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MODIFIED ASYMMETRIC SCHEMES FOR BLACK -SCHOLES EQUATION OF EUROPEAN OPTIONS PRICING SYSTEM

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ABSTRACT: *In this paper, a fast numerical scheme for computing the European call options pricing problems governed by the Black Scholes equation was developed. Error analysis algorithm was developed from the proposed scheme. It is proved that the proposed scheme has second order accuracy in both time and space under some restrictions, the stability of the scheme in the sense of Non –Neumann analysis is presented. It is shown that the proposed scheme has a good performance in the sense of Errors, the computational cost and storage capacities compare to the Crank- Nicolson. Also the accuracy of the purposed scheme is better than the Semi- implicit scheme in most cases. Numerical examples are conducted to test the validity, efficiency, and accuracy of the proposed scheme.*

KEYWORDS: Black-Scholes equation, European option pricing, Asymmetric scheme, Stability analysis, Numerical experiments.

INTRODUCTION

Derivatives are nowadays widely used in financial markets. A derivative is a financial instruments speculation and arbitrage. The values of a financial asset are uncertain and invest directly in then comprehend a big risk. Derivatives arise as a way to invest in actives with less risk. However, by continuing to depend on the value of the underlying asset, there is a great difficulty in deterring their value. In this paper, the proposed methods for option pricing including the European call options was developed.The traditional approach to price derivative assets or options is to specify an asset price process exogenously by a stochastic diffusion process and then price by no-arbitrage arguments. The seminal example of this is Black and Scholes (1973) on pricing of European –style options. This approach leads to simple, explicit pricing formulas. However, empirical research has revealed that they are not able to explain important effects in real financial market for example the volatility smile (or skew) in option prices. In real financial markets, not only asset returns are subject to risk, but also the estimate of the riskiness is typically subject to uncertainty. To incorporate such additional source of randomness into an asset pricing model, one has to introduce a second risk factor. This also allows to fit higher moments of the asset return distribution. Weng et.al (2014), presented a paper on asymmetric finite difference method for the Black- Scholes equation of European Options discussing the Superiority of the proposed method (asymmetric

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method) over Crank Nicolson (C-N) method. Black and Scholes (1973) , developed the pricing of options and corporate liabilities with relevance to European and American political economy. Yang et. Al (2012), demonstrated the Semi- implicit Difference Algorithm for Solving the Black-Scholes equations with payment of Dividend. It was obvious in his paper that the Semi-implicit difference algorithm is an effective method for solving Black- Scholes equations on options pricing models. Jeong et. al (2009), presented an accurate and efficient numerical method for Black-Scholes equations. Han and Wu (2003), presented a fast numerical method for Back-Scholes equations of America options which shows that prices are not pre- determined. Liao and Khaliq (2009)now presented high order compact scheme for solving nonlinear Black-Scholes equations with transaction cost. Zhang and Yang , (2012) developed a parallel difference numerical methods for solving the payment of dividend of Black- Scholes equations. This paper is organized as follows: Mathematical model is treated in section 2, error analysis in section 3, analysis of stability and convergence in section 4, Numerical results in 5 and finally conclusion.

MATHEMATICAL MODEL

This paper begins with the introduction of Mathematical model for European call option. Assume that V is the call option value, S is the assert price, t is the time. Let $r,\ \sigma,\ q$ denote the risk- free interest rate, the volatility of the assert price and the continuous dividend yield respectively. Then the model is based on the Black Scholes partial differential equations.

$$\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

Where the explicit expression of Black Scholes equation (1) can be written as

$$V(S,t)=Se^{-q(q-t)N(d_1)}-ke^{-r(\tau-t)N(d_2)} \tag{2}$$

The boundary condition of the Black- Scholes is given by

$$V(S,T)=\text{Max }(S-K,0)$$

Introducing the change variables

$$x=\log S,\quad \tau=T-t$$

Then equation (1) can be written as

$$\frac{\partial V}{\partial \tau}-\frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}-(r-q-\frac{\sigma^2}{2})\frac{\partial v}{\partial x}+rv=0 \tag{3}$$

with the associated boundary condition is given by

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$$\lim_{n\to\infty} v(x,\tau)=0, \qquad \lim_{n\to\infty} v(x,\tau)=e^{x}-k$$

assume that the solution region is:

$$\sum\nolimits_0 = \{-\infty \le x < +\infty,\ 0 \le \tau \le T\}$$

In order to construct a two- step asymmetric difference scheme, firstly discretized the region $\sum_0$ as a uniform grids with the space step h and time k.

