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HARMONIC EQUATIONS DEDUCIBLE FROM THE HYDRODYNAMIC MOTION OF A FLOATING OBJECT

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ABSTRACT: *Elementary analysis of hydrostatics gives an important result usually called Archimedes principle. Further investigation of this principle in hydrodynamic terms reveal some important properties common to an oscillatory system. The major aim of this paper is to give a brief derivation of a differential equation that best describes the harmonic nature of the motion of an object that is partially submerged in a liquid. The fundamental physical law for a floating object is called Archimedes principle stated as "An object that is partially or wholly submerged in a liquid is acted on by an upward force which equals the weight of the liquid displaced". However, additional general description of this principle concerning the motion of the object will be established here in an effort to see if the results will be of any particular significance as a problem of hydrodynamics.*

KEYWORDS: floating object, hydrodynamic, harmonic motion, damping, cylinder, amplitude

INTRODUCTION

The mathematical formulation of numerous physics problems usually results in differential equations, and the solution of these equations normally gives an insight into the nature of the problems and the methods of their solution/interpretations. [1,2] On the motion of floating bodies. In most cases, it is possible to use the equations as a frame-work for the study and analysis of the general phenomenon governing the problem. The free surface interacting plays an important role in the hydrodynamics as seen. [3]

In the case of a floating object, the nature of its motion would be best described if an appropriate differential equation is found to represent it. The most common idea of floating refers to a rigid body partially immersed at a horizontal free surface of liquid at rest under gravity. [4] Smoothness of the motion of a rigid body immersed in an incompressible perfect fluid.

Specification of Problems

Suppose that an object of known dimension is immersed in a liquid at rest. [6] Implementation and Validation of a Potential Model for a mooned floating cylinder under waves. The resultant force on the object first at rest and subsequently in motion will be due entirely to the presence of the liquid. [7] studied the motion of mechanical system found by a fluid and a partially immersed solid. [8] Stability Analysis of a calm floating offshore structure.

Consider a right circular cylinder of radius R, height H, weight W and $mass = \frac{w}{g}$, floating on a liquid of known density ρ.

Basic Assumptions

For standard conditions, assume that:

1. The cylinder is floating with its axis vertical in a liquid of density ρ.
2. That $W < \pi R^2 H\rho g."If\ W \geq \pi R^2 H\rho$ the cylinder will be completely submerged.
3. Let $h = \frac{w}{\pi R^2 \rho g}$ (that is $W = \pi R^2 h\rho g$)

The buoyant force upwards is precisely the weight of the cylinder. Thus, the equilibrium position for the cylinder is when h vertical units lie under the liquid. When at equilibrium, the system experiences a disturbance oscillation occurs in the system as studied [9]

For each time $t \geq 0$ let $g(t)$ be the displacement of the equilibrium position from the liquid surface with the convection that $y(t) > 0$. If the equilibrium position is above the surface and $y(t) < 0$ if the equilibrium position is below the surface as in fig C. The motion of the cylinder will be an oscillation between $y(t) > 0\ and\ y(t) < 0$ of the equilibrium positions from the liquid surface with the initial conditions that $y(0) = y_0\ and\ y^I(0) = V_0$.

Now defining the acceleration of the cylinder to be $a = \frac{d^2y}{dt^2}$ and the mass to be $m = \frac{w}{g}$ where g is a gravitational constant, then the total force f acting on the cylinder will be given by Newton's 2nd law to be $F = \frac{w}{g} \circ \frac{d^2y}{dt^2}$ ... (1)

The total force F can also be described in Archimedes term as

$$F = -W + (Bouyant\ force) \quad \ldots.*$$

Now the volume of the cylinder under the liquid can be expressed as $\pi R^2\big(h - y(t)\big)$ so that the weight of the liquid displaced becomes $\pi R^2 \rho g[h - y(t)]$ and is equal to the Bouyant force.

