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ON THE APPLICATION OF MULTIVARIATE TEST OF HYPOTHESIS TO COMPARE ADVANCED MATHEMATICAL ABILITIES

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ABSTRACT: *The multivariate test of means, using the Hotelling's T^2 distribution, has vast applications in many real-life situations; education inclusive. Prominent amongst them is the test of means between two groups; say treatment and control groups in tests scores given an intervention. The objective of this study was to apply the Hotelling's T^2 distribution to compare the performance of mathematical science students in Kaduna Polytechnics, Nigeria in Advanced Mathematics. In specifics terms, the study wishes to determined the mean difference between the students' Advanced Mathematical abilities for treatment and control group in tests scores using three variables; Linear Algebra, Calculus I and Calculus II. The treatment intervention is an introductory algebra taught for ten weeks. In order to compare only two populations, multiple t-tests may be considered, but such analyses may result in an unacceptably high probability of Type I error. To avoid this problem, a single multivariate hypothesis testing procedure (omnibus test) serves better. This omnibus test of two group means is conducted using the Hotelling's T^2 distribution. After the data analysis using the SPSS, the results have revealed that the treatment group produces better results than the control group in Advanced Mathematical abilities. Furthermore, the descriptive statistics have also affirmed that the treatment group produces better results of Advanced Mathematical performance than the control group. Moreover, this was further confirmed by the individual Analysis of Variance (ANOVA) test. In general, the treatment has enhanced performance in Advanced Mathematics. Hence, the treatment in form of tutorial (In Introductory Algebra) for ten weeks, has significantly improved the performance of students in Advanced Mathematics. By extension, we conclude that the treatment (In Introductory Algebra) is effective.*

KEY WORDS: Multivariate test, omnibus test, variance-covariance matrix, mean vectors, Analysis of Variance (ANOVA) test, covariances, omnibus test, dispersion matrix, mean vectors, univariate test, Hotelling's T^2 distribution.

INTRODUCTION

Multivariate statistical methods are designed to investigate relationships between several variables in a set of data. Multivariate analysis essentially extends univariate analysis for single parameters to vectors or matrices of parameters. Concisely, multivariate techniques are useful

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when observations are obtained for each of a number of subjects on a set of variables of interest. Comparably, univariate statistical methods deal with single variable. The idea is to make statistical inferences about the population distribution of the variable, set confidence intervals for its parameters, and perhaps test hypotheses about the values of the parameters. The data in each variable is one-dimensional while multivariate analysis deals with many variables simultaneously (Usman, 2016).

Studies in the real-life situations usually have more than one outcome variable. Frequently in such studies, the researcher simply performs separate statistical tests on each of the outcome variables. However, there are serious drawbacks to such a separate test. Such tests may result in an unacceptably high Type I error. In other words, if you test many hypotheses separately, each at the 5% level of significance, there is a high probability that at least one of them will be rejected by chance. Thus, it is desirable that a more robust multivariate method be found to test all of the hypotheses simultaneously. It turns out that the procedure for the multivariate test of means is an equivalent of single variable test of means. Hence, a multivariate test procedure (omnibus test) based on the analysis of variation among group means and variation of units within groups may be conducted using the Hotelling's T^2 distribution (Usman, 2016).

The multivariate test of means, using the Hotelling's T^2 distribution, has vast applications in real-life situation; education inclusive. Prominent amongst them is the test of means between two groups; say control treatment and control groups in tests scores. The objective of this study was to apply the Hotelling's T^2 distribution to compare the performance of mathematical science students in Kaduna Polytechnics, Nigeria in Advanced Mathematics. In specifics terms, this study is to determined the mean difference between the students' Advanced Mathematical abilities for treatment and control group in tests scores using three variables; Linear Algebra, Calculus I and Calculus II. The treatment intervention is an introductory algebra taught for ten weeks.

