International Journal of Mathematics and Statistics Studies
Vol.8, No.2, pp.40-50, June 2020
Published by ECRTD-UK
Print ISSN: 2053-2229 (Print)
Online ISSN: 2053-2210 (Online)

ON SOME STOCHASTIC GROWTH MODELS WITH APPLICATIONS

Khairia El-Said El-Nadi, M. EL-Shandidy and Yousria Hamad Omralryany
Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Alexandria Egypt
Abstractkhairia_el_said@hotmail.com, elshandidy@yahoo.com and omralryany@gmail.com

ABSTRACT: *In this paper, we consider some stochastic dynamical systems.The stochastic processes related to these models are some different kinds of tumor cancer cells. The growth and diffusion of brain tumor cancer is also studied by using a non-linear stochastic diffusion model. The considered models are generalization of some deterministic models. In the frame of these stochastic models we are able to study the growth models for tumor cells under the influence of random perturbations. In general mathematical models help to predict the tumour size and optimize the treatment procedure in deteministic form, there are several models including exponential, dynamical systems, logistic, nonlinear diffusion equations that have been used to describe the behaviour of cancer cell growth amd proliferation. By using the stochastic analysis and the Adomian method we can study more general models.*
KEYWORDS: stochastic dynamical systems, stochastic diffusion processes, ito integrals, tumor cancer cells, adomian method.

AMS Subject classifications: 34F05-92B05-37C45-34A60-56C30

.

INTRODUCTION

Different types of dynamical systems of cancer progression and treatment have already been constructed. For example there are many problems written on the subject of mathematical models in cancer chemotherapy such as Hannelore Liset and David Julitz [1], K.R. Fister and Pannetta [2], J.M. Murrary,[3] and many others. We mension also the papers [4-7]. In this paper, we shall study a stochastic dynamical system and a stochastic diffusion process. In section 2, we shall solve the considered stochastic dynamical sestem. Some properties are also studied. In section 3, we shall solve a nonlinear stochastic partial differential equation. In section 4 a nonlinear stochastic diffusion model related to brain cancer is studied. We shall use Adomian decomposition method to find the stochastic solution.

A stochastic dynamical system

Let: (Ω, F, $(\mathcal{A}_t)_{t\geq 0}$, P) be a filtered probability space. Consider the following stochastic dynamical system.

$dX = b(X)\,dt + B(X)dW(t)$, (1)

$t\in[0,a_2^{-1}]$, where; $X = \begin{bmatrix} T \\ N \\ L \end{bmatrix}$,

International Journal of Mathematics and Statistics Studies
Vol.8, No.2, pp.40-50, June 2020
Published by ECRTD-UK
Print ISSN: 2053-2229 (Print)
Online ISSN: 2053-2210 (Online)

$$b(X) = \begin{bmatrix} a_1 T\,(1 - a_2 T) - D(t)\,T \\ a_3 - a_4 N - a_5 NT \\ -a_6 L - a_7 L\,T + a_8 NT \end{bmatrix},$$

$$B(X) = \begin{bmatrix} \sigma_1 T & 0 & 0 \\ 0 & \sigma_2 N & 0 \\ 0 & 0 & \sigma_3 L \end{bmatrix},$$

$$W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \\ W_3(t) \end{bmatrix}, \qquad \sigma_i \in (-\infty, \infty),\ i = 1, 2, 3,$$

W_1 , W_2 and W_3 are standared independent Weiner processes adapted to the filteration $(\mathcal{A}_t)_{t\geq 0}$, See [8].

The stochastic process $\{T(t) : t \geq 0\}$ is the tumor cell population at time t, $\{N(t): t \geq 0\}$ is the total level of natural killer cells effectiveness at time t and $\{L(t): t \geq 0\}$ is the total level of tumor specific CD 8^+ T cell effectveness at time t, see [9 – 12].
The parameters $a_1,\ldots, a_8$ are positive and D(t) is given by

$$D(t) = a_7 \frac{(L/T)^c}{s + (L/T)^c},\ c, s \geq 0.$$

It is supposed that the expectations E $[T^2(0)]$, E $[N^2(0)]$ and E $[L^2(0)]$ exist. It is supposed also that the vector (T(0), N(0), L(0)) is independent of $(W_1(t), W_2(t), W_3(t))$. We shall find now a representation for T(t).

