CONSTRUCTION OF LOCALLY D-OPTIMAL DESIGN FOR POISSON REGRESSION MODEL USING RECURSIVE ALGORITHM

***J. Garba[1], O.E. Asiribo[2], S.I.S. Doguwa[3], H.G. Dikko[4]**

[1,3,4] Department of Statistics, Ahmadu Bello University, Zaria
[2] Department of Statistics, Federal University of Agriculture, Abeokuta.

ABSTRACT: *Designing optimum experiment for nonlinear models is generally challenging due to the dependence of design support point on the unknown parameter(s) value(s). For a locally optimal design, the unknown parameters are replaced by a guess value; if the guesses are not so close to the actual parameter(s) value(s), the resulting design may not be optimal, robust and efficient. One possible way to salvage the situation is to identify a subclass of designs with a simple format, so that one can restrict considerations to this subclass for any optimality problem. With a simple format, it would be relatively easy to derive an optimal design, analytically or numerically. In this paper, we identified a subclass of design with relatively simple format and use functional approach based on implicit function theorem to construct locally D-optimal design for Poisson regression model. The result showed the dependence of optimal design points on values of unknown parameters and on the bound of the design interval. Also the design proved to be minimally supported (saturated) at two design points including B, the upper boundary point. Furthermore, the lower support point was shown to be approximated by a convergent power series using recursive algorithm.*

KEYWORDS: Poisson Regression Model, Minimally Supported/Saturated Design, D-optimal Design, Fisher Information Matrix, Functional Approach, Tylor Series.

INTRODUCTION

Optimal experimental designs for Poisson regression model have received increasing attention in recent years, most especially in the field of Biomedical and clinical trials. One major challenge in the construction and development of design for general nonlinear models (Poisson inclusive) is the dependence of design support points on the unknown parameters of the Fisher information matrix (Zhang, 2006). One possible way to salvage the situation is to identify a subclass of designs with a simple format, so that one can restrict considerations to this subclass for any optimality problem (Yang and Stufken, 2012). With a simple format, it would be relatively easy to derive an optimal design, analytically or numerically. This approach can be meaningful provided the number of support points in the subclass is as small as possible. According to Carathéodory's theorem, we can always restrict our consideration to at most $r(r+1)/2$ design points (where r is the total number of parameters). On the other hand, if we want all parameters to be estimable, the minimum number of support points should be at least r. If the number of support points of a design for an r-parameter model is r, then the design is called minimally supported or saturated. Thus, the ideal situation is that the designs in the subclass should have no more than r points.

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So our goal here is to identify a class of relatively simple designs so that for any design ξ that does not belong to this class, there exist a design ξ^* in the class that has an information matrix that dominates that of ξ in the Loewner ordering. Also ξ^* will not be worse than ξ for most of the commonly used optimality criteria and for many functions of the parameters. Furthermore, for the already identified subclass, D-optimality criterion and recursive algorithm would be used to determine the specific design support point. This paper is organized and divided into the following sections. A brief review of literature on locally optimal designs is presented in Section two. Strategy and basic concept of methodology is presented in section three while section four is devoted to application of the methodology to Poisson regression model. Lastly, a concluding remark is given in section five.

Locally Optimal Designs

The most commonly and simple approach to the problem of experimental designs for nonlinear models is to use a best guess for the unknown parameters which usually comes from the previous experimentation, pilot experiment conducted specially for this purpose, or simply a mere guess by the experimenter. A better design can be chosen base on the selected design optimality criterion function evaluated at the guess value. This approach was introduced by Chernoff (1953) and is termed as *locally optimal design*.

Locally optimal designs have been investigated by several authors. Konstantinous and Dettea (2015) considered the construction of optimal designs for nonlinear regression models when there are measurement errors in the covariates and developed approximate design theory for the estimation of parameters using maximum likelihood and least-squares method of estimation.
Dette *et al(2010)* derived locally D- and EDρ-optimal designs for the exponential, log –linear and three-parameter emax models. For each model the locally D- and EDρ-optimal designs are supported at the same set of points, while the corresponding weights are different. They also demonstrate that locally D- and EDρ-optimal designs for the emax, log-linear and exponential models are relatively robust with respect to misspecification of the model parameters.

Wu and Stufken (2014) considered generalized linear models with a single-variable quadratic polynomial as the predictor under a popular family of optimality criteria and derive explicit expressions for some *D*-optimal designs. For unrestricted design region, optimal designs can be found within a subclass of designs based on a small support with symmetric structure. Gutiérrez et al (2014) gave explicit formulae for the construction of D-optimal designs as a function of the unknown parameters for the logistic regression model.

