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ON THE SUM OF EXPONENTIALLY DISTRIBUTED RANDOM VARIABLES: A CONVOLUTION APPROACH

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ABSTRACT*: In this paper, Exponential distribution as the only continuous statistical distribution that exhibits the memoryless property is being explored by deriving another two-parameter model representing the sum of two independent exponentially distributed random variables, investigating its statistical properties and verifying the memoryless property of the resulting model.*

KEYWORDS: Exponential, Independent, Memoryless, Convolution, Hazard, Cumulant.

INTRODUCTION

Exponential distribution is a continuous probability model that is similar in one way to the geometric distribution (the duo are the only probability models that exhibit memoryless property). It is the only continuous probability distribution that has a constant failure rate (Garcia-Ortega, 2005 and Montgomery and Runger, 2003). It has been used severally for the analysis of Poisson processes and it is perhaps the most widely used statistical distribution for problems in reliability.
It has been established in literatures that if $X_i,\quad i = 1, 2, 3, \ldots, n$ are independently and identically distributed Exponential random variables with a constant mean $\frac{1}{\lambda}$ or a constant parameter λ (where λ is the rate parameter), the probability density function (pdf) of the sum of the random variables results into a Gamma distribution with parameters n and λ .

In this article, it is of interest to know the resulting probability model of Z= $X_1 + X_2$, the sum of two independent random variables X_1 and X_2, each having an Exponential distribution ***but not with a constant parameter***. That is, $X_1 \sim Exp(\lambda_1)$ and $X_2 \sim Exp(\lambda_2)$. Besides, we seek to know if the resulting model will still exhibit the memoryless property of the Exponential distribution and to investigate some of the statistical properties of the new model.

The technique of Convolution of random variables which has notably been used to derive the Convoluted Beta-Weibull distribution (Nadarajah and Kotz, 2006; Sun, 2011) and Convoluted Beta-Exponential distribution (Mdziniso, 2012; Shitu et al., 2012) shall be adopted.

METHODOLOGY

If X_1 and X_2 are iid Exponential random variables with parameters λ_1 and λ_2 respectively, Then,

$f(x_1) = \lambda_1 e^{-\lambda_1 x} \quad ,\lambda_1 > 0, x \geq 0$

Let $Z = X_1 + X_2$, then, $X_2 = Z - X_1$

$f(x_1, z - x_1) = \lambda_1 \lambda_2 e^{-\lambda_2 z} e^{(\lambda_2 - \lambda_1)x_1} \quad ,\lambda_1 > 0, \lambda_2 > 0, z > 0$

By the concept of Convolution of random variables,

$$f(z) = \int_0^z f(x_1) f(z - x_1)\, dx_1$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 z} \int_0^z e^{(\lambda_2 - \lambda_1)x_1}\, dx_1$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 z} . \frac{1}{\lambda_2 - \lambda_1} \left[e^{(\lambda_2 - \lambda_1)x_1} \right]_0^z$$

$$f(z) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 z} - e^{-\lambda_2 z} \right) \qquad (1)$$

$\lambda_1 > 0, \lambda_2 > 0, z \geq 0$

The model in Equation (1) above represents the probability model for the sum of two iid Exponential random variables.

Validity of the model $f(z)$

For the model $f(z)$ to be a valid model, it suffices that $\int_0^\infty f(z) dz = 1$

$$\int_0^\infty f(z) dz = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \int_0^\infty \left(e^{-\lambda_1 z} - e^{-\lambda_2 z} \right) dz$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left[\frac{1}{\lambda_2} e^{-\lambda_2 z} - \frac{1}{\lambda_1} e^{-\lambda_1 z} \right]_0^\infty$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} . \left[\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2} \right]$$

$$= 1$$

Cumulative Density Function (CDF)

By definition, the cdf is derived by; $F(z) = P(Z \leq z)$

$$= \int_0^z f(t) dt$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \int_0^z \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) dt$$

$$F(z) = 1 + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 z} - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 z} \qquad (2)$$

We can deduce from Equation (2) that $\lim_{z \to \infty} F(z) = 1$

2.3 Shape of the model

Differentiating $f(z)$ in Equation (1) with respect to z and equating the result to zero;

$$f'(z) = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_2}e^{-\lambda_2 z} - \frac{1}{\lambda_1}e^{-\lambda_1 z}\right) = 0$$

Solving for z,

$$z = \frac{1}{\lambda_2 - \lambda_1}[\ln\lambda_1 - \ln\lambda_2] > 0 \quad , \qquad for\ \lambda_1 \neq \lambda_2$$

This value indicates that the model has one mode (Unimodal)

