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SIGNED SINGULAR MAPPINGS TRANSFORMATION SEMIGROUP

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ABSTRACT: *The semigroup of all singular self mappings of $X_n \longrightarrow X_n$ denoted by $Sing_n = T_n \setminus S_n$ was studied by Howie (1966). Also Howie and Schein (1973) investigated the elements of $PT_n \setminus S_n$. In like manner, the singular mappings in signed transformation, $SSing_n$ to be defined on the set $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$ where $X_n = \{1,2,3,\cdots,n\}$ and $X_n^* = \{-n,\cdots,-3,-2.-1.0,1,2,3,\cdots,n\}$. In this paper the order, idempotent, nilpotent and chain decomposition of signed singular mapping were investigate.*

KEYWORDS: singular mapping, idempotent, nilpotent, chain decomposition, semigroup, signed full transformation.

INTRODUCTION AND PRELIMINARIES

The origins of group theory are in the study of permutations, and the symmetric group, S_n the group of all permutation of a set is rightly an object of importance within the abstract study. The corresponding object in semigroup is the full transformation semigroup, the semigroup of all selfmaps of set X. The full transformation semigroup, T_n and some special sumsemigroups of T_n, has been much studied over last fifty years. (see Howie (1966); Higgins, Howie, Mitchell and Ruskuc (2003)) .

We begin by recalling some of notations and definition that will be useful in the paper. For standard terms and concepts in semigroup theory we refer the reader to Umar (1992, 2007). Consider $X_n = \{1,2,3,\cdots,n\}$, then a (partial) transformation $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subseteq X_n$ is said to be full or total if $Dom(\alpha) = X_n$ otherwise it is called strictly partial. The set of full transformation of X_n denote by, T_n more commonly known as the full transformation is also known as the full symmetric semigroup or monoid with composition of mappings as the semigroup operator. It is well known that S_n, T_n and PT_n have order $n!$, n^n and $(n+1)^n$ respectively. Garba (1990), showed the number of idempotent in full and partial transformation semigroups are $|E(T_n)| = \sum_{r=1}^{n} \binom{n}{r} r^{n-r}$ and $|E(PT_n)| = \sum_{r=1}^{n+1} \binom{n}{n+r} r^{n+1-r}$. An element $e \in S$ is an idempotent if $e^2 = e$. The set of all idempotent of S is denoted by $E = E(S)$. Also an element $\alpha^m = 0$ (the zero mapping) for some integer $m \geq 1$ this is denoted by $N(S)$.
Howie (1995) define the semigroup of singular selfmaps of X_n i.e. $Sing_n = T_n \setminus S_n = \{\alpha \in T_n: |\, lm(\alpha) \leq n-1|\}$ and investigated that the order and number of idempotent of $Sing_n$ as compiled by Umar (2017).

Table 1: Combinatorial results in transformation semigroup

S	$\lvert S\rvert$	$\lvert E(S)\rvert$	$\lvert N(S)\rvert$
S_n	$n!$		
T_n	n^n	$\sum_{r=1}^{n} \binom{n}{r} r^{n-r}$	
PT_n	$(n+1)^n$	$\sum_{r=0}^{n} \binom{n}{r} (r+1)^{n-r}$	$\sum_{r=0}^{n-1} \binom{n}{r} S(n,r+1) r! = (n+1)^{n-1}$
$Sing_n$	$n^n - n!$	$\sum_{r=0}^{n-1} \binom{n}{r} r^{n-r}$	
$PT_n \setminus S_n$	$(n+1)^n - n!$	$\sum_{r=0}^{n} \binom{n}{r} (r+1)^{n-r}$	$(n+1)^{n-1}$

The signed symmetric and signed transformation semigroup was studied by Mogbonju (2015). Table 2 contains the summary of the results obtained

Table 2: Combinatorial results in signed transformation semigroup

S	$\|S\|$	$\|E(S)\|$	$\|N(S)\|$	$\|H_n\|$	$\|E_n\|$
SS_n	$2n! - (2^n - 2)n!$	1	$-$	$2n^2$	n
ST_n	$2n^n + n^n(2^n - 2)$	5^{n-1}	$-$	$2n^2$	$2n^2 - 2$
SPT_n	$(2n + +)^n$		5^{n-1}	$2n^2 + 2$	$2n^2$
SI_n	?	2^n	?	$2n^2 + n$	$2n$

