ON A CLOSED-FORM ESTIMATOR OF THE SHAPE PARAMETER OF THE THREE-PARAMETER WEIBULL DISTRIBUTION

Ogbonna Uchenna Austine
Dr Obiora-Ilouno Happiness

ABSTRACT: *The shape parameter of the three-parameter Weibull distribution (β) was considered in this study. Known estimation methods like the maximum likelihood, method of moment and maximum product of spacing do not have closed-form estimators for the shape parameter of the three-parameter Weibull distribution rather they involve iterative procedures which may be time-consuming and are less tractable. Dubey (1967), Goda et al (2010) and Teimouri and Gupta (2013) have proposed closed-form estimators for β. In this study, a closed-form estimator for β is proposed and the proposed estimator is compared with the existing closed-form estimators proposed by the authors mentioned above. To compare the accuracy of the estimators, Monte Carlo simulation is performed. Simulated data from the Weibull distribution are used to check the accuracy of the estimators and the root mean square error (RMSE) is used as a metric for accuracy. The results show that in general, the proposed estimator performs better than the other three closed-form estimators that were compared.*

KEYWORDS: weibull, estimators, tractability, accuracy, parameter

INTRODUCTION

The Weibull distribution is of great interest to theory-oriented statisticians because of its great number of special features. It is also of great interest to practitioners because of its ability to fit data from various fields, ranging from life data to weather data or observations made in economics and business administration and this is because the Weibull distribution can behave like some other distributions depending on the value of its shape parameter.

The pdf of the three-parameter Weibull distribution is given by;

$$f(x) = \frac{\beta}{\theta}\left(\frac{x-\gamma}{\theta}\right)^{\beta-1} e^{-\left(\frac{x-\gamma}{\theta}\right)^{\beta}} \qquad x > \gamma,\ \beta > 0,\ \theta > 0 \qquad [1]$$

The mean of the distribution can be expressed as

$$\text{Mean} = \gamma + \theta\Gamma\left(1 + \frac{1}{\beta}\right) \qquad [2]$$

The variance of the distribution can be expressed as

$$\text{Variance} = \theta^2\left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right] \qquad [3]$$

The quantile function can be expressed as

$$x(F) = \theta\left[-\ln\left(1 - F(x)\right)\right]^{1/\beta} + \gamma \qquad [4]$$

where $\beta,\ \theta\ and\ \gamma$ are the shape parameter, scale parameter and location parameters respectively. The shape parameter is also known as the slope parameter. As the name implies, it determines the

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shape of the distribution (that is different values of β give different shapes of the distribution). It also shows the skewness of the distribution. Some values of the shape parameter will cause the probability density function of the Weibull distribution to reduce to that of other distributions. For example, when β=1, the probability density function (PDF) of a three-parameter Weibull distribution reduces to that of a two-parameter exponential distribution, (Rinne, 2008).

However, to model data using a probability distribution requires estimation of the parameters of the distribution. Maximum Likelihood Method works efficiently if each contribution to the likelihood function is bounded above. Though the MLE having such nice properties and better applicability, it also has some weaknesses. Its greatest weakness is that it cannot work for 'heavy-tailed' continuous distribution with unknown location and scale parameters. The Maximum Likelihood Method does not always provide precise estimates for certain distributions such as the gamma, Weibull, and log-normal distributions. This is because in these cases the critical difficulty is that there are paths in parameter space with the location parameter tending to the smallest observation along which the likelihood becomes infinite, Pitman (1979). According to Teimouri and Gupta (2013), the three-parameter Weibull distribution does not satisfy the regularity condition requirement of the MLE, and also its estimator of the shape parameter of the three-parameter Weibull distribution is not available in closed form.