Set $T^* = T^{-1}$ and using the stochastic chain rule of Ito, we get:

$$dT^* = [a_1 a_2 + (\sigma_1^2 - a_1 + D(t))\,T^*]\,dt - \sigma_1 T^*\,dW_1(t).$$

Thus;

$$T^*(t) = T^{-1}(0) \exp\left[\sigma_1 W_1(t) - \frac{1}{2}\sigma_1^2\, t + \int_0^t f(\theta)\,d\theta\right] +$$

$$a_1 a_2 \int_0^t exp\ \left[\sigma_1 \{W_1(t) - W_1(\theta)\} \frac{1}{2}\sigma_1^2\,(t - \theta) + \int_\theta^t f(r)\,dr\right]\ d\theta,$$

Where; $f(r) = \sigma_1^2 - a_1 + D(r)$

Notice that $T^*(t)$ is almost surely, a.s., positive,

Thus $T(t) = T^{*-1}(t)$

The total level of the natural killer cells N(t), is given by:

$$N(t) = N(0) \exp\left[\sigma_2 W_2(t) - a_4 t - \frac{1}{2}\sigma_2^2\, t - a_5 \int_0^t T(\theta)\right] dt$$

$$+ a_3 \int_0^t exp\ \left[\sigma_2 \{W_2(t) - W_2(\theta)\} - a_4(t-\theta) - \frac{1}{2}\sigma_2^2\,(t - \theta) - a_5 \int_\theta^t T(r)\ dr\right]dt$$

The total level L(t) of tumor specific CD 8^+ is given by:

$$L(t) = L(0) \exp\left[\sigma_3 W_3(t) - a_7 \int_0^t T(\theta)\ dt - a_6 t - \frac{1}{2}\sigma^2 t\right]$$

$$+ \int_0^t a_8\, N(\theta)\, T(\theta) \left[\exp \sigma_3 (W_3(t) - W_3(\theta)) - a_7 \int_\theta^t T(r)\ dr - a_6(t - \theta) - \frac{1}{2}\sigma^2\,(t-\theta)\right] d\theta\,.$$

Let us study the case when;

s = 0 , Notice that the processes

International Journal of Mathematics and Statistics Studies
Vol.8, No.2, pp.40-50, June 2020
Published by ECRTD-UK
Print ISSN: 2053-2229 (Print)
Online ISSN: 2053-2210 (Online)

$W_1(t)$, $W_1(t) - W_1(s)$, $T(0)$ are independent and

$$E\left[e^{\sigma_i W_1(t)}\right] = \exp\left[\frac{\sigma_i^2}{2} t\right],$$

$$E\left[e^{\sigma_i (W_1(t) - W_i(s)}\right] = \exp\left[\frac{(t-s)\,\sigma_i^2}{2}\right], \qquad i = 1, 2, 3$$

Thus;

$$E[T^{-1}(t)] = e^{(a_7 - a_1 + \sigma_1^2)t}\, E[T^{-1}(0)] + \frac{a_1 a_2}{a_7 - a_1 + \sigma_1^2}\left[e^{(a_7 - a_1 + \sigma_1^2)t} - 1\right].$$

It is clear that

$$T(t) \le T(0) \exp\left[\sigma_1 W_1(t) + \frac{1}{2}\sigma_1^2 t - \int_0^t f(\theta)\, d\theta\right].$$

Thus;

$$E[T(t)] \le E(T(0))\, e^{(a_1 - a_7)t}.$$

If $a_2 = 0$, we get

$$T(t) = T(0) \exp\left[\sigma_1 W_1(t) + \frac{1}{2}\sigma_1^2 t - \int_0^t f(\theta)\, d\theta\right].$$

If $a_2 = s = 0$, we get

$$E(T^2(t)) = E(T^2(0)) \exp[\sigma_1^2 t + 2(a_1 - a_7)t].$$

Notice that $T(t)$ and $W_2(t)$ are independent, thus

$$E[N(t)] = e^{-ta_4} E(N(0)) E\left[\exp\left\{-a_5 \int_0^t T(\theta)\, d\theta\right\}\right] + a_3 \int_0^t e^{-a_4 (t-\theta)} E\left[\exp\left\{-a_5 \int_\theta^t T(r)\, dr\right\}\right] d\theta.$$