The graph in Fig. 1 shows the shape of the distribution for $\lambda_1 = 0.5, \lambda_2 = 0.1$ and $\lambda_1 = 1, \lambda_2 = 0.5$

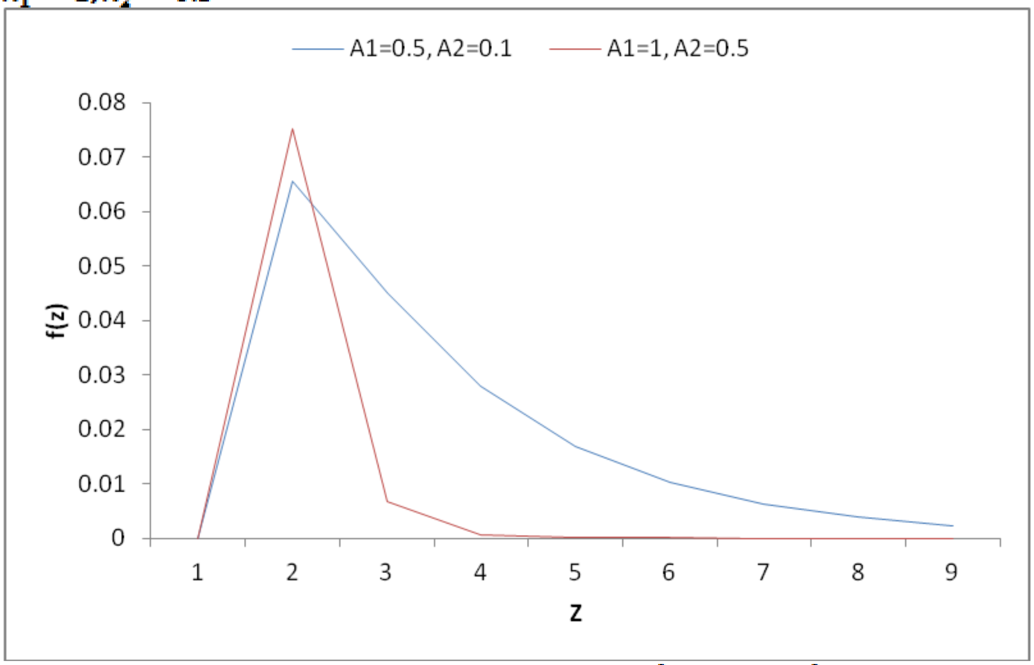

Fig. 1: Graph for the pdf of variable Z (where A1=λ_1 and A2=λ_2)

It can be deduced from Fig. 1 that the parameters λ_1 and λ_2 are both shape parameters. Increase in the value of λ_1 results in increase in the peak of the graph and increase in the value of λ_2 results in increase in the width of the graph.

Parameter Estimation

Let $z_1, z_2, \dots, z_n$ denote random sample from 'n' independent and identically distributed random variables each having the pdf derived in Equation (1) above. Using the method of maximum likelihood estimation, the likelihood function is given by;

$$L(Z|\lambda_1, \lambda_2) = \left(\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\right)^n . \prod_{i=1}^{n}(e^{-\lambda_1 z_i} - e^{-\lambda_2 z_i})$$

Let $l = log L(Z|\lambda_1, \lambda_2)$

$$l = n\log(\lambda_1\lambda_2) - n\log(\lambda_2 - \lambda_1) + \sum_{i=1}^{n}\log[e^{-\lambda_1 z_i} - e^{-\lambda_2 z_i}]$$

$$\frac{\partial l}{\partial \lambda_1} = \frac{n}{\lambda_1} - \frac{n}{(\lambda_2 - \lambda_1)} + \sum_{i=1}^{n} \frac{z_i e^{-\lambda_1 z_i}}{(e^{-\lambda_1 z_i} - e^{-\lambda_2 z_i})} \quad (3)$$

$$\frac{\partial l}{\partial \lambda_2} = \frac{n}{\lambda_2} - \frac{n}{(\lambda_2 - \lambda_1)} + \sum_{i=1}^{n} \frac{z_i e^{-\lambda_2 z_i}}{(e^{-\lambda_1 z_i} - e^{-\lambda_2 z_i})} \quad (4)$$

Setting $\frac{\partial l}{\partial \lambda_1}$ and $\frac{\partial l}{\partial \lambda_2}$ to zero and solving for λ_1 and λ_2 gives the maximum likelihood estimates of the parameters.