METHODOLOGY

The Hotelling's T^2 distribution is used to test of hypothesis and inferences concerning two mean vectors rather than individual means while preserving the omnibus inference and covariance structure. It is the multivariate equivalent of the t-test used in univariate test of hypothesis (Morrison, 2005). In order to demonstrate the Hotelling's T^2 distribution, start by recalling the univariate theory for the test of the null hypothesis $\mathbf{H_0 : \mu_1 = \mu_2}$ against, say, a two-sided alternative hypothesis $\mathbf{H_1 : \mu_1 \neq \mu_2}$ using the univariate independent sample t-test. Where; $\boldsymbol{\mu_1}$ and $\boldsymbol{\mu_2}$ are the means from two normal populations. If $\bar{X}_1$ and $\bar{X}_2$ denote the means of independent random samples from the two respective normal population, the appropriate test statistic is as follows.

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$$t = \frac{\bar{X}_1 - \bar{X}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

This test statistic follows the t-distribution with $n_1 + n_2 - 2$ degrees of freedom. The null hypothesis is rejected if the observed $|t|$ exceeds the critical value of t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

Where;

$$\bar{X}_{ij} = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \quad for \quad i = 1, 2, \cdots, p; \quad j = 1, 2, \cdots, n$$

$$s_{ij}^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2 \quad for \quad i = 1, 2, \cdots, p; \quad j = 1, 2, \cdots, n$$

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 1}$$

Rejecting H_0 when $|t|$ exceeds $t_{\alpha/2, n_1+n_2-2}$ is tantamount to rejecting the null hypothesis when:

$$t^2 = \frac{\left(\bar{X}_1 - \bar{X}_2\right)^2}{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \frac{n_1 n_2}{n_1 + n_2}\left(\bar{X}_1 - \bar{X}_2\right)\left(s_p^2\right)^{-1}\left(\bar{X}_1 - \bar{X}_2\right) > t_{\alpha/2, n_1+n_2-2}^2$$

The variable t^2 above is the squared distance from the sample means $\bar{X}_1$ and $\bar{X}_2$. The units of distance are expressed in terms of the pooled standard deviation of $\bar{X}_1$ and $\bar{X}_2$. (Marcoulides, & Hershberger, 2012). The moment $\bar{X}_1$, s_1^2 and $\bar{X}_2$, s_2^2 are observed, then the null hypothesis is rejected, at α/2 level of significance, if:

$$t^2 = \frac{n_1 n_2}{n_1 + n_2}\left(\bar{X}_1 - \bar{X}_2\right)\left(s_p^2\right)^{-1}\left(\bar{X}_1 - \bar{X}_2\right) > t^2_{\alpha/2, n_1+n_2-2}$$

Where; $t_{\alpha/2, n_1+n_2-2}$ denotes the upper 100α of the *t*-distribution with $n_1 + n_2 - 2$ degrees of freedom. If the null hypothesis is not rejected, it is then concluded that the values of the normal population means are approximately equal. (Pituch, & Stevens, 2016). A natural generalization of the squared distance in its multivariate analogue is as follows:

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}\left(\bar{\mathbf{X}}_{\mathbf{1}} - \bar{\mathbf{X}}_{\mathbf{2}}\right)^{\mathbf{T}} \mathbf{s}_{\mathbf{p}}^{\mathbf{-1}}\left(\bar{\mathbf{X}}_{\mathbf{1}} - \bar{\mathbf{X}}_{\mathbf{2}}\right)$$

Where;

$$\underset{(\mathbf{p}\times\mathbf{1})}{\bar{\mathbf{X}}_{\mathbf{j}}} = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \quad for \quad i = 1, 2, \cdots, p; \quad j = 1, 2, \cdots, n$$

$$\underset{(\mathbf{p}\times\mathbf{p})}{\mathbf{S}} = \frac{1}{n-1}\sum_{j=1}^{n}\left(\mathbf{X}_{\mathbf{ij}} - \bar{\mathbf{X}}_{\mathbf{i}}\right)\left(\mathbf{X}_{\mathbf{ij}} - \bar{\mathbf{X}}_{\mathbf{i}}\right)^{\mathbf{T}} \quad for \quad i = 1, 2, \cdots, p; \quad j = 1, 2, \cdots, n$$

