

## ON APPLICATION OF LERAY-SCHAUDER FIXED POINT AND SCHEAFER'S LEMMA IN SOLUTION OF DUFFING'S EQUATION

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**ABSTRACT:** *In this paper, we considered the application of Leray-Schauder fixed point theorem with further augmentations by Scheafer's Lemma for an integrated mode for investigating apriori-bounds of solutions of Duffing's Equation. The results obtained showed boundedness of the solution with different apriori-bounds pointing to optimal solution which is continuous, closed, bounded and integrable. We concluded that the bounded solution of Duffing's Equation has a cyclic relationship other qualitative properties of solutions such as stability, periodicity and uniform convergence. This has opened the window for further research, using Abstract Implicit function theorem in Banach spaces.*

**KEYWORDS:** Apriori-Bounds, Leray-Schauder Fixed Point, Scheafer's Lemma, Cauchy – Schwartz's Inequality, Duffings Equation.

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### INTRODUCTION

As a consequence of the previous paper on Stability, Asymptotic Stability, Boundedness and Periodicity of Solutions of Duffing's Equation, [Eze E.O, Ogbu H.M. and Aja R.O (1)] the need to determine the apriori bounds and proof of the existence of such bounds have arisen. This will be undertaken when we invoke the Leray-Schauder fixed point theorems and a further argumentations by the Scheafers Lemma for the proof of apriori bounds of  $2\pi$ -periodic solutions of Duffings Equations. Leray-Schauder defined continuous mapping from a matrix space A into a matrix space B as a continuous mapping on A which takes bounded subsets of A into relatively compact ones of B. when a continuous mapping takes A into a relatively compact subset, it is nowadays said to be compact on A. J Mawhin[2]. Schauder identified an improper class of non-linear operators in a Banach space, the completely continuous perturbations of identity for which he could generalize two important results of Brouwer in finite dimensional topology; the fixed point theorem and Invariance of domain theorem. Schauder applied the first Schauder extensions nowadays called fixed point theorem [J. Schauder[ 3],[ 4], [5]. Leray- Schauder theorem, like many variance and extension of Schauder fixed point theorem discuss the case of mappings between spaces of different "dimensions". The theory of multi-valued mappings, Nilesens fixed points theorem, asymptotic fixed point theorem, the computation of Leray Schauder degree and its use is the critical point theory. In the whole paper, if X is a metric space  $I = [0,1]$   $A \subset X \times I$  and  $\lambda \in I$ , we write  $A_\lambda = \{x \in X: (x, \lambda) \in A\}$ . For  $a \in X$  and  $r > 0$ ,  $B(a, r)$  denotes the open ball of centre  $a$  and radius  $r$ [J. Mawhin;[ 2].

The classical importance of fixed point theory in functional analysis is due to its usefulness in the theory of ordinary and partial differential equations. The existence or construction of a solution in a differential equation is often reduced to the existence or location of a fixed point for an operator defined on a subset of a space of functions. Fixed point theorems have also been used to determine the existence of periodic solutions for functional differential equations when

solutions are already known to exist. Apart from this deep involvement in the theory of differential equation, fixed points have also been extremely useful in such problems as finding zeros of non-linear equations and proving Surjectivity theorems partly as a consequence of the importance of its application, fixed point theory has developed into an area of independent research[C.E. Chidume(6)] On the Existence and Uniqueness of the solutions of Duffings equations, previous works on them have been carried by Eze E.O, Ogbu H.M and Ugbene I.J [7], Sana Gasmi and Alain Haraux [8], Agarwal and O. Regan [9], Pedro F. Tores [10], DivakarViswanath [11].

In this paper we consider Duffings Equation of the form

$$\ddot{x} + a\dot{x} + cx + bx^2 + 2x^3 = p(t) \tag{1.1} \quad \text{where } a, b, c \text{ are real constants and } P:[0, 2\pi]-E^1 \text{ is continuous}$$

Subject to the initial conditions

$$\left. \begin{aligned} x(0) &= x(2\pi) \\ \text{and} \\ \dot{x}(2\pi) &= \dot{x}(2\pi) \end{aligned} \right\} \tag{1.2}$$

The Existence and Uniqueness of  $2\pi$ -periodic solutions of equations (1.1) and (1.2) has been investigated [see Eze E,O et al[ 7]. Finally using Scheafers Lemma (Scheafer[ 12] established that

$$\left. \begin{aligned} |x|_{\infty} &\leq C_6 \\ |\dot{x}|_{\infty} &\leq C_3 \end{aligned} \right\} \tag{1.3}$$

where the C's are the apriori bounds of our equation (1.1)-(1.2).