$$x_j = jh, \quad j=0,\pm 1,\pm 2,\ldots N, \quad \tau_n = nk,\ n=0,1,2,\ldots M\,(M=[T/K])$$

Let V_j^n denote the numerical approximation of the solution (x_i,τ_n). Then the time derivative $\frac{\partial v}{\partial \tau}$ at each grid point $V(x_j,\tau_n)$ can be approximated by the backward finite difference as

$$\frac{\partial v}{\partial \tau} \approx \frac{v_j^{n+1}-v_j^n}{k}$$

For the construction of the second order asymmetric finite difference method, the first and second derivative for spatial variables can e approximated by

$$\frac{\partial v}{\partial x} \approx \frac{v_{j+1}^n - v_j^n + v_j^{n+1} - v_j^{n+1}}{2h}$$

$$\frac{\partial^2 v}{\partial x^2} \approx \frac{v_{j+1}^n - v_j^n - v_j^{n+1} + v_{j-1}^{n+1}}{h^2}$$

Here rv is approximated by $r\left(\frac{v_j^{n+1}+v_j^n}{2}\right)$. Hence, equation (1) can be rewritten as follows:

$$\frac{v_j^{n+1}-v_j^n}{k} = \frac{\sigma^2}{2}\frac{v_{j+1}^n - v_j^n - v_j^{n+1} + v_{j-1}^{n+1}}{} + (r-q-\frac{\sigma^2}{2})\frac{v_{j+1}^n - v_j^n + v_j^{n+1} - v_{j-1}^{n+1}}{} - r(\frac{v_j^{n+1}+v_j^n}{2}) \quad (4)$$

and the above equation can be simplified as

$$v_j^{n+1} = a_1 v_{j+1}^n + b_1 v_j^n + c_1 v_{j-1}^n \quad (5)$$

where,

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$$a_1 = \frac{k\sigma^2 + kh(r - q - \frac{\sigma^2}{2})}{2h^2 + k\sigma^2 + kh^2 r - kh(r - q - \frac{\sigma^2}{2})}$$

$$b_1 = \frac{2h^2 - k\sigma^2 - kh^2 r - kh(r - q - \frac{\sigma^2}{2})}{2h^2 + k\sigma^2 + kh^2 r - kh(r - q - \frac{\sigma^2}{2})}$$

$$c_1 = \frac{k\sigma^2 - kh(r - q - \frac{\sigma^2}{2})}{2h^2 + k\sigma^2 + kh^2 r - kh(r - q - \frac{\sigma^2}{2})}$$

Similarly, if the first and the second derivative for spatial variables are approximated as :

$$\frac{\partial v}{\partial x} \approx \frac{v_{j+1}^{n+1} - v_j^{n+1} + v_j^n - v_{j-1}^n}{2h}$$

$$\frac{\partial^2 v}{\partial x^2} \approx \frac{v_{j+1}^{n+1} - v_j^{n+1} - v_j^n + v_{j-1}^n}{h^2}$$

Then, the equation (1) can be rewritten as

$$\frac{v_j^{n+1} - v_j^n}{k} = \frac{\sigma^2}{2}\frac{v_{j+1}^{n+1} - v_j^{n+1} - v_j^n + v_{j-1}^n}{h^2} + (r - q - \frac{\sigma^2}{2})\frac{v_{j+1}^{n+1} - v_j^{n+1} + v_j^n - v_{j-1}^n}{2h} - r(\frac{v_j^{n+1} + v_j^n}{2}) \quad (6)$$

Also the above equation can be simplified as

$$v_j^{n+1} = a_2 v_{j-1}^n + b_2 v_j^n + c_2 v_{j+1}^n \quad (7)$$

where,

$$a_2 = \frac{k\sigma^2 - kh(r - q - \frac{\sigma^2}{2})}{2h^2 + k\sigma^2 + kh^2 r + kh(r - q - \frac{\sigma^2}{2})}$$

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$$b_2 = \frac{2h^2 - k\sigma^2 - kh^2 r + kh(r - q - \frac{\sigma^2}{2})}{2h^2 + k\sigma^2 + kh^2 r + kh(r - q - \frac{\sigma^2}{2})}$$

$$c_2 = \frac{k\sigma^2 + kh(r - q - \frac{\sigma^2}{2})}{2h^2 + k\sigma^2 + kh^2 r + kh(r - q - \frac{\sigma^2}{2})}$$

Finally, the asymmetric scheme for the model problem is designed as follows; at each time levels the approximation A_j^{n+1} are obtained by using the scheme in equation (5) from left boundary and B_j^{n+1} are calculated by the scheme in equation (7) from the right boundary.

