International Journal of Mathematics and Statistics Studies
Vol.9, No.3, pp.49-64, 2021
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

HARMONIC C^*-CATEGORIES OF LONGITUDINAL PSUDODIFFERENTIAL OPERATORS OF HIGHER ORDER OVER FLAG VARIETY

Dr.Safa Ahmed Babikir Alsid

[1] Department of Mathematics, Omdurman Islamic University, Khartoum Sudan, Department of Mathematics

ABSTRACT: *The main aim of this paper is to use Gelfand- Tsetlin bases to show that the set of longitudinal psudo differential operators of order zero* $\psi^0_{\mathcal{F}_i}$ *or higher on* K *−homogenuose vector bundles* E, E^* *is the subset of simultaneous multiplier category* $\mathcal{A} = \bigcap_{s\subseteq\sum} \mathcal{A}_s$, *for* C^**-categories* $\mathcal{A}_s$ *and* $\mathcal{K}_s$ *operators between* $K-$ *spaces, with simple roots* $\propto_1, \propto_2$ *of Lie group* $SL(3;\mathbb{C})$ *by using the Lie algebra* $sl(3;\mathbb{C})$ *weight;* $s \subseteq \sum = \{\propto_1, \propto_2\}$.

KEYWORDS: *Gelfand-Tsetlin pattern; harmonic analysis on flag variety; longitudinal psudodifferential operators; Lie algebras and Lie group.*

INTRODUCTION

In this paper we show that the set of longitudinal psudodifferential operators of order zero on K −homogenuose vector bundles over flag variety $\mathbb{X}$ is the subset of simultaneous multiplier category, this matter requires some lengthy computations in noncommutative harmonic analysis. The important progress connected with the idea of the harmonic C^*-categories and with Bernstein- Gelfand-Gelfand complex and Kasparov theory for the action of the group $SL(3;\mathbb{C})$ [1]. As in [22], they key computation will be made using Gelfand-Tsetline bases (GT), it is possible to relate each (GT) pattern of integers array with a vector in the irreducible representation with highest weight. These vectors form an

Vol.9, No.3, pp.49-64, 2021
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

orthogonal basis for this representation. and our work depends on the expository paper [17] together with some remarks of [22]. We specialize the case of $sl(3;\mathbb{C})$. Furthermore, in [22] another construction of regarding harmonic analysis on flag manifolds for $SL(n;\mathbb{C})$. We are particularly interested in $n = 3$ here. For the definition and basic properties of longitudinal psudodifferential operators, we refer the reader to [18]. For $SL(3;\mathbb{C}) = G$, the simple roots $\propto_i$ $(i = 1,2)$ there aer G-equivariant fibrations $\mathbb{X} \to \mathbb{X}_i$ where $\mathbb{X}_i$ is the Grassmannians of lines and planes in the complex Bernstein, Gelfand and Gelfand made a homological complex by assembling interwiners between Verma modules, [6] for details. The C^*- algebra $\mathbb{K}_{\propto_i}$ of operators on the L^2-section space of any homogenous line bundle over flag variety $\mathbb{X}$ associated to each of these fibration. The fiberation is tangent to the longitudinal psudodiferential operators.the intersection of C^*- algebra $\mathbb{K}_{\propto_i}$ of compact operators is the important property. For more information, see [1]

Notations and Preliminaries

First we introduce some notation. let $K = SU(3)$ be the maximal compact subgroup of Lie group $SL(3;\mathbb{C})$. we denote the set of longitudinal psudodifferential operators of order at most P by $\psi^{p}_{\mathcal{F}_i}$. and $\mathcal{A}$ is the simultaneous multiplier category of C^*-algebra $\mathcal{K}_{\propto_i}$ $(i = 1,2)$ of operators on the L^2- section space of any homogenous vector bundles over $\mathcal{X}$ [1]. We are only interested in $P = 0$ and for $i \neq j$ we will answer the question that: $\psi^{0}_{\mathcal{F}_i}(E, E^*) \subset \mathcal{A}_j$

Let $n\colon \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \to \sum_i n_i\, t_i$ be a map , weights for $gl(3,\mathbb{C})$ match the $n = (n_1, n_2, n_3)$ via the pervious map. The triples $n_1 \geq n_2 \geq n_3$ correspond the dominant weights. We have a triangular array of integers with conditions:

$$\mu_{(k+1,j)} \geq \mu_{(k,j)} \quad \text{and} \quad \mu_{(k,j)} \geq \mu_{(k+1,j+1)} \qquad (1)$$

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

International Journal of Mathematics and Statistics Studies
Vol.9, No.3, pp.49-64, 2021
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

which is the Gelfand-Tsetline pattern (or GT-pattern):

$$\Lambda = \begin{pmatrix} \mu_{3,1} & & \mu_{3,2} & & \mu_{3,3} \\ & \mu_{2,1} & & \mu_{2,2} & \\ & & \mu_{1,1} & & \end{pmatrix}.$$

There are vectors ξ_Λ form an orthogonal basis for the irreducible representation $\pi_{\mathrm{n}}$with highest weight n= $(\mu_{31}, \mu_{32}, \mu_{33})$, such that to each GT-pattern there is associated a vector ξ_Λ in π_{n}. ξ_Λ is a weight vector, with weight $(s_1 - s_0, s_1 - s_2, s_3 - s_2)$. And the sum of the entries of the kth row is $s_{\mathrm{k}} = \sum_{j=1}^{k} \mu_{k,j}$; we obtain the GT-pattern $l_{k,j} = \mu_{k,j} - j + 1$; and $\Lambda \mp \delta_{k,j}$ from Λ by adding ∓ 1 to the (k, j)-entry.

$$\pi(X_1)\xi_\Lambda = -(l_{11} - l_{21})(l_{11} - l_{22})\, \xi_{\Lambda +} \delta_{11},$$

We switch the longest element $\omega_\rho \in W$ to the element

$$\omega_\rho = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \in K \tag{2}$$

Conjugation by ω_ρ interchanges the subgroups K_1 and K_2. We define vectors form as an alternative orthogonal basis for π_m with related pro3perties $\eta_\Lambda = \pi_m(\omega_\rho)\xi_\Lambda$.