In this case, the two multivariate normal populations are the percentage tests scores in treatment and control groups using three variables; scores in Linear Algebra, Calculus I and Calculus II. In order to test the null hypothesis concerning two multivariate normal populations where the difference is zero; we proceed as follows (Everitt, & Hothorn, 2011).

$$\mathbf{H_0} : \boldsymbol{\mu}_{\mathbf{1}} = \boldsymbol{\mu}_{\mathbf{2}}$$
$$\mathbf{H_1} : \boldsymbol{\mu}_{\mathbf{1}} \neq \boldsymbol{\mu}_{\mathbf{2}}$$

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In this case; $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are the mean vectors are tests scores from the treatment and control groups respectively. Based on random samples of sizes each for treatment and control groups examined respectively; the multivariate test statistic is as follows:

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)^{\mathbf{T}} \mathbf{s}_{\mathbf{p}}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)$$

The formula above is the two-sample Hotelling's T^2 distribution; if the observed T^2 exceeds the critical value or the *p-value* less than the level of significance, the null hypothesis is rejected, (Timm, 2002).

Where; $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ denote the sample mean vectors which are tests scores from the treatment and control groups respectively are defined as follows.

$$\bar{\mathbf{X}}_1 = \begin{pmatrix} \bar{X}_{1(1)} \\ \bar{X}_{2(1)} \\ \bar{X}_{3(1)} \end{pmatrix} \text{ and } \bar{\mathbf{X}}_2 = \begin{pmatrix} \bar{X}_{1(2)} \\ \bar{X}_{2(2)} \\ \bar{X}_{3(2)} \end{pmatrix}$$

The three variables in each mean vector are scores in scores in Linear Algebra, Calculus I and Calculus II respectively. Furthermore, $\mathbf{S}_{\mathbf{P}}$ is the pooled sample dispersion matrix defined as follows.

$$\mathbf{s}_{\mathbf{p}} = \frac{(n_1 - 1)\mathbf{s}_{\mathbf{1}} + (n_2 - 1)\mathbf{s}_{\mathbf{2}}}{n_1 + n_2 - 1}$$

The sample dispersion matrix are values of tests scores from the treatment and control groups respectively are defined as follows.

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$$\mathbf{S_1} = \begin{pmatrix} S_{11(1)} & S_{12(1)} & S_{13(1)} \\ S_{21(1)} & S_{22(1)} & S_{23(1)} \\ S_{31(1)} & S_{32(1)} & S_{33(1)} \end{pmatrix} \text{ and } \mathbf{S_2} = \begin{pmatrix} S_{11(2)} & S_{12(2)} & S_{13(2)} \\ S_{21(2)} & S_{22(2)} & S_{23(2)} \\ S_{31(2)} & S_{32(2)} & S_{33(2)} \end{pmatrix}$$

The elements on the main diagonal of the dispersion matrices above are the variances of scores of Linear Algebra, Calculus I and Calculus II respectively while the off-diagonal elements are the covariances. The data would be analyzed using the SPSS.

RESULTS

Based on random samples of sizes 150 each from treatment and control groups examined respectively; by using the following Hotelling's T^2 test statistic:

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)^{\mathbf{T}} \mathbf{s}_{\mathbf{p}}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)$$

Where; $\bar{\mathbf{X}}_1$ and $\bar{\mathbf{X}}_2$ denote the sample mean vectors from the treatment and control groups respectively are defined as follows.

$$\bar{\mathbf{X}}_1 = \begin{pmatrix} \bar{X}_{1(1)} \\ \bar{X}_{2(1)} \\ \bar{X}_{3(1)} \end{pmatrix} = \begin{pmatrix} 67.0 \\ 66.3 \\ 60.2 \end{pmatrix}$$

$$\bar{\mathbf{X}}_2 = \begin{pmatrix} \bar{X}_{1(2)} \\ \bar{X}_{2(2)} \\ \bar{X}_{3(2)} \end{pmatrix} = \begin{pmatrix} 53.8 \\ 48.9 \\ 44.8 \end{pmatrix}$$