The total force in terms of Archimedes becomes $F = -w + \pi R^2 g\rho[h - y(t)]$ .... (2)

Which reduces to $F = \pi R^2 \rho y(t) g$ .... (3)

$$\frac{w}{g}\frac{d^2y}{dt^2} = -\pi R^2 \rho y(t) g \quad \ldots \quad (4)$$

Therefore, the initial value problem (IVP) describing the motion of the cylinder is given by the 2nd Order differential equations (ODE) with constant coefficient of the form

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$$\frac{d^2y}{dt^2} = -\frac{\pi R^2 \rho g^2}{W} y(t)$$

$$\frac{d^2y}{dt^2} + \frac{\pi R^2 \rho g^2}{w} y(t) = 0 \quad \ldots \qquad (5)$$

With $y(0) = y_0,\quad \frac{dy}{dx}\downarrow_{t=0} = V_0$

Where y_0 is the initial position and V_0 is the initial velocity of the cylinder

Letting $\alpha^2 = \frac{\pi R^2 \rho g^2}{W} \quad \Rightarrow \ \alpha = R\sqrt{\frac{\pi \rho g^2}{w}} \ = Rg\sqrt{\frac{\pi \rho}{W}}$

Equation 5 becomes $\quad \frac{d^2y}{dt^2} + \alpha^2 = 0 \qquad . \ . \ . \ (6)$

Equation (1) is the Simple Harmonic Motion and so the motion of the cylinder is harmonic in nature and represent a force oscillation about the equilibrium position. Applying the oscillation principle Cutnell and Johnson K.W. (2019)

The general solution of the differential equation will give the positional displacement of the cylinder about the equilibrium position in the form

$$y(t) = B\cos\alpha t + D\sin\alpha t \ when\ the\ displacement\ is$$

$$y(t) = y_0 \cos\alpha t + \frac{V_0}{\alpha}\sin\alpha t \qquad . \ . \ . (7)$$

Where $B = y_O \ and \ D = \frac{V_0}{\alpha}$ which is the initial condition.

Eqn (7) is the solution of the IVP and can be simplified into the form

$$y(t) = \sqrt{{y_0}^2 + \frac{{V_0}^2}{\alpha^2}}\, sin(\alpha t + \emptyset) \qquad . \ . \ . \ (8)$$

Where $\emptyset = tan^{-1}\left(\frac{y_0\alpha}{V_0}\right)$

Eqn (8) is equivalent to the standard harmonic displacement of the form

$$y = Asin(\alpha t + \in) \quad . \ . \ . \ (9)$$

Thus the motion of the cylinder is periodic and the period T and amplitude A are given by

$$T = \frac{2\pi}{\alpha} = \frac{2\sqrt{\pi W}}{R\sqrt{\rho g^2}} \qquad . \ . \ . \qquad (10)$$

$$A = \sqrt{{y_0}^2 + \frac{{V_0}^2}{\alpha^2}} \quad or \ \sqrt{{y_0}^2 + \frac{W{V_0}^2}{R^2\pi\rho g^2}} \qquad . \ . \ . \ (11)$$

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Notes on Results:

1. The amplitude depends on both the initial position and velocity but the period is independent of the initial conditions.
2. The motion is independent of the height H of the cylinder so long as the amplitude is smaller than ${}^{H}/_{2}$.

Condition for Damping

Suppose the motion is perturbed by damping factors which include the viscosity of the liquid, then the velocity of the cylinder will diminish with the positional displacement, which will represent an exponential decay of the form $y = y_0 e^{-kt} \quad . \; . \; . (12)$

$\therefore \quad the\ y = Asin(\alpha t + \in)$ which represent the amplitude decaying exponentially.