Harmonic analysis for longitudinal psudodifferntial operators

Let $m = (m, 0, -m)$ be the π_m representation with highest weight such that $m \in \mathbb{N}$.

The Gelfand-Tsetlin vectors spans the 0-weight space of $V^{(m,0-m)}$

$\xi_{m,j} = \xi_\Lambda$, with $\Lambda = \begin{bmatrix} m & & 0 & & -m \\ & j & & -j & \\ & & 0 & & \end{bmatrix}$, for $j = 0, \ldots, m$. The vectors span the $(0, -1, 1)$-weight space

$\xi'_{m,j} = \xi_\Lambda$, for $\Lambda = \begin{bmatrix} m & & 0 & & -m \\ & (j-1) & & -j & \\ & & 0 & & \end{bmatrix}$, $j = 1, \ldots, m$. Via the The Gelfand-Tsetlin formulas that mentioned above

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

$$\pi_m(X_2^*)\xi_{m,j} = \frac{j}{2j+1}\xi'_{m,j} + \frac{j+1}{2j+1}\xi'_{m,j+1} \quad (3)$$

$$\pi_m = (X_2)\xi'_{m,j} = \frac{1}{2}((m+1)^2 - j^2)\xi_{m,j-1} + \frac{1}{2}((m+1)^2 - j^2)\xi_{m,j}$$
(4)

$$\pi_m = (X_2)\pi_m(X_2^*)\xi_{m,j} = \frac{j}{2(2j+1)}((m+1)^2 - j^2)\xi_{m,j-1} + \frac{1}{2}\big((m+1)^2 - (j^2+j+1)\big)\xi_{m,j} + \frac{j+1}{2(2j+1)}((m+1)^2 - (j+1)^2\xi_{m,j+1}$$
(5)

$$\pi_m(X_1^*)\pi_m(X_1)\xi_{m,j} = j(j+1)\xi_{m,j} \quad (6)$$

The vectors norms are

$$\|\xi_{m,j}\|^2 = \frac{1}{2j+1}m!^2\,(2m+1)! \quad (7)$$

$$\|\xi'_{m,j}\|^2 = \frac{1}{2j}((m+1)^2 - j^2)m!^2\,(2m+1)! \quad (8)$$

We define vectors

$\eta_{m,j} = \pi_m(\omega_\rho)\ \xi_{m,j} \quad (0 \le j \le m), \quad \eta'_{m,j} = \pi_m(\omega_\rho)\ \xi'_{m,j} \quad (1 \le j \le m).$

with weights

$\omega_\rho.0 = 0$ has norm $\|\eta_{m,j}\| = \|\xi_{m,j}\|$ and

$\omega_\rho.(0,-1,1) = (1,-1,0) = \alpha_1$ has norm $\|\eta'_{m,j}\| = \|\xi'_{m,j}\|$.

"Eq. (3)" and "Eq. (6)" yield

$$\pi_m(X_1^*)\eta'_{m,j} = \frac{1}{2}((m+1)^2 - j^2)\eta_{m,j-1} + \frac{1}{2}((m+1)^2 - j^2)\eta_j$$
(9)

$$\begin{aligned}\pi_m(X_1^*)\pi_m(X_1)\eta_{m,j} \\ &= \frac{1}{1(2j+1)}((m+1)^2 - j^2)_{\eta_{m,j-1}} \\ &+ \frac{1}{2}((m+1)^2 - (j^2+j+1))_{\eta_{m,j}} \\ &+ \frac{j+1}{2(2j+1)}((m+1)^2 - (j+1)^2)_{\eta_{m,j+1}} \quad (10)\end{aligned}$$

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

We define

$$a_{m,j,k} = \frac{(-1)^j \, \overline{\omega_m}}{m!^2(2m+1)!} \langle \xi_{m,j}, \eta_{m,k} \rangle. \tag{11}$$

If $0 \leq j, k \leq m$ does not hold we write $a_{m,j,k} = 0$.

Lemma 3.1 [1] for any $m \in \mathbb{N}$, $\eta_{m,0} = \omega_m \sum_{j=0}^{m} (-1)^j \frac{2j+1}{m+1} \xi_{m,j}$, $|\omega_m| = 1$, where $\omega_m \in \mathbb{C}$ is some phase factor.

Lemma 3.2 for $0 \leq j, k \leq m$, we have the recurrence relation in k with initial condition $a_{m,j,0} = \frac{1}{(m+1)}$.

$$k((m+1)^2 - k^2)_{a_{m,j,k-1}} + (2k+1)\big((m+1)^2 - (k^2+k+1) - 2j(j+1)\big)_{a_{m,j,k}} + (k+1)((m+1)^2 - (k+1)^2)_{a_{m,j,k+1}} = 0 \tag{12}$$

Proof we should apply “Eq. (6)” and “Eq. (10)”
to the equality

$$\langle \pi_m(X_1^*)\pi_m(X_1)\xi_{m,j}, \eta_{m,k} \rangle = \langle \xi_{m,j}, \pi_m(X_1^*)\pi_m(X_1)\eta_{m,k} \rangle$$

Give us

$$j(j+1)\langle \xi_{m,j}, \eta_{m,k} \rangle = \frac{k}{2(2k+1)}((m+1)^2 - k^2)\langle \xi_{m,j}, \eta_{m,k-1} \rangle + \frac{1}{2}\big((m+1)^2 - (k^2+k+1)\big)\langle \xi_{m,j}, \eta_{m,k} \rangle + \frac{k+1}{2(2k+1)}((m+1)^2 - (k+1)^2)\langle \xi_{m,j}, \eta_{m,k+1} \rangle,$$