Other methods have been proposed, like The Modified Maximum likelihood estimators (MLE) proposed by Cohen and Whitten (1982), the method of moments and modified method of moments estimators presented by Cohen and Whitten (1982), the maximum product of spacing method presented in Cousineau (2008) the method proposed by Goda et al (2010) based on L-skewness and the modifications to the procedure for MLE proposed by Yang et al (2019). However, in these methods, the shape parameter of the three-parameter Weibull distribution is estimated by iterative procedures making them less tractable, more complicated and time-consuming to compute.

Existing closed-form estimators of the shape parameter

To solve the problem of tractability when estimating the shape parameter of the three-parameter Weibull distribution, several authors like Dubey (1967), Goda et al (2010) and Teimouri and Gupta (2013) have proposed closed-form estimators for the shape parameter.

Dubey (1967) proposed the percentile method which is an estimator based on the 17th and 97th percentiles for the shape parameter of the three-parameter Weibull distribution. Taking $P_1 = 0.167$ and $P_2 = 0.9737$ and defining $K_1 = log\left(-log(1-P_1)\right) - log(-log(1-P_2))$. Also making Y_1 and Y_2 represent the 100(P_ith) percentile from a sample given sample, then,

$$\hat{\beta} = \frac{-K_1}{\log(Y_1)-\log(Y_2)} \qquad [5]$$

Goda et al (2010) an estimator gotten by fitting a polynomial to the L-skewness of the three-parameter Weibull distribution to estimate the shape parameter.

$$\text{L-skewness} = \frac{\left(1-\frac{3}{2^{1/\beta}}+\frac{2}{3^{1/\beta}}\right)}{\left(1-\frac{1}{2^{1/\beta}}\right)} \qquad [6]$$

$$\hat{\beta} = 285.3\pi_3{}^6 - 658.6\pi_3{}^5 + 622.8\pi_3{}^4 - 317.2\pi_3{}^3 + 98.52\pi_3{}^2 - 21.256\pi_3 + 3.516 \qquad [7]$$

Teimouri and Gupta (2013) used a theorem to construct a simple, consistent and closed-form estimator for β. The theory states that: suppose $x_1, x_2, \dots, x_n$ is a random sample from a Weibull distribution. Let ρ denote the sample correlation coefficient between x_i and their ranks. Then,

$$\rho = COV(x_i, R_i) = \left[\left(\int_{-\infty}^{\infty} xF(x)\, dF(x) - \frac{\mu_x}{2}\right)\sqrt{\frac{12(n-1)}{\sigma_x^2(n+1)}}\right] \qquad [8]$$

Where μ_x= mean of the distribution and $xF(x)$ is the quantile function of the distribution. Teimouri and Gupta (2013) stated that for the three-parameter Weibull distribution, equation [8] becomes;

$$\rho = \left(\frac{\mu_x - \gamma}{\sigma_x}\right)\left(\frac{1}{2} - \frac{1}{2^{1+\frac{1}{\beta}}}\right)\sqrt{\frac{12(n-1)}{n+1}} \qquad [9]$$

where $\mu_x = E(x)$, $\sigma_x =$ standard deviation and β is the shape parameter of the distributuon Solving for β in equation [9] gives;

$$\beta = \frac{-\ln 2}{\ln\left(1 - \frac{\rho}{\sqrt{3}}\left(\frac{\mu_x}{\sigma_x} - \frac{\gamma}{\sigma_x}\right)^{-1}\sqrt{\frac{(n+1)}{(n-1)}}\right)} \qquad [10]$$

Teimouri and Gupta (2013) stated that a good estimator for γ is $x_{(1)} - \frac{1}{n}$, where $x_{(1)} = min\{x_1, x_2, \dots, x_n\}$.