It is easy to see that

$$E[N(t)] \le e^{-ta_7} E[N(0)] + \frac{a_3}{a_3}[1 - e^{-ta_4}].$$

The product $N(t)T(t)$ and $W_3(t)$ and also $T(t)$ are indepent, thus

$$E[L(t)] = E[L(0)]\, e^{-a_6 t} E\left[-a_7 \int_\theta^t T(\theta)\, d\theta\right] + a_8 \int_\theta^t E\left[N(\theta)\, T(\theta) \exp\left\{-a_7 \int_\theta^t T(r)\, dr\right\}\right] e^{a_6(t-\theta)}\, d\theta.$$

A nonlinear stochastic diffusion model

In this section, we shall study the following nonlinear stochastic diffusion model;

$$u(x, t) = f(x) + \int_0^t a(x)\, u(x, s) \frac{\partial}{\partial x}\left[\frac{1}{u(x, s)} \frac{\partial u(x, s)}{\partial x}\right] ds + \int_0^t \left[\lambda \ln \frac{\mu}{u(x,s)} - g(v(s))\right] u(x, s)\, ds + \sigma \int_0^t u(x, s)\, d W(s)$$

Where $W(t)$ is a standard Wiener process adapted to the filteration $(\mathcal{A}_t)_{t\ge 0}$, $\sigma \in (-\infty, \infty)$, λ and μ are positive parameters, $f(x)$ and $a(x)$ are given continuous bounded functions defined on $(-\infty, \infty)$. It is supposed that $a(x)$ is positive for all $x \in (-\infty, \infty)$.

It is supposed also that g and v are maps from $(0, \infty) \times \Omega$ to $(0, \infty)$, that are measurable and adapted to the filtration $(\mathcal{A}_t)_{t\ge 0}$ and are a.s., locally bounded.

International Journal of Mathematics and Statistics Studies
Vol.8, No.2, pp.40-50, June 2020
Published by ECRTD-UK
Print ISSN: 2053-2229 (Print)
Online ISSN: 2053-2210 (Online)

The stochastic process $\{u(x, t): -\infty < x < \infty, t > 0\}$ may be thought as the concentration of brain cancer tumor cells at a location x and time t, $u(x,0) = f(x)$ represents the initial deterministic state.

Consider the following two Brownian motions:

$X_1(t) = \exp[-\sigma W(t)]$, $X_2(t) = \exp[\sigma W(t)]$.

It is easy to see that:

$$d X_1(t) = \frac{\sigma^2}{2} X_1(t)\, dt - \sigma X_1(t)\, dW(t),$$

$$d X_2(t) = \frac{\sigma^2}{2} X_2(t)\, dt + \sigma X_2(t)\, dW(t),$$

$X_1(0) = X_2(0) = 1$.

Set $u^*(x, t) = X_1(t)\, u(x,t)$ and applying Ito's product rule formula, we get

$du^*(x, t) = X_1\, du(x,t) + u(x,t)\, dX_1 - \sigma^2 X_1\, u(x,t)\, dt$.

Thus ;

$$\frac{1}{u^*(x,t)} \frac{\partial u^*(x,t)}{\partial t} = a(x) \frac{\partial}{\partial x}\left[\frac{1}{u^*(x,t)} \frac{\partial u^*(x,t)}{\partial x}\right] + F(t) - \lambda \ln(u^*(x,t)),$$

$u^*(x, 0) = f(x)$,

Where;

$$F(t) = \frac{-\sigma^2}{2} + \lambda \ln \mu - g(v(t)) - \lambda \sigma W(t).$$

Set $\upsilon(x,t) = e^{\lambda t} u^*(x,t)$, one gets:

$$\frac{\partial \upsilon(x,t)}{\partial t} = a(x) \frac{\partial^2 \upsilon(x,t)}{\partial x^2} + e^{\lambda t} F(t),$$

$\upsilon(x, 0) = \ln f(x)$

Thus; $\upsilon(x, t)$ is given by

$$\upsilon(x,t) = \int_{-\infty}^{\infty} G(x, y, t) \ln f(y)\, dy +$$

$$+ \int_0^t \int_{-\infty}^{\infty} G(x, y, t-s)\, e^{\lambda s} F(s)\, dy\, ds.$$

.