Signed transformation semigroups is the set of all mapping from $X_n \longrightarrow X_n^*$.The signed (partial) transformation semigroup is defined in the form $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$ where $X_n = \{1,2,3,\cdots,n\}$ and $X_n^* = \{-n,\cdots,-3,-2.-1.0,1,2,3,\cdots,n\}$. The domain may be empty. The signed partial one – one transformation semigroups is defined in the form of signed partial transformation semigroup as $\alpha: dom(\alpha) \subseteq X_n \longrightarrow X_n^*$ but strictly one – one.

The signed singular mappings is the set of all mapping on the set $\alpha : dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$.

METHODOLOGY

Matrix notations

The following notations will be used in representing elements in transformation semigroups.

Let $\alpha \in T_n$; this elements can be represent $\alpha(j) = i$ by placing a 1 in (i,j) entry of an $n \times n$ matrix.

Example 1

If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 5 & 6 & 3 & 7 \end{pmatrix} \in T_7$

Which can be written in matrix notation as;

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 2

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -1 & 3 & -4 & 2 & -6 & -5 \end{pmatrix} \in ST_6$

One can represents these elements in matrix form by placing ± 1 in the (i,j) entry to indicate $j \longrightarrow \pm i$.

So,

$$\alpha = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Example 3

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -3 & \emptyset & 4 & -5 & \emptyset & -3 \end{pmatrix} \in SPT_6$

then the matrix notation

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 2.1

Let S be the partial signed (partial) transformation. Composition of mapping is associative if a, b and c are partial mappings such that the composition $a(b)c$ is define if and only if the composition $(ab)c$ is defined and both equal.

Proposition 2.2
Let $S = Sing_n$ and $\alpha, \beta \in S$ then $\alpha\beta \neq \beta\alpha$ and not commutative.

Proposition 2.3
Let $\beta \in SSing_n$ then $\beta^n \in SSing_n$ where $n \in X_n$
Proposition 2.3 can be illustrated with the following example
Some of the elements of $SSing_2$ are:
$SSing_2 = \left\{, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}$
Let $\beta = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, then

$$\beta^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\beta^3 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Thus $\beta^n \in SSing_n$ for $n \in X_n$
Results / Findings

Cardinality of $SSing_n$ and $SPT_n \setminus SS_n$

Some of the elements of $\boldsymbol{SSing_n}$
When $n = 2$

$$|\, Im(\alpha^+) \,| = \left\{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\right\} = 2$$

$$|\, Im(\alpha^-) \,| = \left\{\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\right\} = 2$$

$$|\, Im(\alpha^*) \,| = \left\{\begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\right\} = 4$$

Table 3.1: Elements in $\boldsymbol{SSing_n}$

$\boldsymbol{n}$	$\boldsymbol{\|\, Im(\alpha^+) \,\|}$	$\boldsymbol{\|\, Im(\alpha^-) \,\|}$	$\boldsymbol{\|\, Im(\alpha^*) \,\|}$	$\boldsymbol{\|\, SSing_n \| = 2^n(n^n - n!)}$
1	**0**	**0**	**0**	**0**
2	**2**	**2**	**4**	**8**
3	**21**	**21**	**126**	**168**
4	**229**	**229**	**3254**	**3712**
5	**3125**	**3125**	**89910**	**96160**

Theorem 3.1
Let $S = SSing_n$, if $\alpha \in S, n \geq 0$, then $|S| = 2^n(n^n - n!)$
Proof
Let $\in SSing$, such that $\alpha: dom(\alpha) \longrightarrow Im(\alpha)$. The number of elements the image of α can assume is $n - 1$. If $|\, Im(\alpha^+) \,|$, then there $n^n - n!$ elements and also $n^n - n!\, for \,|\, Im(\alpha^-) \,|$. Since $Im(\alpha)$ is either $i\ or - i$ for $i = 1,2,3, \ldots, n$, the nature of $Im(\alpha)$ is such that $Im(\alpha) \subset X_n^*$. Also for each n the $Im(\alpha)$ occurs $n^n - n!$ times 2^n different groups. Hence the results.

