ON THE MODIFIED EXTENDED GENERALIZED EXPONENTIAL DISTRIBUTION

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ABSTRACT: *A modification on the extension of generalized exponential distribution due to Olapade (2014) is presented in this paper and some of its properties, such as; The cumulative distribution function, the survival function, the hazard function and it properties, the reverse hazard function, the moment generating function, the kth moment about the origin, the median, the percentile point and the associated initial-value problem (IVP) for ordinary differential equation (O.D.E.) are established.*

KEYWORDS: Generalized Exponential Distribution, Moment Generating Function, Median, Percentile, Mode, Initial-Value Problems.

INTRODUCTION

Exponential distributions have proven to be one of the outstanding continuous distributions over time, probably due to its simplicity nature and useful in modeling life time data and various problems related to waiting time events. If X is a random variable denoting the waiting time between successive events which follows the poisson distribution with mean λ, then X follows the exponential distributions with probability density function (pdf)given by

$$f_X(x;\lambda) = \lambda e^{-\lambda x} \quad x,\lambda > 0 \qquad (1.1)$$

And it associated cumulative distribution function (cdf) is given by

$$F_X\,(x;\lambda) = 1 - e^{-\lambda x}\,,\ x,\lambda > 0 \qquad (1.2)$$

respectively.

Over the years, generalization of distribution functions has attracted a lots of attention and interest, after the introduction of a generalised (exponentiated) exponential (GE) distribution by Gupta and Kundu (1999), several researchers have focused on improving this distribution function in different directions with the aim of obtaining a distribution function that will be more robust, and applicable in modeling different types of data. Gupta and Kundu (1999) introduced a new distribution, named, "Generalized Exponential (GE) distribution " with pdf, cdf, survival function, and hazard function given by;

$$f_X(x;\alpha,\lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1}e^{-\lambda x} \qquad (1.3)$$

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$$F_X\,(x;\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha} \tag{1.4}$$

$$S_X\,(x;\alpha,\lambda) = 1-(1-e^{-\lambda x})^{\alpha} \tag{1.5}$$

and

$$h_X\,(x;\alpha,\lambda) = \frac{\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}}{1-(1-e^{-\lambda x})^{\alpha}}\,, x,\alpha,\lambda > 0 \tag{1.6}$$

where α, λ are the shape parameter, scale parameter respectively.

Gupta and Kundu (2000, 2001, 2003, 2004, 2007) also continued in this direction; by studying the properties of Generalized Exponential (GE) distribution, also in relation to weibull and Gamma distribution. They observed that this distribution can be used in place of gamma and weibull distributions, since the two parameters of the gamma, weibull and generalized exponential distribution have the same properties of increasing and decreasing hazard function if their shape parameter is greater than one and less than one, and they have a constant hazard function if their shape parameter is equal to. In the likes of this, several researchers have done the same , the logistic distribution has enjoyed the same trend of generalization in many forms as could be seen in Balakrishnan and Leung (1988), Jong-Wuu, Hung Wen-Liang, and Lee Hsiu-Mei, (2000), George et'al (1980) and Olapade (2004,2005,2006) worked on several types of generalized logistic distribution.

Motivated by the works of Gupta and Kundu and several authors, Olapade (2014) extended the generalized exponential distribution by introducing an additional parameter as an improvement to the existing model, and called it extended generalized exponential (EGE) distribution given by

$$f_X\,(x;\mu,\alpha,\beta,\lambda,) = \frac{\alpha\lambda(\beta-e^{-\lambda(x-\mu)})^{\alpha-1}e^{-\lambda(x-\mu)}}{[\beta^{\alpha}-(\beta-1)^{\alpha}]};\; x,\mu,\lambda > 0;\; \alpha,\beta > 1 \tag{1.7}$$

REMARK 1.1 It is important to note that the introduction of the parameter β into equation (1.3) by Olapade (2014) to obtain equation (1.7) has no effect on;

(a) the hazard function at the modal point,

(b) the properties of the hazard function of the original model by Gupta and Kundu (1999),

(c) the determining factor for existence of mode.

