. This is not a process that is entirely dependent on students, but the teacher can also help in facilitating the complete elimination of the misconstructions. The teacher can help do so by providing students with an enabling classroom environment that can help them develop both procedural and conceptual knowledge such that they can construct correct conceptions from the start.

It was also noted that, the follow-up question which the researcher gave during the interview process to allow the students explain their thinking, helped the researcher to get better insight into the reasons for the students' misconstructions. When a teacher listens to students, they would understand the diversity of students' understanding. At the same time students would revise and refine their own mathematical thinking. Therefore individual attention to the students was necessary as it would reveal a lot of inadequacies on the part of the students' understanding of algebraic concepts.

It was also observed that students made similar procedural errors in more than one conceptual area. It could be seen from the findings that both knowledge of procedures and concepts are important for students to correctly handle problems in Algebra. This means that it is important for students not only to have procedural knowledge (how procedures and algorithms work) but they should also develop conceptual knowledge so that they should be able to explain why certain procedures and algorithms work. The study revealed that both types of knowledge are

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important to prevent students from making many errors and misconstructions. Further, when teaching a new concept, giving examples as well as non-examples is vital as it will help students get a better understanding of concepts, facts and procedures.

By committing errors and looking to understand their origins, Students may achieve a stronger conceptual basis than if they had never committed the errors in the first place. Macgregor and Stacey (1997) reinforced this idea by saying that rational errors should not be a hindrance to the mathematical learning process but should serve as constructive and adaptive tools for promoting mathematical understanding. In the process of correcting or searching for the origins of errors, Students may reach a better understanding of their own mathematical reasoning.

CONCLUSION

There were seven new error types that came out from this study and these include: misinterpreting the product of two variables, giving answers in the form of false equations, incomplete simplification and incorrect simplification. Others were incorrect quantitative comparisons, numbers as labels and misinterpreting of the elimination method when solving equations.

The symbols in Algebra have different meanings and interpretations in different situations. Students had incorrect and incomplete perceptions about the letters, numbers and signs. The overall image that emerged from the findings was that the misunderstanding of the concept of variables did have a clear bearing on their errors and misconstructions.

With regards to algebraic expressions, it was discovered from the study that the main problem which students encountered was the lack of understanding of the structural features in this conceptual area which led students to use many wrong procedures. Also students modified or misapplied rules which were inappropriate in certain situations.

With regard to equation solving, the misuse of the equal sign out of its accepted meaning was obvious. In most cases, students used the 'equal sign' in a single sense, that is, to do the operation to the left and get the answer to the right or vice versa. Finally, the students misused the elimination and substitution method.

The study revealed that students' errors and misconstructions were more in algebraic expressions than in variables, equations and word problem. It was revealed that students had difficulties with algebraic expressions because there were abstract in nature. The fact that algebraic expressions involve the use of letters as opposed to numbers made students feel uncomfortable with the operations. The other factor was that there is interference from the arithmetic methods which they wrongly apply in algebraic expressions.

In many instances, students appear to overcome a misconstruction only to have some misconstructions resurface later. This is probably as a result of the fact that, when students construct knowledge, they become attached to the notions they have constructed. Therefore, one important requirement in eliminating misconstructions is that students must actively participate in the learning process.

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By committing errors and looking at their origin, students may have a stronger basis for reasoning correctly than if they never committed the errors in the first place. Finally, the students' errors and misconstructions are largely attributed to the lack of conceptual knowledge in solving algebraic problems.

There is no difference in errors committed between males and females in algebra. Tertiary students had more experience when dealing with algebra since they are more exposed to mathematical problems than their counter- parts at the Senior High school level. This made tertiary students commit fewer errors than the Senior High School students.

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