Substituting $x_{(1)} - \frac{1}{n}$ for γ in equation [10] gives as an estimator for β that is independent of the other two parameters.

$$\hat{\beta} = \frac{-\ln 2}{\ln\left[1 - \frac{\rho}{\sqrt{3}}\left(\frac{1}{C} - \frac{x_{(1)} - \frac{1}{n}}{S}\right)^{-1}\sqrt{\frac{n+1}{n-1}}\right]} \qquad [11]$$

where C is the coefficient of variation ($C = \frac{\sigma_x}{\mu_x}$)

A closed-form estimator of the shape parameter based on the first two L-moments

The L-moments of distributions are an analogy to the conventional moments, but they are based on linear combinations of the rank statistics, i.e. the L-statistics. Using the L-moments is theoretically more appropriate than the conventional moments because the L-moments characterize a wider range of the distribution. When estimating from a sample, L-moments are more robust to the existence of the outliers in the data. The experience shows that in comparison with the conventional moments the L-moments are more difficult to distort and in finite samples, they converge faster to the asymptotical normal distribution, (Bilkova 2012).

We derive a closed-form estimator for the shape parameter of the three-parameter Weibull distribution based on L-moments. According to Hosking (1990), Let $x_{1:n} \le x_{2:n} \le \dots \le x_{n:n}$ be

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the order statistics of a random sample of size n drawn from the distribution of X. The r^{th} population L−moments of X are defined as the quantile;

$$\alpha_r = r^{-1}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(x_{r-k:r}) \qquad where\ (r = 1, 2 \dots) \qquad [12]$$

Substituting r =1 and r = 2 in equation [12], the first two L-moments can be expressed in the following forms;

$$\alpha_1 = E(x) \qquad [13]$$

$$\alpha_2 = \frac{1}{2}[E(x_{1:2}) - E(x_{2:2})] \qquad [14]$$

According to Goda et al (2010), the “L” in L−moments emphasizes the fact that α_r is a linear function of the expected order statistics $\hat{\alpha}_r$ based on a sample of data is a linear combination of the ordered data values. Therefore, the expectation of an order statistic is

$$\therefore E(x_{k:r}) = \frac{n!}{(k-1)!(n-k)!}\int x\left[F(x)^{k-1}\big(1-F(x)\big)^{n-k}\right] dF(x) \qquad [15]$$

According to David (1981), expanding the binomials in F(x) and summing the coefficients of each power of F(x) gives;

$$\alpha_r = \int_0^1 x(F)\,P^{\circ}_{r-1}(F)dF \qquad (\ r = 1,2,\dots) \qquad [16]$$

where x(F) is the quantile function,

$$P^{\circ}_{r-1}(F) = \sum_{k=0}^{r-1}P_{r-1,k}F^k \quad and\ P_{r-1,k} = (-1)^{r-1-k}\binom{r-1}{k}\binom{r-1+k}{k}$$

Substituting r =1 and r = 2 in equation [17] gives;

$$\alpha_1 = E(x) = \int_0^1 x(F)\,P^{\circ}_{1-1}(F)dF = \int_0^1 x(F)\,P^{\circ}_0(F)dF = \int_0^1 x(F)\sum_{k=0}^{0}(-1)^{0-k}\binom{1}{k}\binom{k}{k}F^k\,dF$$

$$\alpha_1 = \int_0^1 x(F)(-1)^0\binom{1}{0}\binom{0}{0}F^0\,dF$$

$$\alpha_1 = \int_0^1 x(F)dF = E(x) \qquad [17]$$

$$\alpha_2 = \int_0^1 x(F)\,P^{\circ}_1(F)dF = \int_0^1 x(F)\left[(-1)^1\binom{1}{0}\binom{1}{0}F^0 + (-1)^0\binom{1}{1}\binom{2}{1}F^1\right]dF$$

$$\alpha_2 = \int_0^1 x(F)[2F-1]\,dF \qquad [18]$$

Recall from equation [4] that the quantile function of the three-parameter Weibull distribution is given by;

$$x(F) = \theta\big[-\ln\big(1-F(x)\big)\big]^{1/\beta} + \gamma$$

Substituting $\theta\big[-\ln\big(1-F(x)\big)\big]^{1/\beta} + \gamma$ for $x(F)$ in equation [17] gives;