Apart from the fact that these properties of the hazard function is very important in inferential statistics of life time data, Gupta and Kundu used these properties to relate the similarities of generalizes exponential distribution with some other (Gamma and Weibull) life time distributions. Hence, the Extended Generalizes Exponential (EGE) Distribution by Olapade (2014) failed to extend the properties of the hazard function of the original model by Gupta and Kundu (1999) and at the modal point. Olapade (2014), (section 6, pp. 1285) claimed that "the only determining factor for the mode of EGE to exist is that $\beta - 2\alpha < 0$ ". This claim does not reduce to the condition that guarantees the determining factor for existence of the mode of GE distribution by Gupta and Kundu (1999).

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Apart from correcting these lapses in extended generalizes exponential (EGE) distribution by Olapade (2014), in this paper, we introduce a new distribution by modifying the extend generalization exponential distribution of Olapade, which we called the modified extended generalized exponential (MEGE) distribution, thereby, improving on the result of Olapade (2014), Gupta and Kundu (1999, 2000).

2 Six-Parameter Exponential Distribution

Definition 2.1 Suppose X is continuous random variable, then we say X follows the (MEGE) distribution if the pdf is given by ;

$$f_X\,(x;\alpha,\beta,\lambda,\eta,\mu,\sigma) = \frac{\alpha\lambda(\beta-\eta e^{-\lambda(\frac{x-\mu}{\sigma})})^{\frac{\alpha}{\eta}-1} e^{-\lambda(\frac{x-\mu}{\sigma})}}{\sigma[\beta^{\frac{\alpha}{\eta}} - \left(\beta-\eta e^{\lambda\frac{\mu}{\sigma}}\right)^{\frac{\alpha}{\eta}}]}; x,\beta,\alpha,\eta,\mu,\lambda,\sigma > 0; \tag{2.1}$$

Where X follows a modify extended generalized exponential distribution. Observe that σ can be absorb by λ, also, there is no loss of generality if we assume that $\mu = 0$. So that;

$$f_X\,(x;\alpha,\beta,\lambda,\eta,0) = \frac{\alpha\lambda(\beta-\eta e^{-\lambda x})^{\frac{\alpha}{\eta}-1} e^{-\lambda x}}{[\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}}]}\;, x,\alpha,\eta,\beta,\lambda > 0 \quad (2.2)$$

Where $\alpha, \beta, \lambda, \eta, \mu, \sigma$, are shape, extension, scale, regularization, location, dispersion parameters respectively and denoted by MEGE$(.;\alpha,\beta,\lambda,\eta)$ (or MEGE for short). We obtain the corresponding cdf, F_X of f_X in equation (2.2) as follows;

$$F_X = \int_0^x f_y(y;\alpha,\beta,\lambda,\eta,0,1)dy$$

$$= K\int_0^x \left(\beta-\eta e^{-\lambda y}\right)^{\frac{\alpha}{\eta}-1} dy\;; \quad for\ K = \alpha\lambda\left(\beta^{\alpha/\eta} - (\beta-\eta)^{\alpha/\eta}\right)^{-1}$$

Let $u = \beta - \eta e^{-\lambda y}$, then $du = \lambda\eta e^{-\lambda y}dy$, we have ;

$$F_X = K\frac{1}{\alpha\lambda}|\left(\beta-\eta e^{-\lambda x}\right)^{\alpha/\eta}|_0^x$$

Hence we have

$$F_X(x;\alpha,\beta,\lambda,\eta,0)=\begin{cases}0 & ; \quad x<0\\ \dfrac{(\beta-\eta e^{-\lambda x})^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}}{\left[\beta^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}\right]} & ; \quad 0<x<\infty\\ 1 & ; \quad x\geq\infty\end{cases} \qquad (2.3)$$

Observe that; $\lim_{x\to\infty} F_X=1(=\int_0^\infty f_x dx)$, confirming that f_x is actually a pdf.

The corresponding survival (reliability) function, hazard function and reverse hazard function are given below as;

$$s_X(x;\alpha,\beta,\lambda,\eta,0)=1-F_X(x;\alpha,\beta,\lambda,\eta,0)=\frac{\beta^{\alpha/\eta}-(\beta-\eta e^{-\lambda x})^{\frac{\alpha}{\eta}}}{\left[\beta^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}\right]} \quad (2.4)$$