Once both approximations are computed, the final approximation v_j^{n+1} are taken with arithmetic mean A_j^{n+1} and B_j^{n+1}. More precisely, the proposed scheme is summarized as

$$\begin{Bmatrix} A_j^{n+1} = a_1 v_{j+1}^n + b_1 v_j^n + c_1 v_{j-1}^{n=1} \\ B_j^{n+1} = a_2 v_{j-1}^n + b_2 v_j^n + c_2 v_{j+1}^{n+1} \end{Bmatrix}$$

$$v_j^{n+1} = \frac{A_j^{n+1} + B_j^{n+1}}{2} \tag{8}$$

ERROR ANALYSIS:

The errors can be computed following this procedure as

$$\begin{aligned} T(k,h) = 2(\frac{v_j^{n+1} + v_j^n}{k}) = \frac{\sigma^2}{2}(\frac{v_j^{n+1} - 2v_j^n + v_{j-1}^n - 2v_j^{n+1} + v_{j-1}^{n+1}}{h^2}) \\ -(r - q - \frac{\sigma^2}{2})(\frac{v_{j+1}^n - v_{j-1}^n}{2h} + \frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2h}) + r(\frac{v_j^{n+1} + v_j^n}{2}) \end{aligned} \tag{9}$$

Applying the Taylor expansion of the term $T(k,h)$ about the point $V(x_j, \tau_n)$, yields

$$\begin{aligned} T(k,h) = 2(\frac{\partial v}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} - (r - q - \frac{\sigma^2}{2})\frac{\partial v}{\partial x} + rv) + k\frac{\partial v}{\partial \tau}(\frac{\partial v}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}) \\ -(r - q - \frac{\sigma^2}{2})\frac{\partial v}{\partial x} + rv + 0(k^2 + h^2) = 0(k^2 + h^2) \end{aligned} \tag{10}$$

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ANALYSIS OF STABILITY AND CONVERGENCE OF THE SCHEME

In this section, the analysis of the stability and convergence of equation (8) will be critically examined. Considering the stability condition of equation (4). Denote $v_j^n = v^n e^{ijQh}$ where $i = \sqrt{-1}$ the imaginary unit is and Q is the wave number.

Then equation (5) becomes,

$$v_j^{n+1} = \frac{b_1 + a_1 e^{iQh}}{1 - c_1 e^{-iQh}} v_j^n \quad , \tag{11}$$

Define $G_1(h,k) = \frac{b_1 + a_1 e^{iQh}}{1 - c_1 e^{-iQh}}$. Based on the Nov Neumann analysis equation (4) is stable if $|G_1(h,k)| \leq 1$. Also define $\alpha = (r - q - \frac{\sigma^2}{2})$ and $\beta = \frac{k\sigma^2}{2h^2}$. Then the stability condition of equation (4) is given by : 1

$$\begin{cases} \text{when } \alpha \leq 0, |G_1(h,k)| \text{ is always less than 1} \\ \text{When } \alpha > 0, \text{ and } 4\beta - \frac{4k\alpha\beta}{h} - \frac{k^2\alpha r}{h} \geq 0, \quad |G_1(h,k)| \leq 1. \end{cases} \tag{12}$$

Similarly, the same idea is to calculate the stability condition of equation (6). Then, the stability is also shown below as :

$$\begin{cases} \text{when } \alpha \geq 0, |G_1(h,k)| \text{ is always smaller than 1.} \\ \text{when } \alpha. < 0, \text{ and } 4\beta + \frac{4k\alpha\beta}{h} + \frac{k^2\alpha r}{h} \geq 0, \quad |G_1(h,k)| \leq 1. \end{cases} \tag{13}$$

Therefore, when $\alpha = 0$, equation (8) for solving the payment of dividend Black Scholes equation is unconditional stable, when $\alpha > 0$ and $4\beta - \frac{4k\alpha\beta}{h} - \frac{k^2\alpha r}{h} \geq 0$, then equation (8) is stable, when $\alpha < 0$ as $4\beta + \frac{4k\alpha\beta}{h} + \frac{k^2\alpha r}{h} \geq 0$, then equation (8) is stable.