Some of the elements of $\boldsymbol{SPT_n \setminus SS_n}$

When $n = 2$

$$SPT_2 \setminus SS_2 = \left\{ \begin{matrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & \emptyset \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 \\ -1 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ \emptyset & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \emptyset & \emptyset \end{pmatrix}, \end{matrix} \right\}$$

Idempotents in signed transformation semigroup

Element $\alpha \in ST_n$ is idempotent if and only if $\alpha^2 = \alpha$. The set of all idempotent in ST_n is denotes by $E(ST_n)$. idempotent elements is usually denote by letter e and $e^2 = e$.

For ST_3 the set of all idempotent elements is

$$E(ST_3) = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{matrix} \right\}$$

Example 3.1

Consider the $\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 3 \end{pmatrix} \in ST_3$ which can be write in matrix form $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

and this element is idempotent via matrix multiplication

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Nilpotents in signed transformation semigroup

A semigroup R with an empty map is said to be nilpotent provided that their exists $t \in P$ such that $R^t = \emptyset$, that is $x_1, x_2, x_3, \dots, x_n = \emptyset$ foe all $x_1, x_2, x_3, \dots, x_n \in R$. If S is nilpotent, then the minimal $t \in P$ such that $S^t = \emptyset$ is called the nilpoteny degree of S and is denoted by $nd(S)$. Nilpotent elements form a subsemigroup, that is, a class, of their own.

For ST_2, the set of all nilpotents elements is as,

$$|E(ST_2)| = \left\{ \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = 5$$

and elements in nilpotent form via matrix multiplication

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Theorem 3.2

Let R be a finite nilpotent subsemigroup where $S = SPT_n$ with an empty map $\emptyset$, then the following conditions are equivalently true (i) S is nilpotent (ii) Each element $\alpha \in S$ is nilpotent.

Proof

Supposed that $R$is nilpotent, hen there exists nilpotency degree m such that $R^m = \emptyset$. This implies that $B^m = \emptyset$. Thus $B^m \in R$ is nilpotent. Conversely, assume that every element $\beta \in R$ is nilpotent. Let $i, j \in lm(\beta)$ such that either βi or $\beta j \in dom(\beta)$ preserve fixing of elements, that is $\beta i = i$ or $\beta j = j$ which contradicts the assumption that every element is nilpotent. Hence βi or $\beta j \notin dom(\beta)$, for every $\beta \in S$.

Table 3.2: Values of Relations in $SSing_n$ and $SPT_n \setminus SS_n$

n	$\lvert SSing_n \rvert$	$\lvert SPT_n \setminus SS_n \rvert$	$\lvert E(SPT_n \setminus SS_n) \rvert$	$\lvert N(SPT_n \setminus SS_n) \rvert$
1	0	1	0	1
2	8	17	4	5
3	168	295	24	25
4	3712	6177	80	125
5	96160	10812	200	625

Theorem 3.3: Let $S = SPT_n \setminus SS_n$, then $|S| = (2n+1)^n - 2^n n!$.
Proof : Let $lm(\alpha)$ is either i or $-i$ for $i = 0,1,2,3,\ldots,n$ where $i = 0$ denotes an empty maps. For each n the $|\, Im(\alpha^+)\,| = |\, Im(\alpha^-)\,|$, it follows from that if $\alpha \in SPT_n \setminus SS_n$ and
$\alpha : dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$ then for each n and $|\, Im(\alpha^+)\,|$ there are $2n+1$ elements. Using the binomial theorem for a positive integer and a case where $|\, Im(\alpha^-)\,|$ we have $(2^n\, n!$ elements. Hence $SPT_n \setminus SS_n = (2n+1)^n - 2^n n!$

Theorem 3.4: Let $S = SSing_n$, and $n > 1$, then $|E(S)| = \frac{n^2(n^2-1)}{3}$.
Proof: Let $\alpha : X_n \longrightarrow X_n^*$ and $\alpha \in S$ for $|lm(\alpha)| = n-1$. If $lm(\alpha) = 1$, the number of elements is n^2 if $m(\alpha) = k$ for $k = 0.1.2.3, \ldots, n$ there are $n^2(n^2-1)$ element. Thus $|E(S)|$ of $S = SSing_n\ is\ \frac{n^2(n^2-1)}{3}$.