$$h_X(x;\alpha,\beta,\lambda,\eta,0)=\frac{f_X(x;\alpha,\beta,\lambda,\eta,0)}{S_X(x;\alpha,\beta,\lambda,\eta,0)}=\alpha\lambda\frac{(\beta-\eta e^{-\lambda x})^{\frac{\alpha}{\eta}-1}e^{-\lambda x}}{[\beta^{\frac{\alpha}{\eta}}-(\beta-\eta e^{-\lambda x})^{\frac{\alpha}{\eta}}]} \quad (2.5)$$

$$\varphi_X(x;\alpha,\beta,\lambda,\eta,0)=\frac{f_X(x;\alpha,\beta,\lambda,\eta,0)}{F_X(x;\alpha,\beta,\lambda,\eta,0)}=\frac{\alpha\lambda(\beta-\eta e^{-\lambda x})^{\frac{\alpha}{\eta}-1}e^{-\lambda x}}{(\beta-\eta e^{-\lambda x})^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}} \quad (2.6)$$

In equation(2.2) to equation(2.6), if we take $\eta=1$, we obtain the results for EGE distribution by Olapade (2014). In equation(2.2), if $\eta=\beta=1$, we obtain the GE distribution by Gupta and Kundu (1999) and if $\eta=\beta=\alpha=1$ we obtain the classical exponential distribution with parameter λ $(with\ \mu=0)$ in equation (1.1).

3. The Moment Generating Function for Modified Extended Generalized Exponential Distribution

The moment generating function for the modified extended generalized exponential distribution is given by;

$$M_X(t)=\int_0^\infty e^{tx}f_X(x;\alpha,\beta,\lambda,\eta,0)dx$$

$$=\frac{\alpha\lambda}{\beta^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}}\int_0^\infty e^{tx}(\beta-\eta e^{-\lambda x})^{\frac{\alpha}{\eta}-1}e^{-\lambda x}dx$$

$$M_X(t)=K\beta^{\frac{\alpha}{\eta}-1}\int_0^\infty\left[1-\left(\frac{\eta e^{-\lambda x}}{\beta}\right)\right]^{\frac{\alpha}{\eta}-1}e^{-x(\lambda-t)}\,dx \qquad (3.1)$$

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Where $K = \alpha\lambda(\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}})^{-1}$

Let $p = \frac{\eta e^{-\lambda x}}{\beta}$; $e^{-\lambda x} = \frac{\beta p}{\eta}$; $x = -\lambda^{-1}\ln(\frac{\beta p}{\eta})$ $and\ x = \ln(\frac{\beta P}{\eta})^{\frac{1}{\lambda}}$

$\Longrightarrow dx = -(\lambda p)^{-1}dp.$

Using equation (3.1) we have ;

$$M_X(t) = -K\beta^{\frac{\alpha}{\eta}-1}\int (1-p)^{\frac{\alpha}{\eta}-1}\left(\frac{\beta}{\eta}\right)^{1-\frac{t}{\lambda}}.(p)^{1-\frac{t}{\lambda}}.(\lambda p)^{-1}\,dp$$

$$= -K\eta^{1-t/\lambda}\beta^{\frac{\alpha}{\eta}-\frac{t}{\lambda}}\lambda^{-1}\int_{\eta/\beta}^{0}(1-p)^{\frac{\alpha}{\eta}-1}p^{-t/\lambda}dp$$

$$\therefore M_X(t) = K\eta^{1-\frac{t}{\lambda}}\beta^{\frac{\alpha}{\eta}-\frac{t}{\lambda}}\lambda^{-1}\int_{0}^{\eta/\beta}[1-p]^{\frac{\alpha}{\eta}-1}.(p)^{-t/\lambda}dp \qquad (3.3)$$

By binomial expansion of $(1-p)^{\frac{\alpha}{\eta}-1}$ and substituting for K, equation(3.3) yields ;

$$M_X(t) = \frac{\alpha.\eta^{1-\frac{t}{\lambda}}\beta^{\frac{\alpha}{\eta}-\frac{t}{\lambda}}}{\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}}}\int_{0}^{\eta/\beta}\sum_{r=0}^{\infty}(-1)^r.C\left(\frac{\alpha}{\eta}-1,r\right).p^{r-\frac{t}{\lambda}}dp$$

where $C\left(\frac{\alpha}{\eta}-1,r\right) = \frac{\left(\frac{\alpha}{\eta}-1\right)\left(\frac{\alpha}{\eta}-2\right)\ldots\left(\frac{\alpha}{\eta}-r\right)}{r!} = \frac{\prod_{i=1}^{r}\left(\frac{\alpha}{\eta}-i\right)}{r!}$

$$M_X(t) = \frac{\alpha.\eta^{1-\frac{t}{\lambda}}.\beta^{\frac{\alpha}{\eta}-\frac{t}{\lambda}}}{\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}}}\int_{0}^{\eta/\beta}\sum_{r=0}^{\infty}\prod_{i=1}^{r}\left(i-\frac{\alpha}{\eta}\right)\frac{p^{r-\frac{t}{\lambda}}}{r!}dp \qquad (3.4)$$