$$\alpha_1 = \int_0^1 x(F)dF = \int_0^1\left\{\theta\big[-\ln\big(1-F(x)\big)\big]^{1/\beta} + \gamma\right\}dF$$

$$= \int_0^1 \gamma\,dF + \int_0^1 \theta\big[-\ln\big(1-F(x)\big)\big]^{1/\beta}\,dF$$

$$but\ \int_0^1 \gamma\,dF = \gamma F|_0^1 = \gamma - 0 = \gamma$$

$\therefore \alpha_1 = \gamma + \int_0^1 \theta\left[-\ln\left(1 - F(x)\right)\right]^{1/\beta} dF$

Now let $1 - F = e^{-y}$, which means that $F = 1 - e^{-y}$

$\therefore \frac{dF}{dy} = e^{-y}$ and $dF = e^{-y}dy$

Using this transformation, we have;

$\alpha_1 = \gamma + \int_0^{\infty} \theta[-\ln(e^{-y})]^{1/\beta} e^{-y}\ dy = \gamma + \theta \int_0^{\infty} [y]^{1/\beta} e^{-y}\ dy$

$$\therefore \alpha_1 = \gamma + \theta\Gamma\left(\frac{1}{\beta} + 1\right) = E(x) \qquad [19]$$

Substituting the quantile function in equation [19] gives;

$$\alpha_2 = \int_0^1 x(F)[2F - 1]\, dF = \int_0^1 \left\{\theta\left[-\ln\left(1 - F(x)\right)\right]^{1/\beta} + \gamma\right\}[2F - 1]\, dF$$

$$but \int_0^1 \gamma[2F - 1]\, dF = \left(\frac{2\gamma F^2}{2} - \gamma F\right)\Bigg|_0^1 = \gamma - \gamma = 0$$

$\therefore \alpha_2 = \int_0^1 \theta\left[-\ln\left(1 - F(x)\right)\right]^{1/\beta} [2F - 1]\, dF$

$= \int_0^1 \theta\left[-\ln\left(1 - F(x)\right)\right]^{1/\beta} 2F\ dF\ - \int_0^1 \theta\left[-\ln\left(1 - F(x)\right)\right]^{1/\beta}\ dF$

$= -\ \theta\Gamma\left(\frac{1}{\beta} + 1\right) + \int_0^1 \theta\left[-\ln\left(1 - F(x)\right)\right]^{1/\beta} 2F\ dF$

Using the same transformation as in α_1 gives;

$\alpha_2 = -\ \theta\Gamma\left(\frac{1}{\beta} + 1\right) + \int_0^{\infty} \theta[-\ln(e^{-y})]^{1/\beta} 2(1 - e^{-y})\ e^{-y}\ dy$

$= -\ \theta\Gamma\left(\frac{1}{\beta} + 1\right) + \int_0^{\infty} \theta[y]^{\frac{1}{\beta}}(2 - 2e^{-y})\ e^{-y}\ dy$

$= -\ \theta\Gamma\left(\frac{1}{\beta} + 1\right) + \int_0^{\infty} 2\theta[y]^{\frac{1}{\beta}}\ e^{-y}\ dy\ - \int_0^{\infty} 2\theta[y]^{\frac{1}{\beta}}\ e^{-2y}\ dy$

$= -\ \theta\Gamma\left(\frac{1}{\beta} + 1\right) + 2\theta\Gamma\left(\frac{1}{\beta} + 1\right)\ -\ \frac{2\theta\Gamma\left(\frac{1}{\beta}+1\right)}{2^{\frac{1}{\beta}+1}}$

$= \theta\Gamma\left(\frac{1}{\beta} + 1\right) - \frac{\theta\Gamma\left(\frac{1}{\beta}+1\right)}{2^{\frac{1}{\beta}}}$

$$\therefore \alpha_2 = \theta\Gamma\left(\frac{1}{\beta} + 1\right)\left[1 - \frac{1}{2^{\frac{1}{\beta}}}\right] \qquad [20]$$