The objective of this paper therefore is on the investigation of these bounds using the Leray-Schauder Theorem and Scheafer's Lemma..

**Preliminaries.**

In[ 12] Scheafer formulated a special case of Leray –Schauder continuation theorem in the form of an alternative, and proves it as a consequence of Schauder fixed point theorem.

**Theorem 2.1:** Let  $X$  be the Banach space and  $f: X \rightarrow X$  be completely continuous then either there exists for each  $\lambda \in [0,1]$  at least one small  $x \in X$  such that  $x = \lambda f(x)$  or the set  $\{x \in X: x = \lambda f(x), 0 < \lambda < 1\}$  is bounded in  $X$ [Scheafer H. H;[ 12].

**Theorem 2.2:** Suppose there exists  $a > 0, b > 0$  and  $\beta > 0$  such that:

- i.  $h'(x) < b, \beta^2 - b$
- ii.  $|h(x) - x| > 0$  for all  $x$
- iii.  $|h(x)| \rightarrow \infty$  as  $|x| \rightarrow \infty$
- iv.  $x^2 + y^2 \rightarrow \infty$  as  $|x| \rightarrow \infty, |y| \rightarrow \infty$  then our equation (1.1) through (1.2) has stable bounded and periodic solutions when  $p(t) = 0$

**Theorem 2.3:** Suppose further in theorem (2.2) that the condition (i) is replaced by

- 1.  $h'(x) < b, \beta^2 \neq b, |a\dot{x} - p(t)| > 0$

Then equation (1.1) and (1.2) have stable bounded and periodic solutions when  $p(t) \neq 0$ .

**Theorem 2.4** Leray-Schauder fixed point(Non linear Alternative) : Let  $X$  be a Banach space with  $C$  being a subset of  $X$  closed and convex. Assume  $U$  is a relatively open subset of  $C$  with  $0 \in U$  and  $g: \bar{U} \rightarrow C$  is a compact map. Then either;

- i.  $g$  has a fixed point in  $\bar{U}$  or
- ii. There is a fixed point  $u \in \partial U$  and  $\lambda \in (0,1)$  with  $u = \lambda G(u)$

[J. Dugunji and A. Gramnas;[ 13]

**Lemma 2.5:** Let  $X$  be a Banach space and  $k$  be a cone of  $X$ . Assume  $\Omega_1, \Omega_2$  are open subsets of  $X$  with  $0 \in \Omega_1 \subseteq \Omega_2$  and let  $A: k \cap (\bar{\Omega}_2 - \Omega_1) \rightarrow k$  be a completely continuous operation such that

$$\|Ax\| \leq \|x\| \text{ for every } x \in k \cap \partial\Omega_1$$

$$\|Ax\| \geq \|x\| \text{ for every } x \in k \cap \partial\Omega_2$$

OR

$$\|Ax\| \geq \|x\| \text{ for every } x \in k \cap \partial\Omega_1 \text{ and } \|Ax\| \leq \|x\| \text{ for every } x \in k \cap \partial\Omega_2$$

Then  $A$  has at least one fixed point in  $k \cap (\bar{\Omega}_2 - \Omega_1)$

[Yuji Liu and WeigaoGe;[ 14]

This fixed point will basically coincide with a fixed solution and a unique fixed point.

**Theorem 2.6:** A priori Bounds on Solutions: there exist constants  $M_0, \dots, M_m$  such that if  $y \in \Omega^I$  is a solution of (1)-(2). Then

$$\text{Sup}\{|y(t)|: t \in [t_{k-1}, t_k]\} \leq M_{k-1}; \quad k=1, \dots, m+1$$

## MAIN RESULTS

We consider the more general form of Duffings equation (1.1) as a parameter  $\lambda$  dependent equation

$$\ddot{x} + a\dot{x} + h_\lambda(x) = \lambda p(t) \quad (3.1) \quad \text{where} \quad h_\lambda(x) = (1 - \lambda)bx + \lambda h(x)$$

$$\lambda p(t) \quad (3.2)$$

Where  $\lambda$  is in the range of  $0 \leq \lambda \leq 1$  and  $b$  is a constant

$$\text{satisfying } a > 0, b > 0 \quad (3.3)$$

The equation (3.1) is equivalent to the system

$$\left. \begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay - h_\lambda(x) = p(t) \end{aligned} \right\} \quad (3.4)$$

Let  $X(t)$  be a possible  $2\pi$ -periodic solution of (3.1). The main tool to be used here is the verification is the function  $W(x, y)$  defined by

$$W(x, y) = \frac{1}{2}y^2 +$$

$$H_\lambda(x) = \int_0^x h_\lambda(s) ds \tag{3.5} \quad \text{where} \quad H_\lambda(x) =$$