Interchanging the integral and the summation with the knowledge that the series converges, we obtain;

$$M_X(t) = \frac{\alpha.\eta^{2(1-\frac{t}{\lambda})}.\beta^{\frac{\alpha}{\eta}-1}}{\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}}}\sum_{r=0}^{\infty}\frac{\psi(r)}{r!}.\frac{\eta^r}{\beta^r(r+1-\frac{t}{\lambda})} \qquad (3.5)$$

$$M_X(t) = \frac{\alpha.\eta^2.\beta^{\frac{\alpha}{\eta}-1}}{\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}}}\sum_{r=0}^{\infty}\frac{\psi(r)\eta^r}{\beta^r r!}\eta^{\frac{-2t}{\lambda}}(r+1--\lambda^{-1}t)^{-1} \qquad (3.6)$$

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where $\psi(r;\alpha,\eta) = \begin{cases} \prod_{i=1}^{r}\left(i-\frac{\alpha}{\eta}\right) \; ; & r>0 \\ 1 \; ; & r=0 \end{cases}$

Put

$$C = \frac{\alpha.\eta^2.\beta^{\frac{\alpha}{\eta}-1}}{\beta^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}}\sum_{r=0}^{\infty}\frac{\psi(r)\eta^r}{\beta^r r!}, \qquad u = \frac{-2}{\lambda}, \qquad v = \frac{1}{\lambda}.$$

Then differentiating equation(3.6), k number of times (denoted by $M_X(t)^{(k)}$) we have ;

$$M_X(t)^{(k)} = C\sum_{s=0}^{k}\binom{k}{s}D^s(\eta^{ut})\,D^{k-s}((r+1-vt)^{-1})$$

$$= C\sum_{s=0}^{k}\binom{k}{s}(\ln\eta^u)^s\,\eta^{ut}(k-s)!\,(r+1-vt)^{-(k-s+1)}v^{k-s} \quad (3.7)$$

Evaluating equation(3.7) at $t=0$, we obtain the kth moment obout the origin μ_k of the MEGE

$$\mu_k = \frac{\alpha\eta^2\beta^{\frac{\alpha}{\eta}-1}}{\beta^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}}\sum_{r=0}^{\infty}\sum_{s=0}^{k}\frac{k!\,(-2)^s\,(\ln\eta)^s}{\lambda^k s!\,r!\,(r+1)^{k+1-s}}\;\psi(r)\left(\frac{\eta}{\beta}\right)^r$$

$$= \frac{\alpha\eta^2\beta^{\frac{\alpha}{\eta}-1}}{\beta^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}}\sum_{r=0}^{\infty}\left(\frac{k!}{\lambda^k r!\,(r+1)^{k+1}} + \sum_{s=1}^{k}\frac{k!\,(-2)^s\,(\ln\eta)^s}{\lambda^k s!\,r!\,(r+1)^{k+1-s}}\right)\psi(r)\left(\frac{\eta}{\beta}\right)^r$$

$$= \frac{\alpha\eta^2\lambda^{-k}k!\,\beta^{\frac{\alpha}{\eta}-1}}{\beta^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}}\sum_{r=0}^{\infty}\left(1 + \sum_{s=1}^{k}\frac{(-2(r+1)\ln\eta)^s}{s!}\right)\frac{\psi(r)(\eta\beta^{-1})^r}{r!\,(r+1)^{k+1}} \quad (3.8)$$

From equation(3.8) we can caculate the mean, variance, skewness and kutoses. Also, observe that if we restrict s such that $s=0$, we obtain

$$\mu_k = \frac{\alpha\eta^2\lambda^{-k}k!\,\beta^{\left(\frac{\alpha}{\eta}-1\right)}}{\beta^{\frac{\alpha}{\eta}}-(\beta-\eta)^{\frac{\alpha}{\eta}}}\sum_{r=0}^{\infty}\frac{\psi(r)(\eta\beta^{-1})^r}{r!\,(r+1)^{k+1}} \quad (3.9)$$