Which reduces to (12). Lemma (3.1) gives

$$a_{m,j,0} = (-1)^j \frac{2j+1}{m+1} \frac{1}{m!^2(2m+1)!} \left\| \xi_{m,j} \right\|^2 = \frac{(-1)^j}{(m+1)}.$$

corollary 3.3 for $0 \leq m-2\in, m-\in \leq m, (m-2\in) < (m-\in)$, we have the recurrence relation in $m-\in$ with respect to the initial condition $a_{m,m-2\epsilon,0} = \frac{1}{(m+1)}$,

$$(m-\in)((m+1)^2 - (m-\in)^2)a_{m,m-2\epsilon,m-\epsilon-1} +$$

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

Vol.9, No.3, pp.49-64, 2021
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

$(2(m-\in)+1)\big((m+1)^2-((m-\in)^2+m-\in+1)-2(m-2\epsilon)(m-2\epsilon+1)\big)_{a_{m,m-2\epsilon,m-\epsilon}}+(m-\in+1)((m+1)^2-(m-\in+1)^2{}_{a_{m,m-2\epsilon,m-\epsilon+1}}=0$ (13)

Proof from "Eq. (6)" and "Eq. (10)" we obtain $\langle \pi_m(X_1^*)\pi_m(X_1)\xi_{m,m-2\in,}\eta_{m,m-\in}\rangle=$
$\langle \xi_{m,m-2\in,\pi_m}\ (X_1^*)\pi_m(X_1)\eta_{m,m-\in}\rangle$

$$(m-2\in)(m-2\in+1)\langle \xi_{m,m-2\in},\eta_{m,m-\in}\rangle$$
$$=\frac{m-\in}{2(2(m-\in)+1)}((m+1)^2-(m-\in)^2)\langle \xi_{m,m-2\in},\eta_{m,m-\in-1}\rangle+$$
$$\frac{1}{2}((m+1)^2-((m-\in)^2+m-\in+1))\langle \xi_{m,m-2\in},\eta_{m,m-\in}\rangle+$$
$$\frac{m-\in+1}{2(2(m-\in)+1)}((m+1)^2-(m-\in+1)^2)\langle \xi_{m,m-2\in},\eta_{m,m-\in+1}\rangle,$$

Which reduces to "Eq. (12)". lemma (3.1) gives

$$a_{m,m-2\in,0}=(-1)^{(m-2\in)}\frac{2(m-2\in)+1}{m+1}\frac{1}{m!^2(2m+1)!}\left\|\xi_{m,m-2\in}\right\|^2=\frac{(-1)^{(m-2\in)}}{(m+1)}.$$

For $j>0, and\ (phX_1)\xi_{m,j}=0$, we obtain the next equation from "Eq. (6)" $(phX_1)\xi_{m,j}=X_1.(X_1{}^*X_1)^{-\frac{1}{2}}\xi_{m,j}=\frac{1}{\sqrt{j(j+1)}}X_1\xi_{m,j}$ (14)

We suppose that $y_{m,j}$ and $y'_{m,j}$ are the corresponding orthonormal bases,

$$y_{m,j}=\eta_{m,j}/\left\|\eta_{m,j}\right\|=\frac{1}{m!(2m+1)!^{\frac{1}{2}}}\eta_{m,j} \quad (15)$$

and

$$y'_{m,j}=\eta'_{m,j}/\left\|\eta'_{m,j}\right\|=\frac{1}{m!(2m+1)!^{\frac{1}{2}}}\quad(\frac{2j}{(m+1)^2-j^2})^{1/2}\eta_{m,j} \quad (16)$$

Lemma 3.4

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

$\lim_{m\to\infty} \left\| \left\langle (phX_i) y_{m,0}, {y'}_{m,k} \right\rangle \right\| = \sqrt{2k}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$, for each fixed $k \epsilon \mathbb{N}$.

Proof we use "Eq. (14)" and "Eq. (15)", lemma (3.1), "Eq. (13)", "Eq. (10)", and "Eq. (11)", to compute

$$\left\langle (phX_1) y_{m,0}, {y'}_{m,k} \right\rangle$$

$$= \frac{1}{m!^2\,(2m+1)!}\sqrt{\frac{2k}{(m+1)^2-k^2}}\left\langle (phX_1)\eta_{m,0}, {\eta'}_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2k}{(m+1)^2-k^2}}\sum_{j=0}^{m}(-1)^j\frac{2j+1}{m+1}\left\langle (phX_1)\xi_{m,j}, {\eta'}_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2k}{(m+1)^2-k^2}}\sum_{j=0}^{m}(-1)^j\frac{2j+1}{m+1}\left\langle \frac{1}{\sqrt{j(j+1)}}X_1\xi_{m,j}, {\eta'}_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2k}{(m+1)^2-k^2}}\sum_{j=0}^{m}(-1)^j\frac{2j+1}{m+1}\frac{1}{\sqrt{j(j+1)}}\left\langle X_1\xi_{m,j}, {\eta'}_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\frac{1}{m+1}\sqrt{\frac{2k}{(m+1)^2-k^2}}\sum_{j=0}^{m}(-1)^j\frac{2j+1}{\sqrt{j(j+1)}}\left\langle \xi_{m,j}, {X_1}^{*}{\eta'}_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\frac{\sqrt{2k(m+1)^2-k^2}}{m+1}\sum_{j=0}^{m}(-1)^j\frac{2j+1}{2\sqrt{j(j+1)}}\left\langle \xi_{m,j}, \eta_{m,k-1} + \eta_{m,k} \right\rangle$$

$$= \omega_m\sqrt{2k\left(1-\frac{k^2}{(m+1)^2}\right.}\ \sum_{j=0}^{m}\frac{j+\frac{1}{2}}{\sqrt{j(j+1)}}\left(a_{m,j,k-1}+a_{m,j,k}\right) \tag{17}$$

$$\sum_{j=0}^{m}\frac{j+\frac{1}{2}}{\sqrt{j(j+1)}}\left(a_{m,j,k-1}+a_{m,j,k}\right) \tag{18}$$

$$= \sum_{j=0}^{m}\left(\frac{j+\frac{1}{2}}{\sqrt{j(j+1)}}-1\right)\left(a_{m,j,k-1}+a_{m,j,k}\right) \tag{19}$$