NUMERICAL RESULTS

This section deals with the implementation of numerical simulations to price European call options using Matlab software

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Example :

Considering the European call option where the parameters used in the simulation are $S = 100, k = 100, T = 0.5, \quad \sigma = 0.2, r = 0.05, \quad q = 0.03$. The reference value for the example is 6.029259

In Table 1 &2 , value denotes the European call options obtained by equation (8), where M is the number of time steps and N is the number of spatial steps.

Table 1: The value and error obtained by employing the proposed method with fixed time step size and varying the number of spatial grids.

M	N	VALUE	ERROR	ORDER
1200	128	6.069953	0.040484	-
1200	256	6.047513	0.017784	1.185
1200	512	6.026426	0.003103	2.519
1200	1024	6.02884	0.000725	2.097

Fig 1; Values M=Fixed time, N =Spatial size

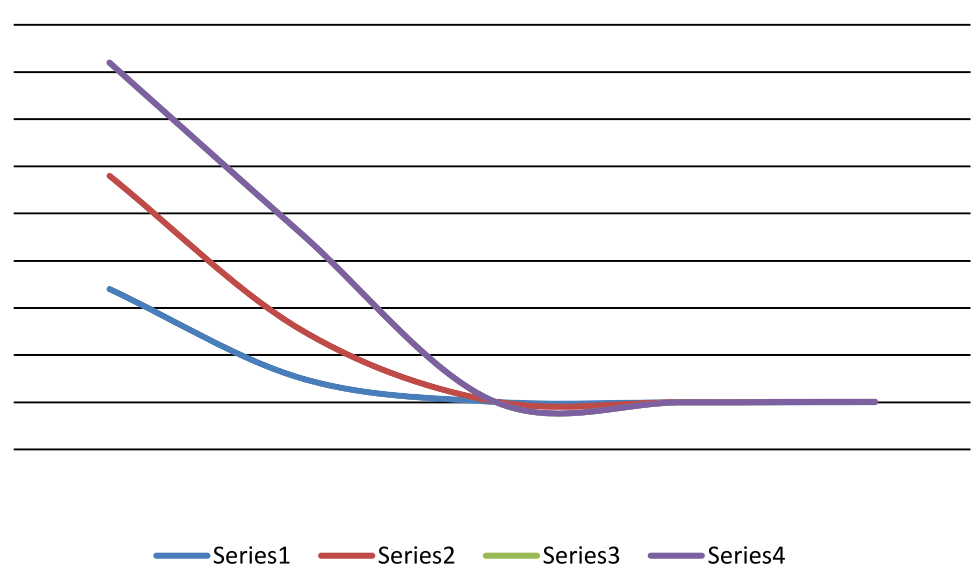

Table2: The value and error obtained by employing the proposed method with fixed spatial discretization and varying the time step size.

N	M	VALUE	ERROR	ORDER
1400	120	6.011806	0.017723	-
1400	240	6.025169	0.004360	2.023
1400	480	6.028496	0.001033	2.077
1400	960	6.029327	0.000202	2.352

As observed in Table 1 and 2, the numerical results shows that the proposed scheme (8) has second order convergence.