Furthermore, if we put $\eta=1$ and $\lambda=1$, then we have

$$\mu_k = \frac{\alpha k!\,\beta^{(\alpha-1)}}{\beta^\alpha-(\beta-1)^\alpha}\sum_{r=0}^{\infty}(-1)^r\binom{\alpha-1}{r}\frac{\beta^{-r}}{(r+1)^{k+1}} \quad (3.10)$$

as a special case, which is the result due to Olapade (2014).

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4. Median of Modified Extended Generalized Exponential Distribution

Here, we seek for $x_m \epsilon \mathbb{R}$ such that;

$$\int_{-\infty}^{x_m} f(x)dx = \frac{1}{2}\ ; \Rightarrow F(x_m) = \frac{1}{2}$$

By equation (2.3), it implies that ;

$$\frac{(\beta - \eta e^{-x_m \lambda})^{\alpha/\eta} - (\beta - \eta)^{\alpha/\eta}}{\beta^{\frac{\alpha}{\eta}} - (\beta - \eta)^{\frac{\alpha}{\eta}}} = \frac{1}{2}$$

Simplifying and making x_m the subject we have;

$$x_m = \frac{-1}{\lambda} \ln \frac{1}{\eta}\left[\beta - \left(\frac{\beta^{\alpha/\eta} + (\beta + \eta)^{\alpha/\eta}}{2}\right)^{\eta/\alpha}\right] \quad (4.1)$$

5. The 100p-Percentage Point of Modified Extended Generalized Exponential Distribution.

Here, we seek for x_p such that;

$$\int_{-\infty}^{x_p} f(x)dx = 100\%\, p\ ; \ \Rightarrow F(x_p) = p.$$

And by equation (2.3), it follows that ;

$$\frac{(\beta - \eta e^{-\lambda x_p})^{\alpha/\eta} - (\beta - \eta)^{\alpha/\eta}}{\beta^{\frac{\alpha}{\eta}} - (\beta - \eta)^{\frac{\alpha}{\eta}}} = p$$

Simplifying and making x_p the subject , we have ;

$$x_m = \frac{-1}{\lambda} \ln \frac{1}{\eta}\left[\beta - \left(p\beta^{\alpha/\eta} - (p-1)(\beta - \eta)^{\alpha/\eta}\right)^{\eta/\alpha}\right] \quad (5.1)$$

6. The Mode of the Modified Extended Generalized Exponential Distribution

Here, we seek for $x^* \in \mathbb{R} \ni f(x^*) \geq f(x)\ \forall\ x > 0$

By equation (2.2) ;

$$f_X(x;.) = K\left(\beta - \eta e^{-\lambda x}\right)^{\frac{\alpha}{\eta}-1} e^{-\lambda x}\ ; \quad K = \frac{\alpha\lambda}{\beta^{\frac{\alpha}{\eta}} - (\beta - \eta)^{\frac{\alpha}{\eta}}}$$

Thus the first derivative of f denoted by $f_X{}'$ is given by

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$$f_X{}' = K\lambda\left[\left(\beta-\eta e^{-\lambda x}\right)^{\frac{\alpha}{\eta}-2} e^{-\lambda x}\left(\alpha e^{-\lambda x}-\beta\right)\right]$$

By optimality condition, we have that $f' = 0$. Hence solving for x;

$$\Longrightarrow x_1 = \infty \text{ or } x_2 = \frac{-1}{\lambda}\ln\left(\frac{\beta}{\eta}\right) \text{ or } x_3 = \frac{-1}{\lambda}\ln\left(\frac{\beta}{\alpha}\right)$$

Are the required solution, where x_2 and x_3 are admissible. We use the second derivative of f_X denoted by $f_X{}''$ given by

$$f_X{}'' = K\lambda^2 e^{-\lambda x}\left(\beta-\eta e^{-\lambda x}\right)^{\frac{\alpha}{\eta}-2}\left[(\alpha-2\eta)\left(\beta-\eta e^{-\lambda x}\right)^{\frac{\alpha}{\eta}-2} e^{-\lambda x}\left(\alpha e^{-\lambda x}-\beta\right) - \left(2\alpha e^{-\lambda x}-\beta\right)\right]$$

