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DIVISION BY ZERO

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ABSTRACT: *There is an exception to the rule that division by zero is undefined or prohibited, which results in a well-defined number, and follows the rules of set theory and algebra. This paper disproves the "proof by contradiction" by using a counterexample, an examination of its logic as a compound statement of two forms of statements that are not accepted as logically equivalent, an explanation of it as a sophism, and by examining some of its underlying assumptions.*

KEYWORDS: Counting Function, Division, Empty Set, Set Theory, Zero.

INTRODUCTION

Knowing that I write math papers, one day a friend asked me for a better explanation of why division by zero is undefined, or prohibited, as I learned it. He said people don't like being told simply that "division by zero is undefined," but would appreciate an explanation.

To answer his question, it may be helpful to recall that division is one of four major operations of arithmetic. These operations are addition, subtraction, multiplication, and division. Division is usually introduced last, since it is an algorithm that involves addition, subtraction, and multiplication.

Of the operations, addition is usually introduced first since it may be used to explain subtraction as a type of reverse operation, and to explain multiplication as an efficient way to add the same number repeatedly.

In other words, since the operations of arithmetic begin with addition, a clear understanding of addition should assist in understanding the rule of algebra that prohibits division by zero, or leaves it undefined.

Counting Function

As a step toward introducing the operation of addition, it may be helpful to introduce the counting function, which counts the number of elements in the set. Counting the number of elements in a set may be likened to unpacking a box of books, and placing them on a bookshelf where they can be sorted, organized, and counted.

As the counting function counts the elements in a set, it yields a number that represents the number of objects or elements in the set. Compared to the usual functions of algebra or calculus, the counting function is a different type of function since it does not use the operations of arithmetic, exponents, or sine or cosine of an angle.

However, like other functions, the counting function constructs a map between two sets that moves in a single direction. It takes an element from a set, usually called the domain, and maps or pairs it with an element from another set, usually called the range, where the element from

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the range is usually determined based on some calculation involving the element from the domain.

The map moves in a single direction. In other words, given a function's result of an element in the range, it is often difficult to trace back the element of the domain that it is associated with, or that was used to calculate it due to either a difficult calculation, or to how more than one element in the domain may give the same result.

For example, after the counting function counts the number of elements in a set and yields its result of a number, it may be difficult to identify or trace back the particular set that the number represents a count of, unless something is known about the domain, or the sets that it is counting.

While it may be thought that functions are easy to reverse in direction, one of the outstanding problems of mathematics is the construction of reverse or inverse functions such as a function that finds the square root of a quadratic equation or cube root of a cubic equation, or the integral of a derivate.

In other words, where y is often expressed as a function of x, or $y = f(x)$, a reverse or inverse function seeks to find the value of x for a given value of y, or relate x in terms of a function of y, which is often called an inverse function.

In contrast to a function, the operations of arithmetic, which perform work between two elements of a set and whose result is another element of a set, are often easy to reverse by simply reversing the flow of their elements, and the position of their resulting element in the equation of operation.

For example, where the equation of addition may be written algebraically as $a + b = c$, where c is an element that represents the sum of two elements, called a and b, its reverse operation, subtraction, takes the sum of c, and subtracts from it one of the other two elements, yielding either $c – a = b$ or $c – b = a$.

Notably, the result of a reverse operation such as subtraction depends heavily on which element, a or b, is chosen to be subtracted from c, since the operation involves a decrease in the count of elements, rather than an increase in the count of elements such as results from the joining of two sets together in a union.

In other words, operations that involve a decrease in the count of elements such as subtraction and division, or the taking of a square root of a number, are non-commutative, meaning the result of the operations depends on the order of its elements. Reverse operations are typically non-commutative.

In contrast, the operations of addition and multiplication are commutative, meaning that their result is independent of the order of their two elements.