The first and second L-moments of the three-parameter Weibull α_1 and α_2 are to be estimated from a finite sample. If we have an ordered sample $x_{1:n} \le x_{2:n} \le \ldots \le x_{n:n}$, it can be characterized better by the estimator of the probability-weighted moments Ω_r. According to Hosking and Wallis (1997), an unbiased estimator of Ω_r based on ordered samples is given as

$$\omega_k = \frac{1}{n}\binom{n-1}{k}^{-1} \sum_{j=1}^{n} \binom{j-1}{k} x_{j:n} \qquad [21]$$

Following the general expression for the population L-moments (equation [16]), the general formula for the sample L-moments is defined as;

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$\hat{\alpha}_r = \sum_{k=0}^{r-1} P_{r-1,k}\, \omega_k$ [22]

$where\ P_{r-1,k} = (-1)^{r-1-k} \binom{r-1}{k} \binom{r-1+k}{k}$

Now, with this we derive estimators of the first two L-moments ($\hat{\alpha}_1$ and $\hat{\alpha}_2$) for a finite sample of size n.

For $r = 1$

$$\hat{\alpha}_1 = \sum_{k=0}^{0} P_{0,k}\, \omega_k \;= P_{0,0}\omega_0 = (-1)^0 \binom{0}{0}\binom{0}{0}\left[\frac{1}{n}\binom{n-1}{0}^{-1} \sum_{j=1}^{n} \binom{j-1}{0} x_{j:n}\right]$$

$$\therefore\ \hat{\alpha}_1 = \frac{1}{n}\sum_{j=1}^{n} x_{j:n} = sample\ mean = \omega_0 \qquad [23]$$

For $r = 2$,

$$\hat{\alpha}_2 = \sum_{k=0}^{1} P_{1,k}\, \omega_k \;= P_{1,0}\omega_0 + P_{1,1}\omega_1$$

$$= (-1)^1 \binom{1}{0}\binom{1}{0}\omega_0 + (-1)^0 \binom{1}{0}\binom{2}{0}\omega_1 = -\omega_0 + 2\omega_1$$

$$\therefore\ \hat{\alpha}_2 = 2\omega_1 - \omega_0 \qquad [24]$$

Equating the population L-moments to their sample estimates gives;

$$\omega_0 = \gamma + \theta\Gamma\left(\frac{1}{\beta}+1\right) \qquad [25]$$

$$2\omega_1 - \omega_0 = \theta\Gamma\left(\frac{1}{\beta}+1\right)\left[1 - \frac{1}{2^{\frac{1}{\beta}}}\right] \qquad [26]$$

From equation [25], $\theta\Gamma\left(\frac{1}{\beta}+1\right) = \omega_0 - \gamma$

Substituting $\omega_0 - \gamma$ for $\theta\Gamma\left(\frac{1}{\beta}+1\right)$ in equation 26 gives;

$$2\omega_1 - \omega_0 = (\omega_0 - \gamma)\left[1 - \frac{1}{2^{\frac{1}{\beta}}}\right]$$

Solving for β;

$$\frac{2\omega_1 - \omega_0}{\omega_0 - \gamma} = 1 - \frac{1}{2^{\frac{1}{\beta}}}$$

$$1 - \frac{2\omega_1 - \omega_0}{\omega_0 - \gamma} = \frac{1}{2^{\frac{1}{\beta}}}$$

$$1 - \frac{2\omega_1 - \omega_0}{\omega_0 - \gamma} = 2^{-\frac{1}{\beta}}$$

$$\ln\left[1 - \frac{2\omega_1 - \omega_0}{\omega_0 - \gamma}\right] = -\frac{1}{\beta}\ln 2$$

$$\frac{-\ln\left[1 - \frac{2\omega_1 - \omega_0}{\omega_0 - \gamma}\right]}{\ln 2} = -\frac{1}{\beta}$$

$$\hat{\beta} = \frac{-\ln 2}{\ln\left[1 - \frac{2\omega_1 - \omega_0}{\omega_0 - \gamma}\right]} \quad \text{where } \omega_k = \frac{1}{n}\binom{n-1}{k}^{-1} \sum_{j=1}^{n} \binom{j-1}{k} x_{j:n} \qquad [27]$$