To determine the natures of x_2 and x_3. Observe that $f_X{}''(x_2) = 0$, hence x_2 is not a maxima (modal) point of f_X and

$$f_X{}''(x_3) = -C(\alpha-\eta)^{\frac{\alpha}{\eta}+2}\,;\, C = K\lambda^2\beta^{\frac{\alpha}{\eta}}\alpha^{\frac{\alpha}{\eta}+1} \qquad (6.1)$$

Thus, x_3 is the mode (maxima point) of f_X provided $\alpha > \eta$, since C is positive constant. This implies in particular, that f_X is unimodal. Now, observe that at x_3, f_X, s_X, h_X and φ_X are given by;

$$(i).\ f_X(x_3) = \frac{\alpha\lambda(\alpha-\eta)^{\frac{\alpha}{\eta}-1}}{\alpha^{\frac{\alpha}{\eta}} - \left(\alpha-\frac{\alpha\eta}{\beta}\right)^{\frac{\alpha}{\eta}}};\ (ii).\ s_X(x_3) = \frac{(\alpha)^{\alpha/\eta} - (\alpha-\eta)^{\alpha/\eta}}{\alpha^{\frac{\alpha}{\eta}} - \left(\alpha-\frac{\alpha\eta}{\beta}\right)^{\frac{\alpha}{\eta}}}$$

$$(iii).\ h_X(x_3) = \frac{\alpha\lambda(\alpha-\eta)^{\frac{\alpha}{\eta}-1}}{(\alpha)^{\alpha/\eta} - (\alpha-\eta)^{\alpha/\eta}};\ (iv).\ \varphi_X(x_3) = \frac{\alpha\lambda(\alpha-\eta)^{\frac{\alpha}{\eta}-1}}{(\alpha-\eta)^{\frac{\alpha}{\eta}} - \left(\alpha-\frac{\alpha\eta}{\beta}\right)^{\frac{\alpha}{\eta}}}$$

Observe in equation(6.1), the condition that guarantees existence of modal point is that $\alpha > \eta$ which is independent of β and if we put $\eta = 1$, we obtain the condition that guarantees the existence of modal point of GE distribution by Gupta and Kundu (1999). We study the behaviour of the hazard function of the MEGE distribution using the following lemma due to Glaser(1980).

Lemma 6.1 Let f be a twice differentiable probability function of a continuous random variable X. Define $\delta(x) = \frac{f'(x)}{f(x)}$, where $f'(x)$ is the first derivative of $f(x)$ with respect to x. Furthermore, suppose the first derivative of $\delta(x)$ exist.

1. If $\delta'(x) < 0$ for all $x > 0$, then the hazard function is monotonically increasing.
2. If $\delta'(x) > 0$ for all $x > 0$, then the hazard function is monotonically decreasing.

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3. If there exist x_0 such that $\delta'(x) < 0$ for all $0 < x < x_0$, $\delta'(x_0) = 0$ and $\delta'(x) > 0$ for all $x > x_0$. In addition, $\lim_{x\to\infty} f(x) = 0$, then the hazard function is upside down bathtub shape.
4. If there exist x_0 such that $\delta'(x) > 0$ for all $0 < x < x_0$, $\delta'(x_0) = 0$ and $\delta'(x) < 0$ for all $x > x_0$. In addition, $\lim_{x\to\infty} f(x) = \infty$, then the hazard function is bathtub shape.

To investigate the behaviour of the hazard function of MEGE distribution, it suffices to define

$$\delta(x) = \frac{d(\log_e f(x))}{dx} = \frac{d\left(\log_e K + \left(\frac{\alpha}{\eta} - 1\right)\log_e\left(\beta - \eta e^{-\lambda x}\right) - \lambda x\right)}{dx}$$

$$= \frac{\left(\frac{\alpha}{\eta} - 1\right)\lambda\eta e^{-\lambda x}}{\beta - \eta e^{-\lambda x}} - \lambda$$

Thus

$$\delta'(x) = \frac{-\beta\lambda^2(\alpha - \eta)e^{-\lambda x}}{(\beta - \eta e^{-\lambda x})^2} \qquad (6.2)$$

Hence, As a consequence of equation(6.1), equation(6.2) and lemma(6.1) we state the following theorem:

Theorem 6.2 Let X be a continuous random variable that is distributed as MEGE, then the following holds:

(i) $\delta'(x) < 0$ for all $x > 0$ if $\alpha > \eta$, then the hazard function of MEGE is monotonically increasing if $\alpha > \eta$,

(ii) $\delta'(x) > 0$ for all $x > 0$ if $\alpha < \eta$, then the hazard function of MEGE is monotonically decreasing if $\alpha < \eta$,

(iii) $\delta'(x) = 0$ for all $x > 0$ if $\alpha = \eta$, then the hazard function of MEGE is constant if $\alpha = \eta$.

(iv) the density of MEGE$(.;\alpha,\beta,\lambda,\eta)$ is log-concave if $\alpha > \eta$ and log-convex if $\alpha < \eta$

Proof. The proof follows from using equation(6.1), equation(6.2) and lemma(6.1).

Observe that if we put $\eta = 1$, we obtain the result due to Gupta and Kundu (1999, 2000, 2001).

7. The ODE of the Modified Extended Generalized Exponential Distribution

Since f_X is a continous monotone real-valued function, we supposed that it is a unique solution to certain initial value problem of an ordinary differential equation (ODE). In fact, the uniqueness of the solution characterizes (single out) the pdf.

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Theorem 7.1 Let g be a continuous real-valued function, then g is a solution to the initial value problem (7.1) and (7.2) if and only if $g = f_X$.

$$\left(\beta - \eta e^{-\lambda x}\right) g'(x) + \lambda\left(\beta - \alpha e^{-\lambda x}\right) g(x) = 0 \qquad (7.1)$$

$$g(0) = \frac{\alpha\lambda(\beta-\eta)^{\frac{\alpha}{\eta}-1}}{\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}}} \qquad (7.2)$$

Proof.

Given that

$$\left(\beta - \eta e^{-\lambda x}\right) g'(x) + \lambda\left(\beta - \alpha e^{-\lambda x}\right) g(x) = 0$$

$$\Leftrightarrow \int \frac{g'(x)dx}{g(x)} = \int \frac{\lambda\left(\alpha e^{-\lambda x} - \beta\right)dx}{(\beta - \eta e^{-\lambda x})} ;$$

$$\Leftrightarrow \int \frac{g'(x)dx}{g(x)} = \lambda\left(\alpha \int \frac{e^{-\lambda x}dx}{(\beta - e^{-\lambda x})} - \beta \int \frac{dx}{(\beta - \eta e^{-\lambda x})}\right)$$

If we take $u = \beta - \eta e^{-\lambda x}$; $\Rightarrow du = \eta\lambda e^{-\lambda x}dx$, so that

$$\Leftrightarrow \ln g(x) = \left(\frac{\alpha}{\eta}\int \frac{du}{u} - \int\left(\frac{1}{u} + \frac{1}{\beta - u}\right)du\right)$$

$$\Leftrightarrow \ln g(x) = \ln\left(u^{\frac{\alpha}{\eta}-1}(\beta - u)C\right) ; \Rightarrow g(x) = u^{\frac{\alpha}{\eta}-1}(\beta - u)C ;$$

$$\Leftrightarrow g(x) = \eta e^{-\lambda x}\left(\beta - \eta e^{-\lambda x}\right)^{\frac{\alpha}{\eta}-1} C \qquad (7.3)$$

Using the initial condition, we have that

$$C = \frac{\alpha\lambda}{\eta\left[\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}}\right]}$$

Substituting for C into (6.1), we obtain

$$g(x) = \frac{\alpha\lambda\left(\beta - \eta e^{-\lambda x}\right)^{\frac{\alpha}{\eta}-1} e^{-\lambda x}}{\beta^{\frac{\alpha}{\eta}} - (\beta-\eta)^{\frac{\alpha}{\eta}}} \qquad (7.4)$$

■

CONCLUSION

A six-parameter modified extended Generalized Exponential Distribution have been introduced and proven to be a pdf. Some statistics characteristics such as ; the cumulative distribution function, the survival function, hazard function and its properties, the reverse hazard function, the moment generating function, the rth moment about the origin ,the median,

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the 100p-Percentage Point, the mode are also established and Finally, an initial-valued ordinary differential equation associated with the new distribution was stated and proved.

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