Where G is the fundamental solution of the equation

$$\frac{\partial u}{\partial t} = a(x) \frac{\partial^2 u}{\partial x^2}.$$

Since $\int_{-\infty}^{\infty} G(x, y, t)\, dy = 1$, it follows that

$$\upsilon(x,t) = \int_{-\infty}^{\infty} G(x, y, t) \ln f(y)\, dy + \int_0^t e^{\lambda s} F(s)\, ds,$$

So;

$$u(x, t) = \exp[\sigma W(t) + e^{-\lambda t} \upsilon(x, t)], \quad (2)$$

For a.s., $\omega \in \Omega$, and all $t \geq 0$, $-\infty < x < \infty$. Thus we can find a subset $\Omega^* \subset \Omega$, with $P(\Omega^*) = 1$, such that formula (2) is valid for all $\omega \in \Omega^*$, $t \geq 0$, $-\infty < x < \infty$.

It is clear that

$$W(t) = \int_0^t e^{\lambda(s-t)}\, dW(s) + \lambda \int_0^t e^{\lambda(s-t)} W(s)\, ds, \quad (3)$$

for all $W \in \Omega^*$.

From (2) and (3) one gets

$$u(x, t) = \exp\left[V(t) - \int_0^t e^{\lambda(s-t)}\, g(v,s)\, ds + H(x, t)\right],$$

Where

International Journal of Mathematics and Statistics Studies
Vol.8, No.2, pp.40-50, June 2020
Published by ECRTD-UK
Print ISSN: 2053-2229 (Print)
Online ISSN: 2053-2210 (Online)

$V(t) = \sigma \int_0^t e^{\lambda(s-t)}\, dW(s)$,

$H(x, t) = e^{-\lambda t} \int_{-\infty}^{\infty} G(x, y, t) \ln f(y)\, dy + \left[\frac{\sigma^2}{2}(1 - e^{-\lambda t})\right] \ln \mu$.

Notice that V(t) is a Gausian process with mean zero. The variance of V(t) is given by

$E[V^2(t)] = E\left[\sigma^2 \int_0^t e^{\lambda(s-t)}\, dW(s)\right]^2 = \sigma^2 \int_0^t e^{2\lambda(s-t)}\, ds =$

$= \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right)$.

Thus;

$E\left[e^{V(t)}\right] = \frac{1}{\sqrt{2\pi\gamma(t)}} \int_{-\infty}^{\infty} e^{x}\, e^{\frac{-x^2}{2\gamma(t)}}\, dx = \exp\left[\frac{\gamma(t)}{2}\right]$,

where;

$\gamma(t) = \frac{\sigma^2}{2\lambda}\left[1 - e^{-2\lambda t}\right]$.

If g and γ are independent of W, then the expected number of tumor cells is given by

$E[u(x,t)] = E\left[\exp\left\{\frac{1}{2}\gamma(t) + H(x, t)\right\}\right] E\left[\exp\left\{-\int_0^t e^{\lambda(s-t)}\, g(\gamma(s))\, ds\right\}\right]$.

It can be also compute all the moments of the stochastic process {u(x,t)}:

$e^{-nH(x,t)} E[u^n(x,t)] =$

$= E\left[\exp\left\{\frac{1}{2} n^2 \gamma(t)\right\}\right] E\left[\exp\left\{\int_0^t n\, e^{\lambda(s-t)}\, g\, ds\right\}\right]$.

A brain cancer model

Let V(R,T) be the concentration of brain tumor cells at a location R and time T. Consider now the Burgers equation;

$$\frac{\partial V(R,T)}{\partial T} = \frac{D}{R^2}\frac{\partial}{\partial R}\left[R^2 \frac{\partial V(R,T)}{\partial R}\right] - (p - k)F(T)V(R,T). \qquad (4)$$

Where D is the diffusion coefficients, (estimated at 0.0013 Cm^2 per day for glioblastoma multiform).

The constants p and k represent proliferations rate and killing rate respectively, (R measures the distance from the origin of glioblastoma). The function F(T) describes to the temporal profile of the treatment.

Equation (5) can be written in the form

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} - c(t)\, u(x,t), \qquad (5)$$

where $t = (p - k)T$, $p > k$,

$$x = \sqrt{\frac{a(p-k)}{D}}R, \qquad c(t) = F(T),$$

$u(x,t) = x\, V(R,T)$.