Hazard Function

By definition, the hazard function for a random variable Z is defined by;

$$h(z) = \frac{f(z)}{1 - F(z)}$$

$$h(z) = (\lambda_1 \lambda_2)\left[\frac{e^{-\lambda_1 z} - e^{-\lambda_2 z}}{\lambda_2 e^{-\lambda_1 z} - \lambda_1 e^{-\lambda_2 z}}\right] \quad (5)$$

Fig. 2 shows the graph of the hazard rate function for the model

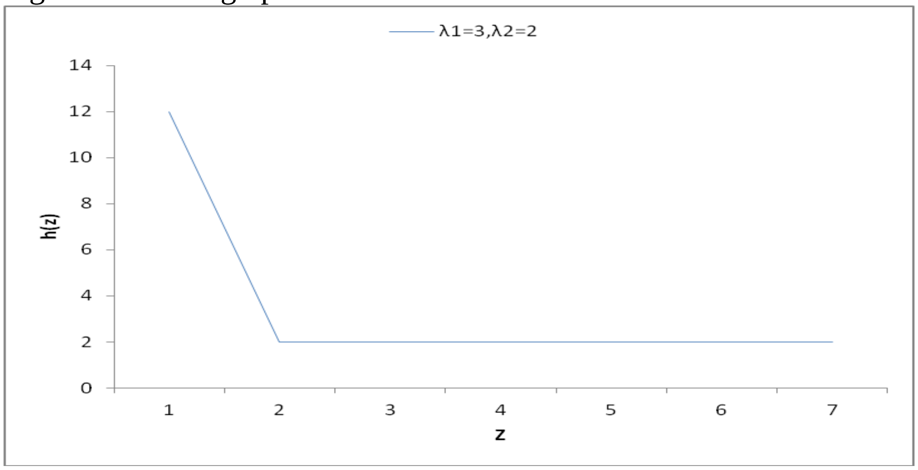

Fig. 2: Graph for the hazard function

It can be deduced from Fig. 2 that the hazard rate decreases at variable Z increases and remains at a constant value at some points. The meaning is that, the model in Equation (1) will be appropriate model events whose risk is high at the early stage, the risk get reduced with time and remains constant at a point.

Asymptotic Behavior

We seek to investigate the behavior of our model in Equation (1) as $z \to 0$ and as $z \to \infty$. This involves considering $\lim_{z \to 0} f(z)$ and $\lim_{z \to \infty} f(z)$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0}\left(\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)\right)$$
$$= 0$$

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty}\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 z}-e^{-\lambda_2 z}\right)\right)$$
$$= 0$$

These results confirm further that the model in Equation (1) has only one mode (Uni-modal)

Moment Generating Function

The moment generating function (m.g.f) of a random variable Z is denoted by $M_Z(t) = E(e^{tz})$. where $Z = X_1 + X_2$,

From the properties of m.g.f,

$$M_{X_1+X_2}(t) = E(e^{tx_1}).E(e^{tx_2})$$

where

$$E(e^{tX_1}) = \frac{\lambda_1}{\lambda_1 - t}$$
$$E(e^{tX_2}) = \frac{\lambda_2}{\lambda_2 - t}$$

$E(e^{tX_1})$ and $E(e^{tX_2})$ are the moment generating functions for a convoluted exponential distribution with parameters λ_1 and λ_2 respectively. Hence,

$$M_{X_1+X_2}(t) = \frac{\lambda_1\lambda_2}{(\lambda_1-t)(\lambda_2-t)} \quad (6)$$

Equation (6) can be re-written as $M_Z(t) = \lambda_1\lambda_2[(\lambda_1 - t)(\lambda_2 - t)]^{-1}$

The Characteristic function $\phi_Z(t) = E(e^{tz})$

$$\phi_Z(t) = \lambda_1\lambda_2[(\lambda_1 - it)(\lambda_2 - it)]^{-1}$$

From the result in Equation (6), we can confidently generalize that if $X_1, X_2, \ldots \ldots \ldots X_n$ are independently and identically distributed random variables, each having Exponential distribution with parameter, the moment generating function of the sum $Z = X_1 + X_2 + \cdots + X_n$ can be expressed as

$$M_Z(t) = \prod_{i=1}^{n} \lambda_i[\lambda_i - t]^{-1} \quad (7)$$

Moments

The rth raw moment of a random variable, say Z is given by;

$$E(Z^r) = \left.\frac{d^r M_Z(t)}{dt^r}\right|_{t=0}$$

As derived in Equation (6), $M_Z(t) = \lambda_1\lambda_2[(\lambda_1 - t)(\lambda_2 - t)]^{-1}$. Therefore, the first four moments are derived below as;