Haines et al (2007) considered the construction of locally D-optimal design for logistic regression model with two explanatory variables, the variables are considered to be doses of two drugs and are thus constrained to be greater than or equal to zero with no interaction term in the model. The result shows that there are two patterns of *D*-optimal design that depend on model parameters based on 3 and 4 support points respectively. Other contributions can be found in Wang et al (2006), Rodriguez-Torreblanca and Rodriguez-Diaz (2007), Russell et al (2009), Dette et al (2001), Chang (2005a and 2005b), Dette *et al*(2006).

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Despite its wide range applications, Poisson regression model have not been given the desirable/considerable attention most especially in relation to the newly improved techniques in the field of optimal experimental design. To this end, we proposed to investigate optimal design problems for this important model. Specifically, functional algebraic-approach would be used to construct locally D-optimal design for the aforementioned model, where Taylor expansion would be used for the numerical calculation of the locally D-optimal designs via recursive algorithm.

Strategy

Suppose we make n observations $y_1, \dots, y_n$; from an experiment with $x_1, \dots, x_n$; explanatory variables. Let's the data be modeled by nonlinear relationship of the form:

$$y_i = \eta(x_i, \theta) + \epsilon_i, \epsilon_i \sim \mathcal{N}(0, \sigma^2), i = 1,2, \dots n \quad (3.1)$$

Where

$\eta(x_i, \theta)$ is the regression function, θ is the vector of $rx1$ unknown parameters and ϵ_i is the random error component. Note that both θ *and* ϵ_i are independent and identically distributed. In addition, suppose we have $t < n$ distinct design point and lets x_i occurs $m_i$times among the points $x_1, \dots, x_n$, in what follows, each of the point x_i is associated with weight coefficient (proportion of observation at i[th] design point), $w_i = m_i/n$. Here we require that $0 \le w_i \le 1; i = 1,2, \dots, t$ and $\sum_{i=1}^{t} w_i = 1$.

Define a discrete probability measure, ξ as

$$\xi = \begin{pmatrix} x_1, & \dots & , x_t \\ w_1, & \dots, & w_t \end{pmatrix} \quad (3.2)$$

Then the matrix,

$$M(\xi, \theta) = \sum_{i=1}^{t} w_i f(x_i, \theta) f^T(x_i, \theta) \quad (3.3)$$

is called Fisher information matrix of the design ξ.

where

$$f(x, \theta) = \left(\frac{\partial \eta(x, \theta)}{\theta_1}, \dots, \frac{\partial \eta(x, \theta)}{\theta_r} \right)^T$$

is the vector of partial derivatives of $\eta(x, \theta)$ with respect to θ

The asymptotic variance-covariance matrix of the least squares estimator, $\hat{\theta}$ is given by

$$V_{\hat{\theta}} = \frac{\sigma^2}{n} \{M(\xi, \theta)\}^{-1}$$

We are interested in the optimal choice of a design among the available classes of designs in the design space, $\chi \subseteq \mathcal{R}$, where $\mathcal{R} = (0,1)$ is the design region/interval such that $V_{\hat{\theta}}$ is minimized.

If an information matrix of any nonlinear model, discrete or continuous can be transform to matrices of the form as in equation 3.4 and 3.5 respectively, then, a subclass of design can be determine with information matrix that dominates that of all other designs in terms of Loewner ordering (Biedermann and Yang, 2015)

$$M(\xi, \theta) = G^T(a, b) H_\xi(a, b) G(a, b) \quad 3.4$$

Where

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$$H_\xi(a,b) = \begin{bmatrix} \sum_{i=1}^{t} w_{i\vartheta_1(h_i)} & \sum_{i=1}^{t} w_{i\vartheta_2(h_i)} \\ \sum_{i=1}^{t} w_{i\vartheta_2(h_i)} & \sum_{i=1}^{t} w_{i\vartheta_3(h_i)} \end{bmatrix} \qquad 3.5$$

is a matrix that depends on design ξ through $w_i{'}s$ and $h_i{'}s$ and on (a,b) through the $h_i{'}s$, while $G(a,b)$is a matrix that depends on (a,b) only.

In this paper we consider Poisson regression model and the information matrix have been verified to conform to the transformed matrices of equation 3.4 and 3.5, details of the verification is presented below.