$= \sum_{j=0}^{m}(a_{m,j,k-1}b_{m,j,k-1}) + (a_{m,j,k} - b_{m,j,k})$ (20)

$= \sum_{j=0}^{m}(b_{m,j,k-1} + b_{m,j,k}).$ (21)

In "Eq. (18)"

$$\left(\frac{j+\frac{1}{2}}{\sqrt{j(j+1)}} - 1\right) = \sqrt{\frac{j^2+j+\frac{1}{4}}{j^2+j}} - 1 \leq \frac{1}{8j^2},$$

Fix k, and by remark[1,(A.6], $a_{m,j,k-1}$and $a_{m,j,k}$ are both $O(m^{-1})$, so (18) goes to 0 as $m \to \infty$. From [1,(A.5)], $(m+1)[C(k-1) + C(k)](m+1)^{-2}$ bounds the equation (19), as $m \to \infty$ also this equation approaches to 0.

The Riemann sum of the integral $(-1^j)\int_{t=0}^{1} P_k\,(2t^2-1)dt$ is

$\sum_{j=0}^{m} b_{m,j,k} = \frac{1}{m+1} P_k(2\left(\frac{j}{m+1}\right)^2 - 1)$ (22)

We substitute $(2t^2-1) = v$, "Eq. (21)"converges to

$$\left(2^{-\frac{3}{2}}\right)\int_{-1}^{1}(1-v)^{-\frac{1}{2}}\, P_k(-v)dv = \frac{(-1)^k}{2k+1}$$

As $m \to \infty$, "Eq. (18)"converges to $(-1)^{k-1}(\frac{1}{2k-1} - \frac{1}{2k+1})$. to complete our proof

We put this into "Eq. (17)":

$\left\langle (phX_1)y_{m,0}, y'_{m,k}\right\rangle = = \omega_m\sqrt{2k(1-\frac{k^2}{(m+1)^2}}\left((-1)^{k-1}(\frac{1}{2k-1} - \frac{1}{2k+1})\right).$

corollary 3.5 ,

$\lim_{m\to\infty}\left\|\left\langle (phX_i)y_{m,0}, y'_{m,(1+\in)}\right\rangle\right\| = \sqrt{2(1+\in)}(\frac{1}{2(1+\in)-1} - \frac{1}{2(1+\in)+1}),$ for each fixed $(1+\in)\epsilon\mathbb{N}$, $\in\geq 0$.

Proof we use "Eq. (15)"and "Eq. (16)", lemma (3.1), "Eq. (14)", "Eq. (11)", and "Eq. (12)", to compute

$\left\langle (phX_1)y_{m,0}, y'_{m,(1+\in)}\right\rangle$

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

$$= \frac{1}{m!^2\,(2m+1)!}\sqrt{\frac{2(1+\in)}{(m+1)^2-(1+\in)^2}}\,\langle (phX_1)\eta_{m,0}, \eta'_{m,(1+\in)} \rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2(1+\in)}{(m+1)^2-(1+\in)^2}}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{m+1}\langle (phX_1)\xi_{m,(1+\in)}, \eta'_{m,(1+\in)} \rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2(1+\in)}{(m+1)^2-(1+\in)^2}}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{m+1}\left\langle \frac{1}{\sqrt{(1+\in)(2+\in)}}X_1\xi_{m,(1+\in)}, \eta\right.$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2(1+\in)}{(m+1)^2-(1+\in)^2}}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{m+1}\frac{1}{\sqrt{(1+\in)(2+\in)}}\langle X_1\xi_{m,(1+\in)}, \eta$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\frac{1}{m+1}\sqrt{\frac{2(1+\in)}{(m+1)^2-(1+\in)^2}}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{\sqrt{(1+\in)(2+\in)}}\langle \xi_{m,(1+\in)}, X_1{}^{*}\eta'_{m}$$

$=$
$\frac{\omega_m}{m!^2(2m+1)!}\frac{\sqrt{2(1+\in)(m+1)^2-(1+\in)^2}}{m+1}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{2\sqrt{(1+\in)(2+\in)}}\langle \xi_{m,(1+\in)}, \eta_{m,\in} +$
$\eta_{m,(1+\in)}\rangle$ $=$
$\omega_m\sqrt{2(1+\in)(1-\frac{(1+\in)^2}{(m+1)^2}}\sum_{(1+\in)=0}^{m}\frac{(\in+\frac{3}{2})}{\sqrt{(1+\in)(2+\in)}}(a_{m,(1+\in),\in} +$
$a_{m,(1+\in),(1+\in)})$ (23)

$\sum_{(1+\in)=0}^{m}\frac{(\in+\frac{3}{2})}{\sqrt{(1+\in)(2+\in)}}\left(a_{m,(1+\in),\in} + a_{m,(1+\in),(1+\in)}\right)$ (24)

$= \sum_{(1+\in)=0}^{m}(\frac{(\in+\frac{3}{2})}{\sqrt{(1+\in)(2+\in)}} - 1)(a_{m,(1+\in),\in} + a_{m,(1+\in),(1+\in)})$
(25)
$= \sum_{(1+\in)=0}^{m}\left(a_{m,(1+\in),\in}\, b_{m,(1+\in),\in}\right) + (a_{m,(1+\in),(1+\in)} - b_{m,(1+\in),(1+\in)})$
(26)
$= \sum_{(1+\in)=0}^{m}\left(b_{m,(1+\in),\in} + b_{m,(1+\in),(1+\in)}\right).$ (27)
In “Eq. (24)”