We, therefore, propose the use of equation [27] to estimate the shape parameter of the three-parameter Weibull distribution.

Comparison of closed-form estimators of β

We compare the proposed estimator with the existing estimators discussed above. To perform the comparison, the Monte Carlo simulation is carried out. Random samples from the three-parameter Weibull distribution are generated with known β values, then the estimators are applied to the samples to estimate β. To determine whether the value of the shape parameter affects the performance of the estimators, the true shape value is varied ($\beta = 0.1, \beta = 0.5, \beta = 1, \beta = 1.5$ $\beta = 2.5,\ \beta = 3.5, \beta = 5$). Sample size is also varied (n= 10, n = 30, n =100, n = 500) to determine if the sample size affects the performance of the estimators. The process is repeated 5000 times for each sample condition. The root mean squared error (RMSE) is used here as the metric for accuracy because it measures how far an estimate is from the true value. The root mean squared error can be expressed as;

$$RMSE = \sqrt{E[(\hat{\beta} - \beta)^2]} \quad [28]$$

where $\hat{\beta}$ is the estimated value and β is the true value.

RESULTS

The tables below show the methods ranked according to their performance based on their RMSE for the categories of simulation.

Table 1: Results for $\beta = 0.1$

	Rank	Method	RMSE
n = 20	1	Proposed estimator	0.09536534
	2	Percentile estimator	0.14401863
	3	Teimouri and Gupta estimator	0.15464661
	4	Goda et al polynomials	9.41301964
n = 50	1	Proposed estimator	0.06929465
	2	Teimouri and Gupta estimator	0.07195404
	3	Percentile estimator	0.11144796
	4	Goda et al polynomials	10.22591168
n = 100	1	Proposed estimator	0.06310422
	2	Teimouri and Gupta estimator	0.07182955
	3	Percentile estimator	0.10899955
	4	Goda et al polynomials	10.11075527
n = 200	1	Teimouri and Gupta estimator	0.03093812
	2	Proposed estimator	0.05300597
	3	Percentile estimator	0.09970828
	4	Goda et al polynomials	10.77889314

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Table 2: Results for $\beta = 0.5$

	Rank	Method	RMSE
n = 20	1	Teimouri and Gupta estimator	0.119806
	2	Proposed estimator	0.12436
	3	Percentile estimator	0.284908
	4	Goda et al polynomial	0.509229
n = 50	1	Teimouri and Gupta estimator	0.07624271
	2	Proposed estimator	0.07780251
	3	Goda et al polynomial	0.1840455
	4	Percentile estimator	0.20938301
n = 100	1	Teimouri and Gupta estimator	0.05395386
	2	Proposed estimator	0.05445809
	3	Goda et al polynomial	0.09415163
	4	Percentile estimator	0.17959128
n = 200	1	Proposed estimator	0.03540425
	2	Teimouri and Gupta estimator	0.03758375
	3	Goda et al polynomial	0.04477563
	4	Percentile estimator	0.16443088