If the influence of drugs is considered, see [4, 5, 6], then a more general mathematical model is given in the form;

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} + \left[\lambda \ln\left(\frac{\mu}{u(x,t)}\right) - g\left(v(s)\right)\right]u(x,t).$$

We perturb the last equation by a multiplicative noise term and consider the following stochastic integral equation;

$$u(x,t) = f(x) + a\int_o^t \frac{\partial^2 u(x,s)}{\partial x^2} ds$$
$$+\int_o^t \left[\lambda \ln\left(\frac{\mu}{u(x,s)}\right) - g\left(v(s)\right)\right] u(x,s)ds + \sigma\int_o^t u(x,s)dW(s). \quad (6)$$

Set h (x, t) = X_1 (t) u (x, t) and applying Ito's formula, we get

$$dh(x,t) = a\frac{\partial^2 h(x,t)}{\partial x^2}dt +$$
$$+\left[\lambda \ln\left(\frac{\mu}{h(x,t)}\right) - g\left(v(s)\right) + \lambda\sigma W(t) - \frac{\sigma^2}{2}\right] h(x,t)dt +$$
$$+\int_o^t f(s)ds - \lambda\int_o^t v(x,s)ds, \quad (7)$$

For almost every w in Ω ,
where

$$f(t) = \lambda \ln\mu - \frac{\sigma^2}{2} - \lambda\sigma W(t) - g\left(v(s)\right).$$

Set $v(x,t) = ln\ [h(x,t)]$. We get

$$v(x,t) = ln\ \phi(x) +$$
$$+a\int_o^t \left[\frac{\partial^2 v(x,s)}{\partial x^2} + \left(\frac{\partial v(x,s)}{\partial x}\right)^2\right] ds$$
$$+\int_o^t f(s)ds - \lambda\int_o^t v(x,s)ds, \quad (8)$$

For almost every w in Ω ,

By a solution of (8), we mean a function ν such that ν, $\frac{\partial v}{\partial x}\ and\ \frac{\partial^2 v}{\partial x^2}$ are continuous on (- ∞, ∞) X [0, T] and satisfies equation (8) on (- ∞, ∞) X [0, T], for almost every ω ε Ω, (see [7,8]).

Theorem (1) . If there exists a solution ν(x,t) of equation (8) such that $\left|\frac{\partial v(x,t)}{\partial x}\right| \le M$ on (- ∞, ∞) x [O, T] for almost every ω ε Ω, where M is a positive constant, then that solution is unique for almost every ω ε Ω.

Proof. Set

$$v_i(x,t) = r^{-\lambda t}V_i(x,t),$$
$$V_3(x,t) = V_1(x,t) - V_2(x,t),$$

where v_1(x,t) and v_2(x, t) are solutions of (8), for almost every ω ε Ω.
It is easy to see that;

$$V_3(x,t) = \int_o^t \int_{-\infty}^{\infty} G(x-y,t,s)\frac{\partial V_3(t,s)}{\partial y}\left[\frac{\partial V_1(y,s)}{\partial y} + \frac{\partial V_2(y,s)}{\partial y}\right] dyds,$$

Where

$$G(x-y,t,s) = e^{-\lambda s}\frac{e^{\frac{-(y-x)^2}{4a(t-s)}}}{\sqrt{4\pi a(t-s)}}$$

Noticing that $\frac{\partial V_1(x,t)}{\partial x}$ and $\frac{\partial V_2(x,t)}{\partial x}$ are bounded on (- ∞, ∞) X [0, T], thus after simple calculations we get

$$Max\left|\frac{\partial V_3(x,t)}{\partial x}\right| \le \frac{M_3^n}{n!}, n = 1,2,3,\ldots..$$

where $M_3 > 0$ is a constant.
Letting $n \to \infty$, we get

$$\left|\frac{\partial V_3(x,t)}{\partial x}\right| = 0 \quad on \quad (-\infty,\infty)X[0,t].$$