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

$$\left(\frac{(\in+\frac{3}{2})}{\sqrt{(1+\in)((2+\in)}}-1\right)=\sqrt{\frac{(1+\in)^2+\in+\frac{5}{4}}{(1+\in)^2+(1+\in)}}-1\leq\frac{1}{8(1+\in)^2},$$

Fix $(1+\in)$, and by remark[1,(A.6], $a_{m,(1+\in),\in}$and $a_{m,(1+\in),(1+\in)}$ are both $O(m^{-1})$, so "Eq. (25)"goes to 0 as $m\to\infty$. From [1,(A.5)], $(m+1)[C(\in)+C((1+\in))](m+1)^{-2}$ bounds "Eq. (26)", as $m\to\infty$ also this equation approaches to 0.

The Riemann sum of the integral $(-1^{(1+\in)})\int_{t=0}^{1}P_{(1+\in)}(2t^2-1)dt$ is

$$\sum_{(1+\in)=0}^{m}b_{m,(1+\in),(1+\in)}=\frac{1}{m+1}P_{(1+\in)}(2\left(\frac{(1+\in)}{\boldsymbol{m+1}}\right)^{\boldsymbol{2}}-\boldsymbol{1})\qquad(28)$$

We substitute $(\boldsymbol{2t^2-1})=\boldsymbol{v}$, "Eq. (27)"converges to

$$\left(\boldsymbol{2}^{-\frac{3}{2}}\right)\int_{-1}^{1}(\boldsymbol{1}-\boldsymbol{v})^{-\frac{1}{2}}\boldsymbol{P}_{(1+\in)}(-\boldsymbol{v})\boldsymbol{dv}=\frac{(-\boldsymbol{1})^{(1+\in)}}{\boldsymbol{2}(1+\in)+\boldsymbol{1}}=\frac{(-\boldsymbol{1})^{(1+\in)}}{\boldsymbol{3}+2\in}$$

As $\boldsymbol{m}\to\infty$, "Eq. (23)"converges to $(-\boldsymbol{1})^{\in}(\frac{\boldsymbol{1}}{1+2\in}-\frac{\boldsymbol{1}}{\boldsymbol{3}+2\in})$. to complete our proof

We put this into "Eq. (22)":

$$\left\langle(\boldsymbol{phX_1})\boldsymbol{y}_{\boldsymbol{m},\boldsymbol{0}},\boldsymbol{y'}_{\boldsymbol{m},(1+\in)}\right\rangle==\boldsymbol{\omega_m}\sqrt{\boldsymbol{2}(1+\in)(\boldsymbol{1}-\frac{(1+\in)^{\boldsymbol{2}}}{(\boldsymbol{m+1})^{\boldsymbol{2}}}}\left((-\boldsymbol{1})^{\in}(\frac{\boldsymbol{1}}{1+2\in}-\frac{\boldsymbol{1}}{\boldsymbol{3}+2\in})\right).$$

Lemma 3.6 on any unitary $\boldsymbol{K}$- representation $\boldsymbol{\mathcal{H}}$ the operators $(\boldsymbol{phX_1}^*)\boldsymbol{p}_{\boldsymbol{\sigma_0}}$, and therefore $\boldsymbol{p}_{\boldsymbol{\sigma_0}}(\boldsymbol{phX_1})$, are in $\boldsymbol{\mathcal{K}}_{\boldsymbol{\beta_2}}(\boldsymbol{\mathcal{H}})$.

Proof. Let $\boldsymbol{U}$ be a unitary representation of $\boldsymbol{K}$ on $\boldsymbol{\mathcal{H}}$. The antilinear map $\boldsymbol{J}:\boldsymbol{\mathcal{H}}\to\boldsymbol{\mathcal{H}}^{\dagger};\boldsymbol{\xi}\to\langle\boldsymbol{\xi},.\rangle$ Intertwines the representations $\boldsymbol{U}$ and $\boldsymbol{U}^{\dagger}$. for any $\boldsymbol{X}$ in the complexification $\boldsymbol{\iota}_{\mathbb{C}}$, $\boldsymbol{J}^{-1}\,\boldsymbol{U}^{\dagger}(\boldsymbol{X})\boldsymbol{J}=-\boldsymbol{U}(\boldsymbol{X})^*$. Since $\boldsymbol{J}$ is anti-unitary, $\boldsymbol{J}^{-1}\,\boldsymbol{ph}\left(\boldsymbol{U}^{\dagger}(\boldsymbol{X})\right)\boldsymbol{J}=-\boldsymbol{ph}(\boldsymbol{U}(\boldsymbol{X}^*))$.

If $\boldsymbol{\xi}\in\boldsymbol{\mathcal{H}}$ has $\boldsymbol{K_2}$ −type $\boldsymbol{\sigma}$, then $\boldsymbol{J\xi}$ has $\boldsymbol{K_2}$ −type $\boldsymbol{\sigma}^{\dagger}$, so $\boldsymbol{p\sigma}=\boldsymbol{J}^{-1}\boldsymbol{p}_{\boldsymbol{\sigma}}^{\dagger}\boldsymbol{J}$. By conjugating by $\boldsymbol{J}$, the estimate $\left\|\boldsymbol{p}_{\boldsymbol{F}}^{\perp}(\boldsymbol{PhX_1})\boldsymbol{p}_{\boldsymbol{\sigma_0}}\right\|<\boldsymbol{\epsilon}$ implies $\left\|\boldsymbol{p}_{\boldsymbol{F}^{\dagger}}^{\perp}(\boldsymbol{PhX_1}^*)\boldsymbol{p}_{\boldsymbol{\sigma_0}}\right\|<\boldsymbol{\epsilon}$, where $\boldsymbol{F}^{\dagger}=\{\boldsymbol{\sigma}^{\dagger}|\boldsymbol{\sigma}\in\boldsymbol{F}\}$.