Table 3: Results for $\beta = 1$

	Rank	Method	RMSE
n = 20	1	Proposed estimator	0.1879985
	2	Teimouri and Gupta estimator	0.1960125
	3	Percentile estimator	0.4401827
	4	Goda et al polynomial	0.484632
n = 50	1	Proposed estimator	0.1172147
	2	Teimouri and Gupta estimator	0.119213
	3	Goda et al polynomial	0.2153293
	4	Percentile estimator	0.2773543
n = 100	1	Proposed estimator	0.08160166
	2	Teimouri and Gupta estimator	0.08255023
	3	Goda et al polynomial	0.13643637
	4	Percentile estimator	0.22150362
n = 200	1	Proposed estimator	0.058323
	2	Teimouri and Gupta estimator	0.058445
	3	Goda et al polynomial	0.094765
	4	Percentile estimator	0.195226

Table 4: Results for $\beta = 1.5$

	Rank	Method	RMSE
n = 20	1	Proposed estimator	0.3316861
	2	Teimouri and Gupta estimator	0.3597133
	3	Goda et al polynomial	0.9502196
	4	Percentile estimator	2.39759
n = 50	1	Proposed estimator	0.1866692
	2	Teimouri and Gupta estimator	0.2150955
	3	Percentile estimator	0.3844049
	4	Goda et al polynomial	0.3869839
n = 100	1	Proposed estimator	0.126544
	2	Teimouri and Gupta estimator	0.1488701
	3	Goda et al polynomial	0.2332038
	4	Percentile estimator	0.296014
n = 200	1	Proposed estimator	0.1005869
	2	Teimouri and Gupta estimator	0.1029094
	3	Goda et al polynomial	0.1585457
	4	Percentile estimator	0.2500391

Table 5: Results for $\beta = 2.5$

	Rank	Method	RMSE
n = 20	1	Proposed estimator	0.900808
	2	Teimouri and Gupta estimator	0.9468059
	3	Percentile estimator	0.9948187
	4	Goda et al polynomial	2.3148297
n = 50	1	Proposed estimator	0.4375576
	2	Percentile estimator	0.6272668
	3	Teimouri and Gupta estimator	0.6671635
	4	Goda et al polynomial	0.9422117
n = 100	1	Proposed estimator	0.3785296
	2	Percentile estimator	0.4664347
	3	Teimouri and Gupta estimator	0.5034881
	4	Goda et al polynomial	0.5685845
n = 200	1	Proposed estimator	0.2775372
	2	Goda et al polynomial	0.3677054
	3	Teimouri and Gupta estimator	0.3857527
	4	Percentile estimator	2.8308696

Table 6: Results for $\beta = 3.5$

	Rank	Method	RMSE
n = 20	1	Proposed estimator	1.676932
	2	Teimouri and Gupta estimator	1.73039
	3	Goda et al polynomial	4.166346
	4	Percentile estimator	5.212321
n = 50	1	Proposed estimator	1.305476
	2	Teimouri and Gupta	1.329823
	3	Goda et al polynomial estimator	1.627909
	4	Percentile estimator	4.342122
n = 100	1	Proposed estimator	0.7221477
	2	Goda et al polynomial	0.9762473
	3	Teimouri and Gupta estimator	1.0945583
	4	Percentile estimator	4.0255074
n = 200	1	Goda et al polynomial	0.6237123
	2	Proposed estimator	0.7083365
	3	Teimouri and Gupta estimator	0.8954279
	4	Percentile estimator	5.440778

Table 7: Results for $\beta = 5$

	Rank	Method	RMSE
n = 20	1	Proposed estimator	2.959233
	2	Teimouri and Gupta estimator	3.018996
	3	Goda et al polynomial	6.335933
	4	Percentile estimator	7.364682
n = 50	1	Goda et al polynomial	2.421429
	2	Proposed estimator	2.497171
	3	Teimouri and Gupta estimator	2.525136
	4	Percentile estimator	6.047462
n = 100	1	Goda et al polynomial	1.500637
	2	Proposed estimator	1.561893
	3	Teimouri and Gupta estimator	2.19721
	4	Percentile estimator	5.6495
n = 200	1	Goda et al polynomial	1.034314
	2	Proposed estimator	1.583582
	3	Teimouri and Gupta estimator	1.902128
	4	Percentile estimator	5.440778