But V_3 (x,0) = 0. Thus
ν_1 (x, t) = ν_2 (x, t) on (- ∞, ∞) X [0, T) and for almost every ω ε Ω .
This means that there exists a unique positive solution u(x,t) of equation (6) for almost every ω ε Ω, (see [8,…,12]).
To solve equation (8), we use the Adomian decomposition method. Assume that the unknown function ν can be represented by an infinite series of the form;
$\nu(x,t) = \sum_{i=o}^{\infty} \quad \nu_i(x,t),$ (9)
and the nonlinear term can be decomposed by an infinite series of polynomials given by

$$\left(\frac{\partial \nu(x,t)}{\partial x}\right)^2 = \sum_{i=0}^{\infty} \quad A_i(x,t).$$

The function ν_i (x, t), I = 0, 1,2, …. will be determined recurrently, and A_i(x,t) are the so-called Adomian polynomials of ν_o, ν_1, ν_2, …… determined by
Determined by

$$A_k(x,t) = \frac{1}{k!}\frac{d^k}{d\eta^k}\left[\frac{\partial}{\partial x}\sum_{i=0}^{\infty} \quad \eta^i \nu_i(x,t)\right]_{\eta=0}^2$$

k = 0, 1, 2, ….
Equation (8) can be written in the form

$$\nu(x,t) = \overset{-\lambda t}{e}\, ln\, \phi\,(x,t) + \int_o^t e^{\lambda(s-t)} f(s)ds$$

$$+a\int_o^t e^{\lambda(s-t)}\left[\frac{\partial^2 \nu(x,t)}{\partial x^2} + \left(\frac{\partial \nu(x,t)}{\partial x}\right)^2\right]ds. \qquad (10)$$

Substituting (9) and (10) into (11) and identifying the zero component ν_o (x, t) by

$$\nu_o(x,t) = \overset{-\lambda t}{e}\, ln\, \phi\,(x) + \int_o^t \quad e^{\lambda(s-t)} f(s)ds,$$

then the remaining components can be determined by using the recurrence relation

$$\nu_n(x,t) = a\int_o^t e^{\lambda(s-t)}\left[\frac{\partial^2 \nu_n(x,s)}{\partial x^2} + A_n(x,s)\right]ds.$$

The Adomian polynomials A_i, i = 0, 1, 2, …… are given by

$$A_o(x,t) = \left(\frac{\partial \nu_o(x,t)}{\partial x}\right)^2,$$

$$A_1(x,t) = 2\frac{\partial \nu_o(x,t)}{\partial x}\frac{\partial \nu_1(x,t)}{\partial x},$$

$$A_2(x,t) = \left(\frac{\partial \nu_1(x,t)}{\partial x}\right)^2 + 2\frac{\partial \nu_o(x,t)}{\partial x}\frac{\partial \nu_2(x,t)}{\partial x},$$

$$A_3(x,t) = 2\frac{\partial \nu_o(x,t)}{\partial x}\frac{\partial \nu_3(x,t)}{\partial x} + 2\frac{\partial \nu_1(x,t)}{\partial x}\frac{\partial \nu_2(x,t)}{\partial x}.$$

Notice that;

$$\sum_{i=o}^{\infty} A_i(x,t) = \sum_{i=o}^{\infty} \left[\frac{\partial v_i(x,t)}{\partial x}\right]^2 + 2\sum_{i\neq j} \frac{\partial v_i(x,t)}{\partial x}\frac{\partial v_i(x,t)}{\partial x}$$

Now the solution of (10) can be represented by;

$v(x,t) = e^{-\lambda t} ln \quad \phi(x) + \int_o^t e^{\lambda(s-t)} f(s)ds + F(x,t),$

where F is a deterministic function, which is given by

$$F(x,t) = a\int_o^t e^{\lambda(s-t)} \left[\sum_{i=1}^{\infty} \frac{\partial^2 v_i(x,t)}{\partial x^2} + A_i(x,s)\right] ds,$$

From the uniqueness of the solution of equation (4) it follows that

$$v(x,t) = \exp\left[v(x,t) + \sigma W(t)\right]$$

$$= exp \quad \left[e^{-\lambda t} ln\, \phi(x) + F(x,t) + \sigma W(t) + \int_o^t e^{\lambda(s-t)} f(s)ds\right].$$

Let us compute the expected number E [u (x, t)] of tumor cells u(x, t) at time t > 0 and location x.