The time derivative  $\dot{W}$  of equation (3.5) along the solution paths of (3.4) is

$$\begin{aligned} \dot{W} &= y\dot{y} + h_\lambda(x)\dot{x} \\ &= -ay^2 - h_\lambda(x)y + \lambda p(t)y + h_\lambda(x)y \\ &= -ay^2 + \lambda p(t)y \end{aligned} \tag{3.6}$$

Integrating equation (3.6) With respect to from  $t = 0$  to  $t = 2\pi$

$$\begin{aligned} \int_0^{2\pi} \dot{W} dt &= \int_0^{2\pi} -a\dot{x}^2 dt + \int_0^{2\pi} \lambda p(t)\dot{x} dt \\ [W(t)]_0^{2\pi} &= - \int_0^{2\pi} a\dot{x}^2 dt + \lambda \int_0^{2\pi} p(t)\dot{x} dt \quad \text{Since} \quad W(0) = \\ W(2\pi) \text{ implies that} \quad [W(t)]_0^{2\pi} &= 0 \end{aligned}$$

because of  $2\pi$  – periodic solution. Thus

$$0 = - \int_0^{2\pi} a\dot{x}^2 dt + \lambda \int_0^{2\pi} p(t)\dot{x} dt$$

$$\int_0^{2\pi} a\dot{x}^2 dt \leq |\lambda| |p(t)| \int_0^{2\pi} \dot{x} dt \tag{3.7}$$

Since  $|\lambda| \leq 1$  and  $p(t)$  is continuous then

$$\int_0^{2\pi} \dot{x}^2 dt \leq C_1 (2\pi)^{\frac{1}{2}} \left( \int_0^{2\pi} \dot{x}^2 dt \right)^{\frac{1}{2}} \tag{3.8}$$

By the hypothesis of Schwartz’s inequality therefore,

$$\left( \int_0^{2\pi} \dot{x}^2 dt \right)^{\frac{1}{2}} \leq C_1 (2\pi)^{\frac{1}{2}} \equiv C_2$$

That is,

$$\left( \int_0^{2\pi} \dot{x}^2 dt \right)^{\frac{1}{2}} \leq C_2$$

Now since  $x(0) = x(2\pi)$ , it is clear that there exists  $\dot{x}(T) = 0$  for  $T \in [0, 2\pi]$ .

Thus using the identity

$$\begin{aligned} \dot{x}(t) &= \dot{x}(T) + \int_0^{2\pi} \ddot{x}(s) ds \\ &= \int_0^{2\pi} \ddot{x}(s) ds \end{aligned}$$

By Schwartz’s inequality, we have

$$\max_{0 \leq t \leq 2\pi} |\dot{x}(t)| \leq \int_0^{2\pi} |\ddot{x}(t)| dt \leq (2\pi)^{\frac{1}{2}} \left( \int_0^{2\pi} \ddot{x}^2(s) ds \right)^{\frac{1}{2}} \tag{3.9}$$

We hereby invoke the Fourier expansion of  $X \sim \sum_{r=0}^{\infty} (\arccos 2\pi l + br \sin 2\pi l)$

For the derivation of this [see Ezeilo and Onyia (15)]. We will now obtain

$$\int_0^{2\pi} \dot{x}^2 dt \leq |\lambda| |p(t)| \int_0^{2\pi} \dot{x} dt$$

Which is

$$\int_0^{2\pi} \ddot{x}^2 dt \leq C_1 (2\pi)^{\frac{1}{2}} \left( \int_0^{2\pi} \dot{x}^2 dt \right)^{\frac{1}{2}}$$

Therefore we have

$$\left( \int_0^{2\pi} \dot{x}^2 dt \right)^{\frac{1}{2}} \leq C_1 (2\pi)^{\frac{1}{2}} \equiv C_2 \quad (3.10)$$

Then from equation (3.10)

$$\max_{0 \leq t \leq 2\pi} |\dot{x}(t)| \leq (2\pi)^{\frac{1}{2}} \cdot C_1 (2\pi)^{\frac{1}{2}} \equiv C_3$$

$$\text{That is } |\dot{x}|_{\infty} \leq C_3 \quad (3.11)$$

Now integrating equation

$\ddot{x} + a\dot{x} + h_{\lambda}(x) = \lambda p(t)$  with respect to  $t$  from  $t = 0$  and  $t = 2\pi$ , we obtain  $t$  from  $t = 0$  and  $t = 2\pi$ , we obtain

$$\int_0^{2\pi} \ddot{x} dt + \int_0^{2\pi} a\dot{x} dt + \int_0^{2\pi} h_{\lambda}(x) dt \equiv \int_0^{2\pi} \lambda p(t) dt \quad (3.12)$$