Moreover, when a reverse operation such as subtraction operates freely over a set instead of simply being used to reverse a sum, it often expands the set over which it operates, just as subtraction operating freely over the set of natural numbers expands that set to include the set of integers. (Hughes, "The Identity Crisis (Element)")

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Counting Theorem of Addition

Using N to describe the counting function, addition may be expressed by the application of the counting function to the union of two sets, A and B, where the number of elements in their union is equal to the number of elements in each set, less the number of elements in their intersection, to avoid double counting the same element, or

N (A U B) = N (A) + N (B) – N (A ∩ B)

If set A and B are disjoint, meaning that their intersection results in the null set, without any elements, the number of elements in their union equals the number of elements in the first set joined together with the number of elements in the second set. In this case, the counting function is applied consecutively from one set to the next, where the total number of elements is independent of the order of A and B, making addition commutative.

If one of the sets A or B is a subset of the other set, the number of elements in their intersection equals the number of elements in the subset, and is subtracted from the total. For example, if A is a subset of B, the equation becomes N (A) + N (B) – N (A), which reduces to N (B). (Hughes, "The Anti-Existence Theorem")

This application of the counting function to the union of sets A and B assumes that the elements in set A and set B share a common trait or characteristic, letting the elements in their union be identified as elements of a larger set, which share a common trait or characteristic.

Set Theory

In other words, a set is a collection or group of objects or elements that share a common trait or characteristic. The common trait or characteristic forms a boundary, which allows the elements in the set to be identified as elements, so the set may be distinguished from other sets that lie outside the set, which possess elements that are dissimilar, or have different traits or characteristics.

In other words, the boundary of a set determines the elements that lie inside the set, and those elements that lie outside the set, as depicted by a geometrical enclosure or figure in a Venn Diagram.

In turn, the boundary of a set is defined a common trait or characteristic of its elements. While this boundary may be a list of the elements in the set, it usually involves a common trait or characteristic, which recognizes the type of elements that comprise the set. It is often related to their location in a given area, time period, or some other feature that identifies an element.

In other words, set theory establishes two step means of classification. It first defines a set universe that encompasses all the elements of its sets. Then it defines a system or way of defining the boundary of a set, using a common trait or characteristic, which allows its elements to be sorted or organized.

With this in mind, the union of two sets, or the operation of addition, is usually assumed to be performed over the same type of set, or a single dimension, so that sets often consist of similar elements. For example, shopping at a grocery store creates a set, the basket or cart that holds all the items a customer plans to purchase.

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This set is clearly distinguished from the baskets or sets of other customers. It is not mixed up with the items other customers are purchasing, so the customer pays an amount equal to the sum of the prices of the items he is purchasing, and represents an arithmetic valuation of the set of items he has decided to purchase.

To prepare for check out, the customer adds apples to apples, and oranges to oranges. In other words, the price of an item is determined based on the type of set the item is drawn from. Apples and oranges are added separately, just as apples and oranges are kept in separate bins at a grocery store, marked at different prices.

In other words, since apples and oranges are different types of fruit, sets of apples and oranges represent different types of elements, so the union of a set of apples with a set of oranges, or the addition of apples with oranges, is generally assumed not to occur, unless explicitly specified.

In other words, some skill is needed to define the set universe over which the operations of set theory, or union and intersection, occur in. Apples and oranges are usually kept in separate sets or universes.

Examples of set universes include classification schemes such as the different kinds of items that are sold at a grocery store, the colors of the rainbow, or the different kinds of plants and animals. Other universes may involve a count such as an inventory of items sold at a grocery store in a given week, or a population census.

If the customer has gone shopping with a child, who was asked to gather some items, the union of two sets may be illustrated by how the family member in charge often examines the items collected by the child, and discards extraneous items such as candy bars and cookies before adding the child's set to the shopping cart.

Returning to the idea of addition as the union of two sets, since the elements in a set are usually viewed as whole or discrete, the counting function is usually introduced by using the set of natural numbers, just as addition is usually introduced by using the set of natural numbers and a number line.

In other words, where counting is often introduced using a number line, which is typically used to represent a dimension as a straight line, the idea of a single dimension may also be introduced by counting the elements in a set.