It is clear that the stochastic process $\sigma\int_o^t e^{\lambda(s-t)}\, dW(s)$ is a zero – mean Gaussian process with variance:

$$E\left[\sigma^2\left\{\int_o^t e^{\lambda(s-t)} dW(s)\right\}^2\right] = \sigma^2 E\left[\int_o^t e^{2\lambda(s-t)} ds\right] = \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right)$$

Consequently

$$E\left[exp \quad \left\{\sigma\int_o^t e^{\lambda(s-t)} dW(s)\right\}\right] = \frac{\sigma^2}{4\lambda}\left(1 - e^{2\lambda t}\right)$$

Supposing that g and ν are independent of the process W(t), we get

$$E[u(x,t)] =$$

$$= \phi^*(x,t) E\left[exp\left\{-\int_o^t e^{\lambda(s-t)} g\left(v(s)\right) ds\right\}\right.$$

Where

$$\phi^*(x,t) = [\phi(x)]^{e^{-\lambda t}} [\mu]^{1-e^{-\lambda t}} exp\left[\frac{\sigma^2}{4\lambda}\left(1 - e^{-2\lambda t}\right) - F^*(x,t)\right],$$

$$F^*(x,t) = \frac{\sigma^2}{2\lambda}\left(1 - e^{-\lambda t}\right) - F(x,t).$$

Theorem 2. If $\phi(x) = e^{\alpha x+\beta}$, where α and β are constants, then the solution of equation (4) is given by;

$$u(x,t) = \Psi(x,t)\, exp \quad \left[\sigma\int_o^t e^{\lambda(s-t)} dW(s) - \int_o^t e^{\lambda(s-t)}\, G(r(s)ds\right],$$

Where

$$\psi(x,t) = [\mu]^{1-e^{-\lambda t}} exp\left[(\alpha x + \beta) e^{-\lambda t} \frac{1}{2\lambda}\left(2a\alpha e^{-\lambda t} - \sigma^2\right)(1 - e^{-\lambda t})\right].$$

Proof. We have

$$v_1(x,t) = a\int_o^t e^{\lambda(s-t)}\left[\frac{\partial^2 v_o(x,s)}{\partial x^2} + A_o(x,s)\right]ds$$
$$= \frac{a\alpha^2}{\lambda}\ {}^{-\lambda t}e\left[1 - {}^{-\lambda t}e\right],$$

$$v_2(x,t) = a\int_o^t e^{\lambda(s-t)}\left[\frac{\partial^2 v_1(x,s)}{\partial x^2} + 2\frac{\partial v_o(x,s)}{\partial x}\frac{\partial v_1(x,s)}{\partial x}\right]ds$$
$$= 0,$$

So $v_n(x, t) = 0$, for $n \geq 2$.
Hence the required result.

Suppose that g and ν are independent of the process W(t). Thus the expected number E [u (x, t)] at time t > 0 and location x is given by :

E(x) = $\psi(x,t) = exp\left[\frac{\sigma^2}{4\lambda}\left(1 - {}^{-\lambda t}e\right) - \int_o^t {}^{-\lambda t}e\, g\left(v(s)\right)ds\right]$.

The Variance V [u (x, t)] of the tumor cells u(x, t) at time t > 0 and location x is given by;

V [u (x, t)] =

$= \psi^2(x,t)\, exp\left[-2\int_o^t e^{\lambda(s-t)}\, g\left(v(s)\right)ds\right] exp[z(t)(z(t)-1)]$,

Where:

$z(t) = \frac{\sigma^2}{2\lambda}\left(1 - {}^{-2\lambda t}e\right)$,

(Comp. [13 – 23]).

CONCLUSION

The suitable mathematical models of stochastic dynamical systems and nonlinear stochastic partial differential equations explore important problems in biology. This tool is an ever increasing towards. The considrered models incomporate tumor immune interaction terms of forms that are qualitively different form these commonly used.