The three variables in each mean vector are scores in Linear Algebra, Calculus I and Calculus II respectively. Furthermore, $\mathbf{S_P}$ is the pooled sample dispersion matrix defined as follows.

$$\mathbf{s_p} = \frac{(n_1 - 1)\mathbf{s_1} + (n_2 - 1)\mathbf{s_2}}{n_1 + n_2 - 1}$$

The sample dispersion matrix from the treatment and control groups respectively are defined as follows.

$$\mathbf{S_1} = \begin{pmatrix} S_{11(1)} & S_{12(1)} & S_{13(1)} \\ S_{21(1)} & S_{22(1)} & S_{23(1)} \\ S_{31(1)} & S_{32(1)} & S_{33(1)} \end{pmatrix} = \begin{pmatrix} 212.9 & 69.2 & 140.5 \\ 69.2 & 217.4 & 85.5 \\ 140.5 & 85.5 & 193.6 \end{pmatrix}$$

$$\mathbf{S_2} = \begin{pmatrix} S_{11(2)} & S_{12(2)} & S_{13(2)} \\ S_{21(2)} & S_{22(2)} & S_{23(2)} \\ S_{31(2)} & S_{32(2)} & S_{33(2)} \end{pmatrix} = \begin{pmatrix} 99.8 & 25.2 & 14.8 \\ 25.2 & 168.5 & -2.9 \\ 14.8 & -2.9 & 114.4 \end{pmatrix}$$

The elements on the main diagonal of the dispersion matrices above are the variances of scores in Linear Algebra, Calculus I and Calculus II respectively while the off-diagonal elements are the pairwise covariances. By using the Hotelling's T^2 test statistic above, the null hypothesis must be rejected if the observed T^2 exceeds the critical value. Alternatively, the null hypothesis must be rejected if the *p-value* is less than the level of significance. From the data using the SPSS, the analysis proceeds as follows:

Hypothesis

H_0: The students' performance in Advanced Mathematics is the same for treatment and control groups.

H_1: The students' performance in Advanced Mathematics differ for treatment and control groups.

Level of Significance

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α=0.05

Test Statistics

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)^{\mathbf{T}} \mathbf{s}_{\mathbf{p}}^{-1}\left(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2\right)$$

Decision Criterion

Reject the null hypothesis if $p < 0.05$

Computations

As carried out by the SPSS as given in the following tables:

Table 1: Multivariate Descriptive Statistics

Variables	Group	N	Mean	Std. Deviation
Scores in Linear Algebra	Treatment group	150	67.0	14.6
	Control group	150	53.8	10.0
	Total	300	60.4	14.1
Scores in Calculus I	Treatment group	150	66.3	14.7
	Control group	150	48.9	13.0
	Total	300	57.6	16.4
Scores in Calculus II	Treatment group	150	60.2	13.9
	Control group	150	44.8	10.7
	Total	300	52.5	14.6

The descriptive statistics above show the means and standard deviations for Scores in Linear Algebra, Calculus I and Calculus II for both the treatment and control groups.

Table 2: Multivariate Tests[a]