In other words, addition takes place over a single dimension since it counts the elements in the same or similar sets within a consistent set universe. Counting elements in a set is like unpacking a box of books, and placing them on a bookshelf, where they can be sorted and counted.

Just as unpacking a box of books requires handling each book, so it can be sorted and counted, and either set aside or placed on a bookshelf, counting elements in a set requires identifying each element as an element of the set, letting it be counted, and sorted or arranged as on a straight line.

As a result of counting the elements in a set, the count of elements may be transposed onto a number line. The count of the first element may be represented by the number one. The count

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of the next element may be represented by the number two, and so on, with the count of the next additional element representing the consecutive ordering of the set of natural numbers.

In other words, where a single dimension is often depicted geometrically by a number line, or a straight line, it may also be depicted by counting the elements in a set, since the count is easily transposed onto a number line.

With this thought in mind (of how a single dimension may be represented by counting the elements in a set), it may be remarked how arithmetic operations such as addition require a minimum of three elements to shows the operation has indeed performed work beyond the use of the identity element. (Hughes, "Beginning Arithmetic Proposition: The Arithmetic Operator Requires Three Elements")

In other words, counting the elements used in an operation is a way of establishing that the operation is non-trivial, just as a line with three points ensures that the interior of the line has substance, or that it consists of more than two endpoints.

Subtraction

Having introduced the idea of addition by counting the number of elements in the union of two sets, the operation of subtraction may be introduced by partitioning a set to create two subsets, and applying the counting function to the set that is partitioned and to each of the two subsets.

The set that is partitioned gives the number that is having a subtraction being performed on it. The number being subtracted represents the count of one of the two subsets while the count of the remaining subset gives the result. This idea of subtraction satisfies the definition of subtraction as taking value away from a number, whose result is another number that is lesser in value.

However, this definition of subtraction does not necessarily reverse the operation of addition in its entirety, defined as the application of the counting function to the union of two sets, since the two original sets are not necessarily disjoint.

In other words, reversing addition as the union of two sets may lose information about their intersection. Moreover, a set may be partitioned in different ways to give the same numerical value or number of elements in two subsets, which may not necessarily replicate the two subsets of the original partition.

In other words, while a reverse operation such as subtraction may reverse a numeric result, it may not necessarily reverse the underlying operation such as a union of two sets, so that information may become lost.

This idea that a reverse operation can lose information is very similar to how the construction of inverse functions is one of the outstanding problems of mathematics, as such functions may require complex calculations, or give multiple answers such as the two roots of a quadratic equation.

Information in arithmetic operations, including division, may also become lost due to rounding. As a result, division is often taught to where the numerator and denominator are first factored, so that like terms or factors, may be canceled before performing the actual division.

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In other words, skill in factoring can save on computation. Even with computers, factoring can avoid errors by minimizing the input of large numbers, while minimizing the amount of computations, while it simplifies the use of hand calculations for the purpose of checking, or making an estimate.

Another example of how information may become lost, or change in its complexion, is the use of a double negative statement in logic. And the use of the double complement in set theory or multiple negative signs opens the door for human error. (Hughes, "Logical Equivalence Failure")

If a set is partitioned to where one of the subsets is larger than the set, the counting function results in a negative integer or number for that subset. This allows the operation of subtraction to be expanded to include negative sets, or sets with a negative count of elements, much like a loan or accounting debit.

Moreover, subtraction may be explained as the union of a set of elements with a set whose elements possess a reverse orientation, or property of anti-existence, or addition of a negative number. (Hughes, "The Anti-Existence Theorem")

Since addition and subtraction are related to each other, using the counting function to reverse their result over a set of elements, they share the same identity element in zero, which represents the numeric value of an empty set.

The idea that zero represents the numeric value of the empty set, or a set without any elements, helps support the use of set theory to explain addition and subtraction since their identity element is explained in terms of a set rather than a point of balance, equilibrium, or origin.

By representing a set without any elements, zero is able to serve as a point of reference for ordering, or counting the number of elements in a set since by lying outside the set, it may serve as an absolute point of reference, and does not possess a numeric value that would be used for counting an element.