References

[1] Hannelore liset and David Julitz, A stochast model for the growth of cancer tumor, STUDIA UNIV. ”BABES-BOLYAI”. MATHEMATICA, Volume LIII, Number 4 December (2008).
[2] K. R. Fister and j. c. Panetta , Optimal control applied to cell -cycle- specific cancer chemotherapy, SIAM Jousnal Applieed Mathematics, 1059-1072, 60 (2000).
[3] M. Murray , Optimal control for a cancer chemotherapy problem with general growth and loss functions ,Mathematics Biosciences,98,273- 287, (1990)
[4] A .J. Coldman and J. M Murray, Optimal control for a stochastic model of cancer chemotherapy, Mathematics Biosciences,187-200, 168 (2002).
[5] G. Albano and V. Giorno, A stochastic model in turom growth, Journal of theoretical Biology, 329-336, 242 (2006).
[6] Boondirek, Y. Lenbury, J. wong-Ekkabut, W. Triampo, I. M. Tang, and P. Picha, A stochastic model of cancergrowth with immune response, Journal of the korean

physical society,1652-1666, 49 (2006).
[7] W. Y. Tan and C. W. Chen, cancer stochastic models, Encyclopediaof statistical sciences I I, published online 15 August (2006) .
[8] N. Komarova, stochastic modeling of drug resistance in cancer, Journal of theoretical Biology, 351-366, 239 (2006).
[9] Sibusiso Moyo and P. G. L. Leach, symmetry methods applied to a math-ematical model of a tumor of the brain, proceeding of Restitute of Mathematical of NAS of Ukraine , 204-210, 50(2004).
[10] T. Revilla and G. Garci a-Ramas, Fighting a virus with a virus, a dynamic model for HIV-1 therapy, Mathematical Bugers Sciences, 191- 203, 185 (2000).
[11] Khairia EL-Said EL-Nadi , On some stochastic parabolic differential equations in a Hilbert space , Journal of Applied Mathematics & stochastic Analysis, 167-175, 2(2006).
[12] Khairia EL-Said EL-Nadi, On some stochastic differential equations and fractional Brownian motion, International Journal of pure and Applied Mathematics , 416-423, 24(2005).
[13] Khairia EL-Said EL-Nadi, Asymptotic methods and some difference frac- tional differential aquatics, Int. J. Contemp. Math, Sciences,3-14, 1 (2006).
[14] Khairia EL-Said EL-Nadi, On the stability of some stochastic integral partial differential equations , Int. J. Contemp. Math, Sciences, 1317- 1326, 2(2007).
[15] Mahmoud M. EL-Borai and Khairia EL-Said EL-Nadi, Onsome fractional parabolic equations driven by fractional Gaussian noise, Special Issue Sci- ence and Mathematics with Applications, Int. J. of Research and reviewers in Applied Sciences, 6(3) (2011).
[16] Mahmoud M. EL-Borai and Khairia EL-Said EL-Nadi, An inverse frac- tional abstract Cauchy problem with non local condition, Life science Journal, 1705-1709, 10(3) (2013).
[17] Mathmoud M. EL-Borai and Khairia EL-Said EL-Nadi, Integrated semi- groups and Cauchy problems for some fractional abstract differential equations, life Science Journal ,793-798, 10 (3) (2013).
[18] Khairia El-Said El-Nadi, Wagdy G. El-Sayed and Ahmed Khdher Qassem, Mathematical model of brain tumor, International Research Journal of Engineering and Technology (IRJET), Volume:02, Aug. 2015, 590-594.
[19] Khairia El-Said El-Nadi, Wagdy G. El-Sayed and Ahmed Khdher Qassem, On some dynamical systems of controlling tumor growth, International Journal of Applied Science and Mathematics, Volume 2, Issue 5, 2015, 146- 151.
[20] Khairia El-Said El-Nadi, L.M. Fatehy, Nourhan Hamdy Ahmed, Marchall Olkin exponential with application on cancer stem cells, American Journal of Theoritical and Applied Statistics, 2017, 6(5-1), 1-7.
[21] Khairia El-Said El-Nadi, Wagdy G. El-Sayed, Mabroka F. Bader, On some lag synchronization and higher order parabolic systems, American Journal of Theoretical and Applied Statistics, 2017; 6(5-1); 23-29.
[22] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, On some stochastic nonlinear equations and the fractional Brownian motion, Caspian Journal of Computational & Mathematical Engineering, 2017, No. 1 , 20-33.
[23] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, On the stability of some

fractional dynamical models related to tumor cancer evaluation, Elixir Adv. 112 (2017) 48894-48899.

Acknoledgment
The authors would like to express their sincere gratitude to the editor-in-chief and the anonymous referees for their valuable comments and suggestions.