Effect		Value	F	Hypothesis df	Error df	Sig.
Intercept	Pillai's Trace	0.972	3479.108[b]	3	256	0.000
	Wilks' Lambda	0.028	3479.108[b]	3	256	0.000
	Hotelling's Trace	35.261	3479.108[b]	3	256	0.000
	Roy's Largest Root	35.261	3479.108[b]	3	256	0.000
Group	Pillai's Trace	0.396	64.764[b]	3	256	0.000
	Wilks' Lambda	0.604	64.764[b]	3	256	0.000
	Hotelling's Trace	0.656	64.764[b]	3	256	0.000
	Roy's Largest Root	0.656	64.764[b]	3	256	0.000

a. Design: Intercept + Group
b. Exact statistic

Table 3: Tests of Between-Subjects Effects

Source	Dependent Variable	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	Linear Algebra Scores	13041.613[a]	1	13041.613	83.412	0.000
	Calculus I Scores	22700.041[b]	1	22700.041	117.637	0.000
	Calculus II Scores	17656.341[c]	1	17656.341	114.646	0.000
Intercept	Linear Algebra Scores	1095172.920	1	1095172.920	7004.524	0.000
	Calculus I Scores	994222.387	1	994222.387	5152.276	0.000
	Calculus II Scores	826297.601	1	826297.601	5365.299	0.000
Group	Linear Algebra Scores	13041.613	1	13041.613	83.412	0.000
	Calculus I Scores	22700.041	1	22700.041	117.637	0.000
	Calculus II Scores	17656.341	1	17656.341	114.646	0.000
Error	Linear Algebra Scores	46592.967	298	156.352		
	Calculus I Scores	57504.352	298	192.968		
	Calculus II Scores	45894.308	298	154.008		
Total	Linear Algebra Scores	1154807.500	300			
	Calculus I Scores	1074426.780	300			
	Calculus II Scores	889848.250	300			
Corrected Total	Linear Algebra Scores	59634.580	299			
	Calculus I Scores	80204.393	299			
	Calculus II Scores	63550.649	299			

a. R Squared = .219 (Adjusted R Squared = .216)
b. R Squared = .283 (Adjusted R Squared = .281)
c. R Squared = .278 (Adjusted R Squared = .275)

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CONCLUSION

From Table 2, since the $p=0.000<0.05$ for all the tests Pillai's Trace, Wilks' Lambda, Hotelling's Trace and Roy's Largest Root, the null hypothesis must be rejected. Hence, the treatment group produces better results than the control group in the Scores in Linear Algebra, Calculus I and Calculus II. Hence, the mean values contained in Table1 have further affirmed that the treatment group produces better results than the control group in Linear Algebra Scores Calculus I and Calculus II. Moreover, this is further confirmed by the individual Analysis of Variance (ANOVA) test for each of the variables on Table 3. That is, from table 3, we have $p=0.000<0.05$ for scores in Linear Algebra, $p=0.000<0.05$ for Calculus I Scores and $p=0.000<0.05$ for Calculus II Scores. Hence, we affirm that the treatment group has better performance in Advanced Mathematics than the control group. In general, the treatment (ten weeks tutorials in introductory Algebra) has enhanced performance in mathematics. Hence, the treatment has significant effect in the performance of students in Advanced Mathematics. By extension, we conclude that the treatment is very effective.

References

Everitt, B.S. & Hothorn, T. (2011). *An introduction to applied multivariate analysis with R.* New York: Springer.

Marcoulides, G.A. and Hershberger, S.L. (2012). *Multivariate statistical methods: a first course*. New York: Routledge; Taylor & Francis Group.

Morrison, D.F. (2005). *Multivariate statistical methods* (4th ed). New York: Duxubury.

Pituch, K.A. & Stevens, J.P. (2016). *Applied multivariate statistics for the social sciences* (6th. Ed). New York: Routledge Press.

Timm, H.N. (2002). *Applied Multivariate analysis.* New York: Springer-Verlag.

Umar, S., Usman, A. & Umar, S.S. (2021). *Multivariate Analysis of the Mathematical Fluid Intelligence of Some Selected Students.* IOSR Journal of Mathematics 17(3), 1-4.

Umar, S., Usman, A. & Umar, S.S. (2021). *Comparing the Trajectory of Mathematical Performance by Gender.* Journal of Research in Applied Mathematics.

Usman, A. (2016). *Introduction to bivariate and multivariate Statistical analysis*. Kaduna: Millennium Printing and Publishing Company Limited.