Since zero may serve as a point of reference for counting the elements in a set, it may also serve as a point of origin for a number line, onto which a count of elements in a set is transposed.

Multiplication

Set theory may also be used to introduce the operation of multiplication, which may be viewed as the application of the counting function to the replication of a set. In essence, the element being multiplied represents a set that is being replicated, which is identified by the number of elements that are in the set.

The other element in multiplication is a multiplier, whose numeric value determines how many times that the set is replicated. Multiplication then applies the counting function to all the replicates of the set, which results in its product.

Unlike addition, which occurs within the same or similar sets, multiplication may occur between different sets, so that its product assumes the identity of the set being replicated, and the identity of the set its multiplier comes from.

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This dual set identity of the multiplication product is based on using one to represent the numeric value of the identity of each set of its two elements, so the identity of its product as a member of a set equals the product of its set identities.

In other words, multiplication can change the identity of its product as a member of a set since its multiplier comes from a set. In other words, numbers possess meaning based on the set that they come from. Multiplication set identity is widely used in scientific and engineering calculations, where it is called dimensional analysis.

Division

Just as set theory may be used to explain multiplication by applying the counting function to the replication of a set, set theory may be used to explain division as the application of the counting function to the partition of a set into equal subsets, defined by the numeric value of its divisor.

Where division splits apart a number or element into pieces, in set theory, division uses a divisor to partition a set into equal subsets, and applies the counting function to count the number of subsets, and any remaining elements in a leftover subset, which is smaller in value than the divisor.

While the operation of division is similar to subtraction in that it generally decreases or makes smaller, it is different from subtraction since it involves the creation of multiple subsets, which are equal in value, size, or area, as defined by its divisor.

Like multiplication, division can occur between different sets, where its result assumes the identity of the set partitioned and its divisor. But unlike multiplication, the identity of its result takes the form of a ratio, where the identity of its numerator is determined by the set that is being partitioned, and the identity of its denominator is determined by the set its divisor comes from.

Just as addition and subtraction share the same identity element in zero, multiplication and division share the same identity element in one. In other words, reverse operations share the same identity element as the operation they reverse.

Analysis

Division by zero may be analyzed, beginning with the definition of an operation, which performs work between two elements and results in another element. It may be observed that this definition generally assumes that the work it performs between two elements results in a change in value of the elements, so its result generally takes the form of another element.

In other words, the work that an operation does, results an element that is different in value from the elements used in the operation. For example, adding one to one results in a sum of two, a different value than either element. However, this rule has an exception for when the operation uses its identity element. In this case, the operation returns the value of the other element.

However, when the operation is non-commutative, in order for this rule to hold, the identity element needs to be used as the action element. For subtraction, this means that its identity element of zero is subtracted from the other element. And for division, this means that its divisor is the identity element of one.

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Otherwise, when the identity element is used as the element of stability, or the element that is acted upon, the operation returns a negative value of the other element in the case of subtraction, or a reciprocal of the other element in the case of division.

When the identity element of the more basic of two related non-commutative operations, such as subtraction and division, is used as the action element for the more sophisticated operation, the result is division by zero.

Division by zero is generally unable to be performed since zero does not possess a numeric value by which it is able to partition a set, or perform the role of a divisor.

In other words, at least for division, an action element generally requires some substance, or numeric value, in order to perform work. In this regard, the identity element of one is able to perform the work of a divisor since it is able to partition a set exactly, according to its existing boundary.

In other words, one partitions a set exactly, according to its existing boundary, or replicates its existing boundary, so it is able to serve as the identity element of division. And since one replicates the existing boundary of a set exactly, it is able to serve as the identity element of multiplication.

With respect to the operation of subtraction, since zero is able to be counted as an empty set, is able to be used in subtraction, which partitions a set into a subset, it can be used as the identity element of subtraction since its partition of a set does not change the boundary of the set, or its numeric value.

In other words, as the empty set, or a set that does not possess any elements, zero does not possess a boundary since it does not encompass any elements. As a result, its use in set theory to change the boundary of a set in a union of two sets by counting its elements does not result in any change in the other set's existing boundary, establishing it as the identity element for addition.