SUMMARY OF FINDINGS

From the results shown in tables 1-7 which was obtained from the simulation experiments, it can be seen that when β = 0.1, the proposed estimator performed better than the other methods that were compared for all sample sizes considered. When β = 0.5, the Teimouri and Gupta estimator performed better for sample sizes n = 20, 50 and 100. However when n = 200, the proposed method performed better. When β = 1, 1.5, and 2.5, the proposed estimator performed better than the other methods that were compared for all sample sizes considered. When β = 3.5, the proposed method performed better than the other methods that were compared for n = 20, 50 and 100 but for n = 200 the proposed estimator is slightly edged out by Goda's polynomial estimator. When β = 5, the proposed method performed better than the other methods that were compared for n = 20, but for n = 50, 100 and 200, the proposed estimator was slightly edged out by Goda's polynomial estimator.

CONCLUSION

In this study, the performance of three existing closed-form estimators (Teimouri and Gupta estimator, Goda et al estimator, percentile estimator) and a proposed closed-form estimator of the shape parameter of the three-parameter Weibull distribution were compared. Based on the findings, we conclude that generally the proposed estimator performs very well compared to the other closed-form estimators compared. Teimouri and Gupta estimator and Goda et al also showed good performances in some cases.

Recommendation

Based on the results of this study, we recommend the use of the proposed estimator for the estimation of the shape parameter of the three-parameter Weibull distribution when tractability and simplicity are required because in general, it performs very well when compared to the other closed-form estimators. We also recommend further study into proposing more accurate closed-form estimators for the three-parameter Weibull distribution for the sake of tractability and estimation speed.

References

Bílková, D. (2014). Alternative Tools of Statistical Analysis: L-moments and TL-moments of Probability Distributions, Pure and Applied Mathematics Journal. 14-25. doi:10.11648/j.pamj.20140302.11.

Cheng, R. C. H. and Amin, N. A. K. (1983). Estimating Parameters in Continuous Univariate Distributions With A Shifted Origin, Journal of the Royal Statistical Society, Series B (Methodological) 45(3), 394–403.

Cohen, A. C and Whitten, B. (1982). Modified Maximum Likelihood and Modified Moment Estimators For The Three-Parameter Weibull Distribution, Communications in Statistics

Theory and Methods 11.

Cousineau. D. (2008). Fitting The Parameters of The Three-Parameter Weibull Distribution: Review And Evaluation Of Existing And New Methods, IEEE Transactions on Dielectrics and Electrical Insulations, 16(1), 281-288.

David, H.A. Some properties of order-Statistics filters. Circuits Systems and Signal Process **11,** 109–114 (1992). https://doi.org/10.1007/BF01189222

Dubey S.D.1967. Some percentile estimators for Weibull parameters.Technometrics 9: 119-129

Goda, Y., Kudaka, M and Kawai, H. (2010). Incorporation Of Weibull Distribution In L-Moments Method For Regional Frequency Analysis Of Peaks-Over-Threshold Wave Heights, Proceedings of the International Conference on Coastal Engineering, http://journals.tdl.org/ICCE/article/viewFile/1154/pdf_42.

Hosking, J. R. M. (1990). L-Moments: Analysis and Estimation of Distributions Using Linear Combinations of Order Statistics, Journal of the Royal Statistical Society, Series (Methodological) 52, 105–124.

Hosking, J.R.M and Wallis, J.R. (1997) Regional Frequency Analysis: An Approach Based on L-moments. Cambridge University Press, UK. http://dx.doi.org/10.1017/cbo9780511529443

Rinne, H. (2008). The Weibull Distribution: A Hand Book, CRC press, London, November 2008.

Teimouri M. and Gupta A.K. (2013). On the three-parameter weibull distribution shape parameter estimation, journal of data science 11, 403-414.