$$E(Z) = M_Z'(0)$$
$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$
$$E(Z^2) = M_Z''(0) = 2[\lambda_1{}^{-2} + \lambda_1{}^{-1}\lambda_2{}^{-1} + \lambda_2{}^{-2}]$$
$$E(Z^3) = M_Z'''(0) = 6[\lambda_1{}^{-3} + \lambda_1{}^{-2}\lambda_2{}^{-1} + \lambda_1{}^{-1}\lambda_2{}^{-2} + \lambda_2{}^{-3}]$$
$$E(Z^4) = M_Z^4(0) = 24[\lambda_1{}^{-4} + \lambda_1{}^{-3}\lambda_2{}^{-1} + \lambda_1{}^{-1}\lambda_2{}^{-3} + \lambda_1{}^{-2}\lambda_2{}^{-2} + \lambda_2{}^{-4}]$$

Hence, we can make the following generalizations;

(1) The mean of the sum of 'n' independent Exponential distribution is the sum of individual means. That is, if $Z = X_1 + X_2 + \cdots + X_n$, then,

$$E(Z) = \sum_{i=1}^{n} \frac{1}{\lambda_i} \qquad (8)$$

(2) The rth moment of Z can be expressed as;

$$E(Z^r) = r! \sum_{i=0}^{r} {\lambda_1}^{-i} {\lambda_2}^{-(r-i)} \qquad (9)$$

Cumulant generating function

By definition, the cumulant generating function for a random variable Z is obtained from,

$$C_Z(t) = \log[M_Z(t)]$$
$$= \log\left\{\frac{\lambda_1\lambda_2}{(\lambda_1 - t)(\lambda_2 - t)}\right\}$$

By expansion using Maclaurin series,

$$C_Z(t) = \sum_{n=1}^{\infty} (n-1)! \left[\left(\frac{1}{\lambda_1}\right)^n + \left(\frac{1}{\lambda_2}\right)^n\right] \frac{t^n}{n!} \qquad (10)$$

From the definition of cumulants, the cumulant K_n of a random variable Z are obtained from the coefficients of $\frac{t^n}{n!}$ In Equation (10).

Hence,

$$K_n = (n-1)! \left[\left(\frac{1}{\lambda_1}\right)^n + \left(\frac{1}{\lambda_2}\right)^n\right] \qquad (11)$$

The first four cumulants are given below as;

$$K_1 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$
$$K_2 = \left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2$$
$$K_3 = 2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3\right]$$
$$K_4 = 6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4\right]$$

The following can be deduced from the results above;

(1) The first cumulant K_1 is the mean of the random variable Z

(2) The second cumulant K_2 is the variance of Z

(3)
$$Skewness = \frac{K_3}{K_2^{3/2}}$$
$$= \frac{2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2\right]^{3/2}}$$

(4)
$$kurtosis = \frac{K_4}{K_2^2}$$

$$= \frac{6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2\right]^2}$$

Memoryless Property

We say that an Exponential distribution exhibits memoryless property because the condition below holds; Given that a bulb has survived *s* units of time, the probability that it survives a further *t* units of time is the same as that of a fresh bulb surviving t unit of time. That is,

$$P(X > s + t | X > s) = P(X > t)$$

where X is a random variable.

$$P(X > s + t | X > s) = \frac{P(X > s + t, X > s)}{P(X > s)}$$

$$= \frac{P(X > s + t)}{P(X > s)} \qquad (12)$$

$$= \frac{\int_{(s+t)}^{\infty} \lambda e^{-\lambda x} dx}{\int_{s}^{\infty} \lambda e^{-\lambda x} dx}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = P(X > t)$$

That is, $P(X > s + t | X > s) = P(X > t)$

Let us now consider the distribution of the sum of two independent Exponential distributions given in Equation (1);

From Equation (12), $P(Z > s + t | Z > s) = \frac{P(Z > s + t)}{P(Z > s)}$

$$= \frac{\int_{(s+t)}^{\infty} \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right) dz}{\int_{s}^{\infty} \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right) dz}$$

$$\neq P(Z > t)$$

Where $P(Z > t) = \frac{1}{\lambda_2 - \lambda_1}\left[\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}\right]$

Hence, we can infer that the memoryless property does not hold for the distribution of the sum of two independent Exponential distributions

CONCLUSION

In this article, we used the concept of convolution to derive a two-parameter distribution representing the sum of two independent Exponential distributions. Some of its statistical properties were also investigated. It was observed that the distribution is positively skewed and has only one mode. Based on the behavior of the hazard function, the model would be appropriate in fitting data of stock returns, death roll in insurgencies, survival of sickle cell patients and survival of a new product in the market among others. It was also observed that the memoryless property does not hold for this model.

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