And since zero represents the empty set, or a numeric element without any value, it is able to serve as the identity element for subtraction since, used as the action element to partition a set, the empty set does not change boundary of another set, or a count of its elements.

In set theory, subtraction and division are related to each other since both operations partition a set, whose result generally decreases its size or count. Division may be treated as the more sophisticated of the two operations (and subtraction as the more basic of the two operations) since it involves an algorithm, which partitions a set, counts the number of partitions, and sets aside any remainder.

In longhand form, division subtracts the nearest multiplication product from the number divided to find a positive remainder, and so incorporates subtraction.

In terms of logic, division by zero may be likened to the supposed equivalence between the contrapositive and a conditional statement, in this sense, that a conditional uses a flow of reasoning from its condition to a conclusion, which the contrapositive reverses like an inverse or a reverse function, and includes a negation.

Although this analogy is inexact, just as zero is a powerful multiplier that always takes a multiplication product to zero, using division by zero to reverse a multiplication product that

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is equal to zero is like a fishing expedition that finds a contradiction from a reverse logic function, whose condition is the negation of the conclusion of a conditional.

In other words, the logic of the contrapositive involves a reverse function, and applies a negation to its reverse logic function to claim it is logically equivalent to the conditional, a situation that, when analyzed, would be rejected, and has been disproved. (Hughes, "Logical Equivalence Failure").

With regard to multiplication and division, which may be considered more sophisticated than addition and subtraction, when their identity element is used in either operation, it is able to change the identity of the other element as a member of set since the identity element has an identity as a member of set.

Notwithstanding, as a general rule, when an operation uses its identity element, its result has a value equal to the value of the other element. Moreover, the operation is performed in its entirety.

In other words, the definition of work in an operation allows for the use of an identity element that returns the value of the other element. With this in mind, zero may be used as a divisor when it is able to perform the work of a divisor, which occurs when the number or element being divided is zero or itself.

In other words, even though zero has no numeric value by which it can partition a set, a set without any elements has no boundary or element over which the normal partition can operate. So, the division of the empty set by the empty set, or an equivalent set without any elements or no numeric value and no boundary, replicates the set exactly, and its result is the identity element of one.

In other words, the empty set or a set without any numeric value is able to form a partition when the set it is partitioning is like itself, in that the set being partitioned is also the empty set, without any numeric value or boundary. In other words, the empty set is able to partition the empty set exactly since the partition that it is able to form has no form or substance as a boundary.

With this in mind, it may be observed that there is no point, element, or object for a boundary to be established around zero, as the empty set, to separate it from other sets or elements. A boundary is used to define a set, to separate its elements from other sets or elements.

Since zero has no elements, it does not require a boundary. In other words, as the empty set, or numeric value of the empty set, zero does not possess a physical location, which a boundary line, such as displayed in a Venn Diagram, would imply.

From another point of view, when a set replicates itself exactly after being partitioned, the numeric value of the result is equal to the identity element of division, or one, as a matter of definition.

In other words, since zero is able to perform a partition that replicates itself exactly as the empty set when the element being divided is itself, or the empty set, which does not have a boundary, the division of zero by zero results in the identity element of one.

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In other words, the classical rule of when the terms in a numerator and denominator are equal, allows for their cancelation, applies to the division of zero by zero so that its result is one, the identity element of division.

In other words, division by zero performs work when the set it is divided into is itself, or the null set, a perfect identity that involves the empty set.

On the other hand, when the numerator is not zero, division by zero is not able to be performed since zero, or the empty set, has no element in common with the set that is being divided or partitioned.

This conclusion that division by zero is not able to be performed for a non-zero numerator may be confirmed by how since zero is such a powerful multiplier, division by zero is unable to trace back the other element that was multiplied by zero, so as a reverse operation of multiplication, division by zero is unable to be performed.

In other words, as reverse operation of multiplication, division by zero is generally not able to be performed since the numeric value or identity of the other element is not able to be determined.

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