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

Vol.9, No.3, pp.49-64, 2021
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

corollary 3.7 on any unitary K- representation $\mathcal{H}$ the operators $(phX_1)p_{\sigma_0}$, and therefore $p_{\sigma_0}(phX_1{}^*)$, are in $\mathcal{K}_{\beta_2}(\mathcal{H})$.

Proof. Let U be a unitary representation of K on $\mathcal{H}$. The antilinear map $J: \mathcal{H} \to \mathcal{H}^\dagger; \xi \to \langle \xi, . \rangle$ Intertwines the representations U. for any X^* in the complexification $\iota_{\mathbb{C}}$, $J^{-1}\ U^\dagger(X)^* J = -U(X)$. Since J is anti-unitary, $J^{-1}\, ph\left(U^\dagger(X^*)\right) J = -ph\big(U(X)\big)$.

If $\xi \in \mathcal{H}$ has K_2 −type σ, then $J\xi$ has K_2 −type $\sigma^\dagger$, so $p\sigma = J^{-1} p_\sigma^\dagger J$. By conjugating by J, the estimate $\left\|p_F^\perp (PhX_1) p_{\sigma_0}\right\| < \epsilon$ implies $\left\|p_{F^\dagger}^\perp (PhX_1) p_{\sigma_0}\right\| < \epsilon$, where $F^\dagger = \{\sigma^\dagger | \sigma \in F\}$.

Lemma 3.8 let v be a weight of K. For any $f \in C(K)$, $[PhX_1, M_f]pv$ and $[PhY_1, M_f]pv$ are in $\mathcal{K}_{\propto_1}(L^2(K))$.

Proof. Assume that f is a weight vector for the right regular representation, i.e, $f \in C(\mathcal{X}; E_{-\mu})$ for some μ. Through Lemma[3.19,1] we have

$$[PhX_1, M_f]: pvL^2(K) \to p_{v+}\mu + \propto_1 L^2(K)$$

Is in $\mathcal{K}_{\propto_1}$, which implies the result. The subspace spanned by these weight vectors contains all matrix units, so is uniformly dense in $C(K)$. A density argument completes the proof. Similarly, $[PhY_1, M_f]pv \in \mathcal{K}_{\propto_1}$.

corollary 3.9 let v be a weight of K. For any $f_j \in C(K)$, $\left[PhX_1, M_{\sum f_j}\right] pv$ and $\left[PhY_1, M_{\sum f_j}\right] pv$ are in $\mathcal{K}_{\propto_1}(L^2(K))$.

Proof. Assume that f_j is a weight vector for the right regular representation, i.e, $f_j \in C(\mathcal{X}; E_{-\mu})$ for some μ. Through Lemma[3.19,1] we have

$$\left[PhX_1, M_{\sum f_j}\right]: pvL^2(K) \to p_{v+}\mu + \propto_1 L^2(K)$$

Is in $\mathcal{K}_{\propto_1}$, which implies the result. The subspace spanned by these weight vectors contains all matrix units, so is uniformly dense in $C(K)$. A density argument completes the proof. Similarly, $\left[PhY_1, M_{\sum f_j}\right] pv \in \mathcal{K}_{\propto_1}$.

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

Theorem 3.10 on any unitary $\boldsymbol{K}$ representation on $\boldsymbol{\mathcal{H}}$, $\boldsymbol{PhX_i}$ and $\boldsymbol{PhY_i}$ are in $\boldsymbol{\mathcal{A}(\mathcal{H})}$ for $\boldsymbol{i=1,2}$.

proof. We first have $\boldsymbol{\mathcal{H}=L^2(K)}$ with right regular representation, and consider $\boldsymbol{PhX_1}$. With respect to lemma[3.7,1], the finite multiplicity of $\boldsymbol{K}$-types in $\boldsymbol{L^2(K)}$ implies that $\boldsymbol{\mathcal{A}_{\Sigma}\left(L^2(K)\right)=\mathcal{L}\left(L^2(K)\right)}$, so $\boldsymbol{PhX_1\in\mathcal{A}_{\Sigma}}$ trivially. Since $\boldsymbol{PhX_1}$ maps the $\boldsymbol{\mu}$-weight space into the $(\boldsymbol{\mu}+\boldsymbol{\propto_1})$-weight space for each weight $\boldsymbol{\mu}$, it is $\boldsymbol{M}$-harmonically proper, so in $\boldsymbol{\mathcal{A}_{\emptyset}}$. Since $\boldsymbol{X_1}$in $(\boldsymbol{\iota_1})_{\mathbb{C}}$, $\boldsymbol{PhX_1}$preserves K_1-types, so $PhX_1\in\mathcal{A}_{\propto_1}$. And eventually we are going to show that $PhX_1\in\mathcal{A}_{\propto_2}$.