Consider Poisson regression model with mean response function given by

$$\eta(x_i,\theta) = \exp(a + bx_i)$$

let $h_i = a + bx_i,\ \vartheta_1(h_i) = \exp 2(h_i),\ \ \vartheta_2(h_i) = h_i \exp 2(h_i),\ \ \vartheta_3(h_i) = {h_i}^2 \exp 2(h_i)$ (3.6)

and

$$G(a,b) = \begin{bmatrix} 1 & -a/_b \\ 0 & 1/_b \end{bmatrix} \qquad (3.7)$$

Suppose the design is run in accordance with (3.2), the information matrix can be derive from (3.3) as

$$M(\xi,\theta) = \sum_{i=1}^{t} w_i f(x_i,\theta) f^T(x_i,\theta) = \sum_{i=1}^{t} w_i \begin{bmatrix} \exp(a+bx_i) \\ x_i \exp(a+bx_i) \end{bmatrix} [\exp(a+bx_i) \quad x_i \exp(a+bx_i)]$$

$$M(\xi,\theta) = \sum_{i=1}^{t} w_i \begin{bmatrix} \exp 2(a+bx_i) & x_i \exp 2(a+bx_i) \\ x_i \exp 2(a+bx_i) & {x_i}^2 \exp 2(a+bx_i) \end{bmatrix} \qquad (3.8)$$

It can be shown that $\sum_{i=1}^{t} w_i f(x_i,\theta) f^T(x_i,\theta)$ and $G^T(a,b) H_\xi(a,b) G(a,b)$ are equivalent, that is to say one is a linear transformation of another.

Consider

$$M(\xi,\theta) = G^T(a,b) H_\xi(a,b) G(a,b)$$

$$M(\xi,\theta) = G^T(a,b) H_\xi(a,b) G(a,b) = \begin{bmatrix} 1 & 0 \\ -a/_b & 1/_b \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{t} w_{i\vartheta_1(h_i)} & \sum_{i=1}^{t} w_{i\vartheta_2(h_i)} \\ \sum_{i=1}^{t} w_{i\vartheta_2(h_i)} & \sum_{i=1}^{t} w_{i\vartheta_3(h_i)} \end{bmatrix} \begin{bmatrix} 1 & -a/_b \\ 0 & 1/_b \end{bmatrix}$$

$$M(\xi,\theta) = \sum_{i=1}^{t} w_i \begin{bmatrix} 1 & 0 \\ -a/_b & 1/_b \end{bmatrix} \begin{bmatrix} \vartheta_1(h_i) & \vartheta_2(h_i) \\ \vartheta_2(h_i) & \vartheta_3(h_i) \end{bmatrix} \begin{bmatrix} 1 & -a/_b \\ 0 & 1/_b \end{bmatrix} \qquad (3.9)$$

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Substituting equation (3.6) in to equation (3.9) we get

$$M(\xi,\theta)=\sum_{i=1}^{t} w_i \begin{bmatrix} 1 & 0 \\ -a/b & 1/b \end{bmatrix} \begin{bmatrix} \exp 2(h_i) & h_i \exp 2(h_i) \\ h_i \exp 2(h_i) & h_i^{\,2} \exp 2(h_i) \end{bmatrix} \begin{bmatrix} 1 & -a/b \\ 0 & 1/b \end{bmatrix}$$

$$=\sum_{i=1}^{t} w_i \begin{bmatrix} \exp 2(h_i) & -a/b \exp 2(h_i) + 1/b\, h_i \exp 2(h_i) \\ -a/b \exp 2(h_i) + 1/b\, h_i \exp 2(h_i) & a^2/b^2 \exp 2(h_i) - 2\, a/b^2\, h_i \exp 2(h_i) + h_i^{\,2}/b^2 \exp 2(h_i) \end{bmatrix}$$

$$M(\xi,\theta)=\sum_{i=1}^{t} w_i \begin{bmatrix} \exp 2(a+bx_i) & x_i \exp 2(a+bx_i) \\ x_i \exp 2(a+bx_i) & x_i^{\,2} \exp 2(a+bx_i) \end{bmatrix} \qquad (3.10)$$

■

Clearly, equation (3.8) and (3.10) are equivalent.

Having successfully verified the equivalences of our design information matrices, we shall now show the mechanism and strategy for a general nonlinear models under certain conditions that there exists a subclass of designs such that for any given design ξ, there exists a design ξ^* in this subclass such that $M_{\xi^*}(\theta) \geq M_{\xi}(\theta)$.

To start with let, $\vartheta_1(h), \vartheta_2(h), \ldots, \vartheta_k(h)$ be k functions defined on [A,B]. Throughout this paper, we have the following assumptions:

(i) $\vartheta_1(h), \vartheta_2(h), \ldots, \vartheta_k(h)$ are infinity differentiable;

(ii) $f_{l,l}$ has no zero value on [A,B].