4.0 Chain decomposition of elements in signed singular mappings

Adeniji(2012) studied chain decomposition of identity difference transformation semigroups. Also Mogbonju(2015) studied chain decomposition of signed full and partial transformation semigroup, he showed that the order of chain decomposition of SS_n, ST_n and SPT_n are respectively; $2n^2, 2n^2$ and $2n^2+2$ respectively.
Chain decomposition δ_β of elements of a semigroup is defined as the set of fragments of each $\beta \in ST_n$, for each $i \in dom(\beta), j \in lm(\beta)$, such that $i\beta = j \Longrightarrow \beta_k : i_k \longrightarrow j_k$ where $k = 1,2,3,\ldots,n$
For example

Let $\beta_1 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ j_1 & j_2 & j_3 & j_4 & j_5 \end{pmatrix}$ then the chain decomposition of

$\beta_1 = \left\{ \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \begin{pmatrix} i_2 \\ j_2 \end{pmatrix}, \begin{pmatrix} i_3 \\ j_3 \end{pmatrix}, \begin{pmatrix} i_4 \\ j_4 \end{pmatrix}, \begin{pmatrix} i_5 \\ j_5 \end{pmatrix} \right\}$

Let $S = SSing_n$, and $S = SPT_n \setminus SS_n$, then,
$|S|$ = the cardinality of S.
$|H_n|$ = the total number of chains in the chain decompositions of all elements of S.
$|E_n|$ = the total number of chains in the chain decompositions of all idempotent elements of S.
$|E(S)|$ = the total number of idempotent in S.
$|N(S)|$ = the total number of nilpotent in S.

Table 4.1 : Values of relations in $SSing_n$ and $SPT_n \setminus SS_n$

n	$\lvert H(SSing_n) \rvert$	$\lvert H(SPT_n \setminus SS_n) \rvert$	$\lvert E_n \rvert$
1	−	1	−
2	10	10	6
3	21	21	18
4	36	36	54
5	55	55	162

The following results emerged from table 4.1 above.
Proposition 4.1: The total number of chains in the chain decompositions of all elements of $SSing_n$ is $2n^2+n$ for $n > 1$.
Proof : Let $X_n = \{1,2,3,\ldots,n\}$ such that $\alpha : dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$ and each $i \longrightarrow j$. The decomposition follows a unique patter such that $\alpha \in |SSing_n|$. If $X_n \longrightarrow X_n^*$ then the mapping is one – one i.e. each elements from X_n taken as the domain could occur n times $2n+1$ ways. Hence $|SSing_n| = 2n^2+n$.

Proposition 4.2 : Let $S = SPT_n \setminus SS_n$, then $|H^*| = 2n^2+n$
Proof : The proof of this is as in Proposition 4.1

Proposition 4.3 : Let $S = SSing_n$, then $|E_n| = 2 \cdot 3^{n-1}$ for all $n > 1$.
Proof : $\alpha \in |SSing_n|$ and $X_n \in X_n^*$ be the base set such that $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha)$ and $Im(\alpha)$ is either i or $-i$ for $i = 1,2,3, \dots, n$ such that each elements from $X_n \in X_n^*$ taken in the domain could occur in $2 \cdot 3$ times 3^n ways for each n

5.0 Summary of the results

The following table contains the summary of various results obtained.
Tables 5.1 contains the summary of the results of the signed singular mappings obtained :

S	$\lvert S \rvert$	$\lvert E(S) \rvert$	$\lvert N(S) \rvert$	$\lvert H_n \rvert$	$\lvert E_n \rvert$
$SSing_n$	$2^n(n^n - n!)$	$\frac{n^2(n^2-1)}{3}$	$-$	$2n^2 + n$	$2 \cdot 3^{n-1}$
$SPT_n \setminus SS_n$	$(2n+1)^n - 2^n n!$	?	5^{n-1}	$2n^2 + n$	?

CONCLUSION

The formula for idempotent of $SPT_n \setminus SS_n$ and nilpotent of $SSing_n$ has not been known.

Future Research

It is hereby recommended that the signed singular mappings of order – preserving and order – decreasing semigroups can also be investigate.

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