Let $\sigma\in\widehat{K}_2$ and let $\psi_{1,\dots,}\psi_n\in C(K)$ be as in lemma [A.10,1]. Then

$$
\begin{aligned}
(PhX_1)p\sigma &= \sum_{j=1}^{n}(PhX_1)M_{\psi j}\,P\sigma_0\,M_{\overline{\psi j}}\\
&= \sum_{j=1}^{n}M_{\psi j}(PhX_1)\,P\sigma_0\,M_{\overline{\psi j}}\\
&+ \sum_{j=1}^{n}[(PhX_1),M_{\psi j}]\,P\sigma_0\,M_{\overline{\psi j}}\,.
\end{aligned}
$$

since $P\sigma_0$ projects into the 0-weight space, lemmas [A.8, 1], (1.6) and [3.11,1], show that $(PhX_1)\,p\sigma\in\mathcal{K}_{\propto_2}$. By analogous computation lemma (1.4) illustrate that $(PhY_1)p\sigma=(PhX_1{}^*)p\sigma\in\mathcal{K}_{\propto_2}$, so $p\sigma(PhX_1)\in\mathcal{K}_{\propto_2}$. By proposition 3.6, $PhX_1\in\mathcal{A}_{\alpha_2}$

then $PhX_1\in\mathcal{A}$. By taking adjoints $PhY_1\in\mathcal{A}$.

conjugationby the longest weyl group element interchanges $\mathcal{A}_{\alpha_1}$ and $\mathcal{A}_{\alpha_2}$ and fixes $\mathcal{A}_{\emptyset}$ and $\mathcal{A}_{\Sigma}$, so fixes $\mathcal{A}$. It also sends X_1 and Y_1 to Y_2 and X_2, respectively. We obtain PhY_2 , $PhX_2\epsilon\mathcal{A}$.

The theorem remains true if $\mathcal{H}$ is a direct sum of arbitrarily many copies of the regular representation. Since every unitary K-representation can be equivariantly embedded into such a direct sum, we are done.

corollary 3.11 on any unitary $\boldsymbol{K_{r-1}}$ representation on $\boldsymbol{\mathcal{H}}$, $\boldsymbol{PhX_i}$ and $\boldsymbol{PhY_i}$ are in $\boldsymbol{\mathcal{A}(\mathcal{H})}$ for $\boldsymbol{i=1,2}$.

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

Vol.9, No.3, pp.49-64, 2021
Print ISSN: 2053-2229 (Print),
Online ISSN: 2053-2210 (Online)

proof. We first have $\boldsymbol{\mathcal{H}} = \boldsymbol{L^2(K_{r-1})}$ with right regular representation, and consider $\boldsymbol{PhX_1}$. With respect to lemma[3.7,1], the finite multiplicity of $\boldsymbol{K_{r-1}}$ -types in $\boldsymbol{L^2(K_{r-1})}$ implies that $\boldsymbol{\mathcal{A}_{\Sigma}}\left(\boldsymbol{L^2(K_{r-1})}\right) = \boldsymbol{\mathcal{L}}\left(\boldsymbol{L^2(K_{r-1})}\right)$, so $\boldsymbol{PhX_1} \in \boldsymbol{\mathcal{A}_{\Sigma}}$ trivially. Since $\boldsymbol{PhX_1}$ maps the $\boldsymbol{\mu}$-weight space into the $(\boldsymbol{\mu} + \boldsymbol{\propto_1})$-weight space for each weight $\boldsymbol{\mu}$, it is $\boldsymbol{M}$-harmonically proper, so in $\boldsymbol{\mathcal{A}_{\emptyset}}$. Since $\boldsymbol{X_1}$in $(\boldsymbol{\iota_1})_{\mathbb{C}}$, $\boldsymbol{PhX}_1$preserves $\boldsymbol{K_r}$-types, so $PhX_1 \in \mathcal{A}_{\propto_1}$. And eventually we are going to show that $PhX_1 \in \mathcal{A}_{\propto_2}$.

Let $\sigma \in \boldsymbol{\widehat{K}_{r+1}}$ and let $\psi_{1,\dots,}\psi_n \in C(\boldsymbol{K_{r-1}})$ be as in lemma [A.10,1]. Then

$$
\begin{aligned}
(PhX_1)p\sigma &= \sum_{j=1}^{n} (PhX_1) M_{\psi j}\, P\sigma_0\, M_{\overline{\psi J}} \\
&= \sum_{j=1}^{n} M_{\psi j} (PhX_1)\, P\sigma_0\, M_{\overline{\psi J}} \\
&+ \sum_{j=1}^{n} [(PhX_1), M_{\psi j}]\, P\sigma_0\, M_{\overline{\psi J}}\,.
\end{aligned}
$$

since $P\sigma_0$ projects into the 0-weight space, lemmas [A.8, 1], (1.6) and [3.11,1], show that $(PhX_1)\, p\sigma \in \mathcal{K}_{\propto_2}$. By analogous computation lemma (1.4) illustrate that $(PhY_1)p\sigma = (PhX_1{}^*)p\sigma \in \mathcal{K}_{\propto_2}$, so $p\sigma(PhX_1) \in \mathcal{K}_{\propto_2}$. By proposition 3.6, $PhX_1 \in \mathcal{A}_{\alpha_2}$

then $PhX_1 \in \mathcal{A}$. By taking adjoints $PhY_1 \in \mathcal{A}$.

conjugationby the longest weyl group element interchanges $\mathcal{A}_{\alpha_1}$ and $\mathcal{A}_{\alpha_2}$ and fixes $\mathcal{A}_{\emptyset}$ and $\mathcal{A}_{\Sigma}$, so fixes $\mathcal{A}$**.** It also sends X_1 and Y_1 to Y_2 and X_2**,** respectively. We obtain PhY_2 , $PhX_2 \epsilon \mathcal{A}$.

The theorem remains true if $\mathcal{H}$ is a direct sum of arbitrarily many copies of the regular representation. Since every unitary K_{r-1} -representation can be equivariantly embedded into such a direct sum, we are done**.**

Theorem 3.12 **let** E, E^* be K −homogenuose vector bundles over flag variety $\mathbb{X}$ then $\psi^0_{\mathcal{F}_i}(E, E^*) \subseteq \mathcal{A}$.