Using equation  $h_{\lambda}(x) = (1 - \lambda)bx + \lambda h(x)$  on equation (3.12) we obtain

$$\int_0^{2\pi} \ddot{x} dt + \int_0^{2\pi} a\dot{x} dt + \int_0^{2\pi} (1 - \lambda)bx dt + \int_0^{2\pi} h(x) dt = \int_0^{2\pi} \lambda p(t) dt$$

Then equation (3.12) yields

$$\int_0^{2\pi} (1 - \lambda)bx dt + \int_0^{2\pi} h(x) dt = \int_0^{2\pi} \lambda p(t) dt \quad (3.13)$$

The continuity of  $p(t)$  assures us of boundedness and the fact that  $0 \leq \lambda \leq 1$ , the right-hand side of equation (3.13) is bounded. That is

$$\left| \int_0^{2\pi} (1 - \lambda)bx dt + \int_0^{2\pi} h(x) dt \right| \leq C_4 \quad (3.15)$$

Therefore given  $\alpha > 0$ , there exists  $\eta > 0$  such that  $T \in [0, 2\pi]$

$$|x(T)| \leq C_5 \quad (3.16)$$

Put  $T = 0$  and we are done.

$\Leftarrow$  Suppose NOT ie.  $x(T) \neq 0$  for any  $T$  then equation (3.14) yields

$$\int_0^{2\pi} (1-\lambda)b|x|dt + \int_0^{2\pi} |\lambda||h(x)|dt > \int_0^{2\pi} (1-\lambda)b\eta dt + \int_0^{2\pi} \lambda\alpha(x)dt$$

$$> 2\pi(1-\lambda)b\eta + 2\pi\lambda\alpha \quad (3.17)$$

But equation (3.17) implies that

$\int_0^{2\pi} (1-\lambda)bxdx + \int_0^{2\pi} \lambda h(x)dx$  is no more bounded. This is a negation of our equation (3.14).

Thus  $|h(x)| \rightarrow \infty$  as  $|x| \rightarrow \infty$  and equation (3.16) holds.

The identity

$$X(t) = X(T) + \int_T^t \dot{x}dt \text{ holds}$$

$$\text{Thus } \max_{0 \leq t \leq 2\pi} |\dot{x}(t)| \leq |x(T)| + \int_0^{2\pi} |\dot{x}(t)|dt \leq C_5 + (2\pi)^{\frac{1}{2}} \left( \int_0^{2\pi} \dot{x}^2 dt \right)^{\frac{1}{2}}$$

Then by Schwartz's inequality

$$\leq C_5 + (2\pi)^{\frac{1}{2}} C_2 \quad (\text{by equation 3.8})$$

$$\text{Thus } |x|_{\infty} \leq C_5 + (2\pi)^{\frac{1}{2}} C_2 \equiv C_6$$

So we finally obtain

$$|x| \leq C_6 \quad (3.18)$$

## DISCUSSION

1. The apriori bounds show exactly the limit of the intervals where the solution of our differential system will lie and the distance between these limit points is the limit supremum of the difference between these points. This distance is a convex set of points. The diameter is finite for boundedness to occur. This interval is a continuous closed, bounded and thus integrable.
2. The Leray-Schauder fixed point theorem helps us to determine the unique point within these apriori bounds where the solution of our differential solution is to exist and this unique point is the optimal point which coincides with the solution of our Duffing's equation.
3. On the use of Cauchy-Schwartz inequality which is a special case of Holder's inequality shows that the space where the unique solution is to exist is in  $L^2$ -space and the limit point where the solution lies is where  $n = 2, p = 2$ .
4. The behavior Duffing's equation shows that it is stable, asymptotically stable bounded and therefore periodic.
5. The use of Scheafer's lemma determines the actual lower limit point and upper limit points which are the actual apriori bounds.

**CONCLUSION**

Thus concluding our search for apriori bounds for our Duffing's equation

$$\ddot{x} + a\dot{x} + bx + cx^2 + 2x^3 = p(t)$$

which can be reduced to a more generalized form of a second non-linear equation  $\ddot{x} + a\dot{x} + h(x) = p(t)$ .

By our equations (3.11) and (3.18), our equation (1.3) is established and an affirmation of boundedness and thus pairing of solutions  $X(t)$  and  $\dot{x}(t)$ . The way for further research on the use of other theorems to investigate the apriori bound of solutions of Duffing's Equation.

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