The motion force on the cylinder of mass m is proportional to the displacement towards the centre, so that $\quad m\frac{d^2y}{dt^2} + \alpha^2 y = 0 \quad . \; . \; . \quad (14)$

And under normal viscous damping conditions, the resistance to motion is proportional to the velocity $\beta \frac{dy}{dx}$, so that combining both we get the equation of the damped harmonic motion as

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + \alpha^2 y = 0 \quad . \; . \; . \quad (15)$$

Dividing through by m and letting $w^2 = \frac{\alpha^2}{m}, \;\; 2k = {}^{\beta}/_{m},$ we get

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + w^2 y = 0 \quad . \; . \; . \quad (16)$$

With the initial condition that $y(0) = y_0\ and\ \frac{dy}{dx}|_{t=0} = 0$

The solution of this equation is given by

$$y(t) = y_0 e^{-kt} sin\left(\sqrt{w^2 - k^2 t + {}^{\pi}/_{2}}\right) \quad . \; . \; . \; (17)$$

Also if $k = W$, the motion is critically damped and the displacement decays exponentially without oscillating that is

$$y = y_0 e^{-kt} \quad . \; . \; . \quad (18)$$

While if $k > w$, the rate of return towards the equilibrium position is even slower.

Now, suppose that the cylinder is subjected to a periodic driving force.

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$$P = F_0 \sin w_1 t \quad . \ . \ . \quad (19)$$

Then the R.H.S of equation (19) will be $f_1 sinw_1 t$ instead of zero where $f_1 = {}^{f_0}/_{M}$.

This implies that the central damped harmonic equation becomes

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + w^2 y = f_1 sinw_1 t \quad . \ . \ . \quad (20)$$

Equation (20) has a solution which can be split into two parts. The 1st part being the integral of the non-homogenous system and secondly the complimentary function of the associated homogenous system.

1st Part gives the solution

$$Yx(t) = \frac{f1}{(w^2-w_1^2)^2 + 4k^2w_1^2} \, Sin(w_1 t - \sigma) \ldots\ldots(21)$$

When $\sigma = tan^{-1}(\frac{2kw_1}{w^2-w_1^2})$

Now

If,

$w_1 < w_2$, tanσ is $+ve$, σ is between 0 & $\frac{\pi}{2}$

$w_1 = w_2$, tanσ is infinite, σ is $\frac{\pi}{2}$

$w_1 > w_2$, tanσ is $-ve$, $\sigma$is between $\frac{\pi}{2}$ & π

This implies that the amplitude has its maximum value when $w_1 = w$. This is a condition called Resonance.

Now, to find the solution of the second part, consider f_1, to be negatively small, we note that the system once displaced will tend to oscillate with its free oscillation period at first although this would quickly delay giving way to forced oscillation.

∴ the complete solution of the equation

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + \omega^2 y = f_1 \sin \omega_1 t \ldots..(22)$$

also includes the situation of the associated homogenous system

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$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + \omega^2 y = 0$….(23)

The solution is of the form

$y_{2(t)}$=$y_0 e^{-kt}\sin(\sqrt{\omega^2 - k^2 t + \frac{\pi}{2}})$ …….(24)

Now the second part of the solution gives an oscillation at the natural frequency of the system, which quickly dies away and does not affect the steadily maintained forced oscillation at all.

∴ The complete solution of the system of the forced damped harmonic motion of the cylinder is given by the sum of the solutions of the two part as

$y(t) = y_1(t) + \ y_2(t)$

CONCLUSION

It must be noted that the harmonic equation deduced from the motion of a floating cylinder is also applicable to the motions of other floating objects of regular dimensions, and can as well be at least be estimated.

As an applicable example, suppose that a rectangular box with width s_1, length s_2 and height.

It is floating in a liquid of density ρ, letting y its denote the position of the box relative to equilibrium and the weight of the box be w, then we can show that the position y(t) is a solution to the IVP given by

$\frac{d^2y}{dt^2} + \frac{S_1S_2Pg^2}{W}y = 0, y(0) = y_0\,, y^1(0) = y_0$

Where y_0 is initial position and v_0 is the initial velocity.

The period and amplitude can be determined,

$T = \frac{2\pi\sqrt{\omega}}{\sqrt{s_1\, s_2\delta\, g^2}}\,, A = \sqrt{y_0^2 + \frac{v_0^2\, w}{s_1\, s_2\delta\, g^2}}$

Which by comparing with values obtained in a cylinder are the same.

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