@ECRTD-UK https://www.eajournals.org/
https://doi.org/10.37745/ijmss.13

Proof. It is clear that $\overline{\psi_{\mathcal{F}_i}^{-1}}(E_0) \subseteq \mathcal{A}$. $let\ E_0 = E = E^* = E_\rho = E_\sigma$, the trivial line bundle over flag variety $\mathbb{X}$.

$$0 \to \overline{\psi_{\mathcal{F}_i}^{-1}}(E_0) \to \overline{\psi_{\mathcal{F}_i}^{0}}(E_0) \to C(S^*\mathcal{F}_i) \to 0$$

According to the Stone-Weierstrass theorem and since the points in different fibres of $S^*\mathcal{F}_i$ are separated by the function algebrac we prove that $C(S^*\mathcal{F}_i) = C$ by showing that it separates the points of $S^*\mathcal{F}_i$.

The multiplication operator $M_f \in \psi_{\mathcal{F}_i}^0(E_0) \cap \mathcal{A}$ for any $f \epsilon\ C(\mathbb{X})$. Let $\emptyset \in C(\mathbb{X}, E_\sigma)$ be any nonezero smooth section of E_σ at the identity coset.

$M_\emptyset phx_i \in \psi_{\mathcal{F}_i}^0(E_0) \cap \mathcal{A}$ and its principal symbol seprates points of the fiber at the identity coset.

Lemma 2.1 shows that $\emptyset_1, \dots, \emptyset_n \in C(\mathbb{X}, E_\mu)$, $\emptyset_1{}', \dots, \emptyset_m{}' \in C(\mathbb{X}, E_v)$. For each j, μ and $k \in K$, $\overline{M_{\emptyset_j'}}\ A\ M_{\emptyset k} \in \overline{\psi_{\mathcal{F}_i}^0}(E_0) \subseteq \mathcal{A}$, for $A \in \overline{\psi_{\mathcal{F}_i}^0}(E, E^*)$.

References

[1] Yuncken R., *The Bernstein–Gelfand–Gelfand complex and Kasparov theory for SL (3, C)*, Advances in Mathematics, **226**(2), pp. 1474-512, 2011 Jan 30.

[2] M.F. Atiyah. & I.M. Singer., *The index of elliptic operators. IV,* Ann. of Math, **93**(2), pp. 119–138, 1971.

[3] P. Baum, A. & Connes, N. Higson., *Classifying space for proper actions and K-theory of group C∗-algebras, in: C∗- Algebras*, San Antonio, TX, in: Contemp. Math, vol. 167, pp. 1943–1993, 1993. & Amer. Math. Soc., Providence, RI, pp. 240–291, 1994.

[4] I. Bernstein, I. & Gel'fand, S. Gel'fand., *Differential operators on the base affine space and a study of g-modules, in: Lie Groups and Their Representations*, Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971. & Halsted, New York, pp. 21–64, 1975.

[5] Bruce Blackadar., *K-Theory for Operator Algebras, second ed,* Math. Sci. Res. Inst. Publ, Cambridge University Press, Cambridge, vol. 5, 1998.

[6] Andreas Cap, Jan Slovák & Vladimír Sou ˇ cek, *Bernstein–Gelfand–Gelfand sequences*, Ann. of Math, (2) **154** (1), 97–113, 2001.
[7] Michel Duflo, *Représentations irréductibles des groupes semi-simples complexes*, *in: Analyse harmonique sur les groupes de Lie*, Sém., Nancy–Strasbourg, 1973–1975, in: Lecture Notes in Math., vol. 497, Springer, Berlin, pp. 26–88, 1975.
[8] I.S. Gradshteyn & I.M. Ryzhik, *Table of Integrals, Series, and Products*, fourth edition prepared by Ju.V. Geronimus and M.Ju. Ceˇıtlin, translated from the Russian by Scripta Technica, Inc., translation edited by Alan Jeffrey, Academic Press, New York, 1965.
[9] Nigel Higson, *The Baum–Connes conjecture*, in: Proceedings of the International Congress of Mathematicians, vol. II, Berlin, pp. 637–646, 1998.
[10]] Pierre Julg, *La conjecture de Baum–Connes à coefficients pour le groupe Sp(n, 1)*, C. R. Math. Acad. Sci. Paris **334** (7), 533–538, 2002.
[11] P. Julg & G. Kasparov, *Operator K-theory for the group SU(n, 1)*, J. Reine Angew. Math. **463**, 99–152, 1995.
[12] G. Kasparov& Lorentz groups: *K-theory of unitary representations and crossed products*, Dokl. Akad. Nauk SSSR, **275** (3), 541–545, 1984.
[13] G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math, **91** (1), 147–201, 1988.
[14] V. Lafforgue, *K-théorie bivariante pour les algèbres de Banach et conjecture de Baum–Connes*, Invent. Math. **149,** 1–95, 2002.
[15] V. Lafforgue, *Un renforcement de la propriété (T)*, Duke Math. J. **143** (3), 559–602, 2008.
[16] E.C. Lance, *Hilbert C∗-Modules: A Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge University Press, Cambridge, 1995.
[17] A.I. Molev, *Gel'fand–Tsetlin bases for classical Lie algebras*, Handbook of Algebra, in: M. Hazewinkel (Ed.), Elsevier, pp. 109–170, 2006.

[18] Calvin C. Moore & Claude L. Schochet, *Global Analysis on Foliated Spaces, second ed.*, Math. Sci. Res. Inst. Publ., vol. 9, Cambridge University Press, New York, 2006.

[19] M. Puschnigg, *Finitely summable Fredholm modules over higher rank groups and lattices*, preprint, http://arxiv.org/ abs/0806.2759, 2008.

[20] Michael E. Taylor, *Pseudodifferential Operators*, Princeton Math. Ser., vol. 34, Princeton University Press, Princeton, NJ, 1981.

[21] R. Yuncken, *Analytic structures for the index theory of SL(3,C),* PhD thesis, Penn State University, 2006.

[22] Robert Yuncken, *Products of longitudinal pseudodifferential operators on flag varieties*, J. Funct. Anal, **258** (4), 1140–1166, 2010.