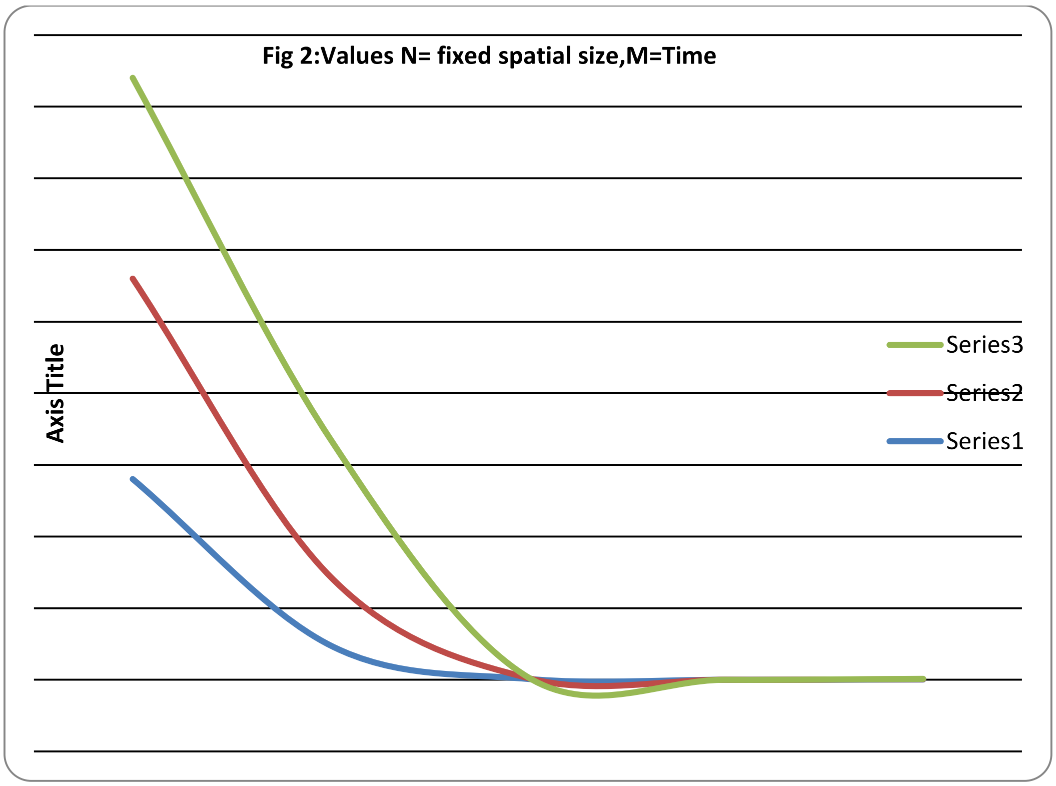

Table 3: Comparison between the proposed method and the Crank Nicolson (C-N) method by varying the time step sizes and the number of spatial grids.

CPU Time in Seconds

M	N	ADM	C-N
200	512	0.069040	0.192794
400	1024	0.123246	2.309442
800	2048	0.862435	14.059381

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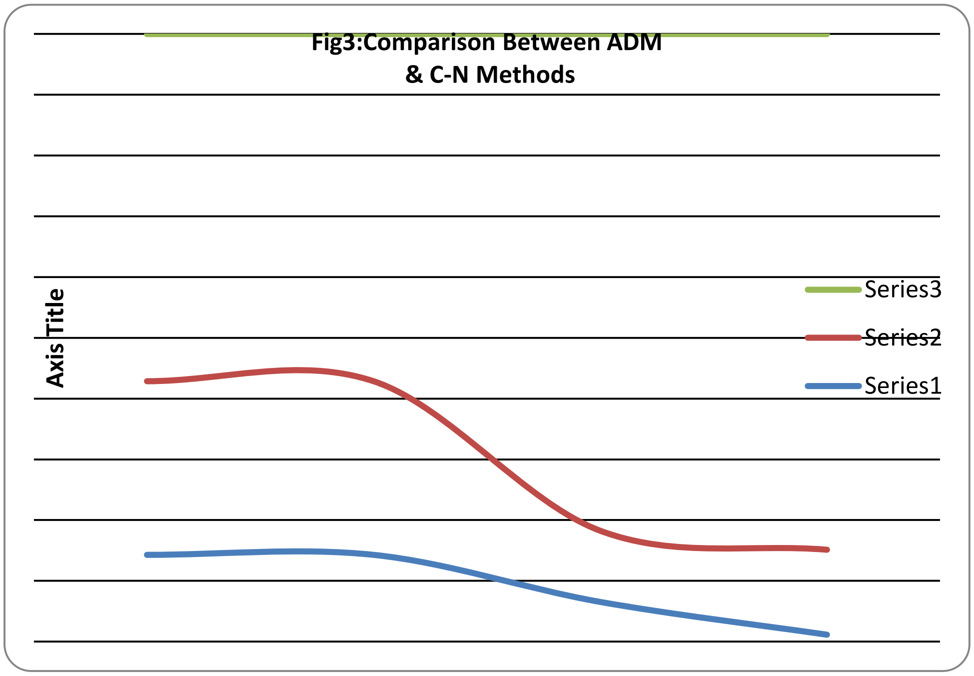
Table 3 , shows the comparison of the computation time between the second order asymmetric difference method (shot for ADM) and the Crank- Nicolson method (shot for C-N).

Table 3 shows that ADM is superior to C-N methods.



CONCLUSION

In this paper, Mathematical model of second order for asymmetric method for solving the European call options was formulated . The accuracy of the asymmetric schemes is better and superior over Crank –Nicolson method considering error differences and computational cost, time and memory storage capacities.

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