Here, $f_{l,t}$ $1 \leq t \leq k, t \leq 1 \leq k$ are defined as follows:

$$f_{l,t}(h)=\begin{cases} \vartheta_l'(h), & t=1, l=1,\ldots,k \\ \left(\dfrac{f_{l,t-1}(h)}{f_{t-1,t-1}(h)}\right)', & 2 \leq t \leq k, t \leq l \leq k \end{cases}$$

Theorem 3.1

For any nonlinear regression model, suppose the information matrix can be expressed as in (3.4) and $\mathrm{h_i} \in [\mathrm{A}, \mathrm{B}]$. Rename all distinct $\vartheta_{\mathrm{lt}}, 1 \leq \mathrm{l} \leq \mathrm{t} \leq \mathrm{p}$ to $\vartheta_1, \vartheta_2, \ldots, \vartheta_{\mathrm{k}}$ such that (i)ϑ_{k} is one of $\vartheta_{\mathrm{ll}}, 1 \leq \mathrm{l} \leq \mathrm{p}$ and (ii)there is no $\vartheta_{\mathrm{lt}} = \vartheta_{\mathrm{k}}$ for $1 < \mathrm{t}$. Let $\Psi(\mathrm{h}) = \prod_{\mathrm{l}=1}^{\mathrm{k}} \mathrm{f}_{\mathrm{l,l}}(\mathrm{c}), \mathrm{h} \in [\mathrm{A}, \mathrm{B}]$. For any given design ξ, there exist a design ξ^*, such that $\mathrm{M}_{\xi^*}(\theta) \geq \mathrm{M}_{\xi}(\theta)$. Note that ξ^* depends on different situations.

i. When k is odd and $\Psi(h) < 0, \xi^*$ is based on at most $(\mathrm{k}+1)/2$ points including point A.
ii. When k is odd and $\Psi(h) > 0, \xi^*$ is based on at most $(\mathrm{k}+1)/2$ points including point B.
iii. When k is even and $\Psi(h) > 0, \xi^*$ is based on at most $(\mathrm{k}/2)+1$ points including points A and B.
iv. When k is even and $\Psi(h) < 0, \xi^*$ is based on at most $\mathrm{k}/2$ points.

For a two parameter model with $k = 3$, the conditions in (i) and (ii) of theorem 3.1 above can be narrowed down to the following important result.

Lemma 3.1: The functions $\vartheta_1(h), \vartheta_2(h), \vartheta_3(h)$ are called type I functions on $[A, B]$ if the following conditions holds

i) $\vartheta_1(h), \vartheta_2(h),\ \vartheta_3(h)$ are continuous functions on $[A,B]$ that are three times differentiable on $(A,B]$

ii) $\vartheta_1'(h)\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\left(\left(\frac{\vartheta_3'(h)}{\vartheta_1'(h)}\right)' \Big/ \left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\right)' < 0\ for\ h \in (A,B] and$

iii) $lim_{h\downarrow A}\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\left(\vartheta_1(A) - \vartheta_1(h)\right) = 0$

Where A is finite and B could be +∞.

Also we can say that the functions $\vartheta_1(h), \vartheta_2(h),\ \vartheta_3(h)$ are functions of type II on $[A,B]$ if

i) $\vartheta_1(h), \vartheta_2(h),\ \vartheta_3(h)$ are continuous functions $[A,B]$ that are three times differentiable on $[A,B)$

ii) $\vartheta_1'(h)\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\left(\left(\frac{\vartheta_3'(h)}{\vartheta_1'(h)}\right)' \Big/ \left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\right)' > 0\ for\ h \in [A,B) and$

iii) $lim_{h\downarrow B}\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\left(\vartheta_1(B) - \vartheta_1(h)\right) = 0$

In this case A could be $-\infty$ and B is finite.

Application

4.1.1 Determination of Required Number of Support Points for Poisson Model

To identify the number of support points needed for the construction of optimal design, we have to check the form of the function as specified in lemma 3.1. Basically, there are three conditions for verifying whether a function is of type I or type II. To verify the first condition, it means to show that $\vartheta_1(h),\ \vartheta_2(h),\ \vartheta_3(h)$ are continuous functions on $[A,B]$ that are three times differentiable on $[A,B)$

Now,

$$\vartheta_1(h_i) = \exp(2h_i)$$

$$\Rightarrow \vartheta_1'(h_i) = 2\exp(2h_i), \quad \vartheta_1''(h_i) = 4\exp(2h_i), \quad \vartheta_1'''(h_i) = 8\exp(2h_i)$$

also

$$\vartheta_2(h_i) = h_i\exp(2h_i)$$

$$\Rightarrow \vartheta_2'(h_i) = \exp(2h_i) + 2h_i\exp(2h_i)$$

$$\vartheta_2''(h_i) = 4\exp(2h_i) + 4h_i\exp(2h_i)$$

$$\vartheta_2'''(h_i) = 12\exp(2h_i) + 8h_i\exp(2h_i)$$

and

$$\vartheta_3(h_i) = {h_i}^2\exp(2h_i)$$

$$\Rightarrow \vartheta_3'(h_i) = 2{h_i}^2\exp(2h_i) + 2h_i\exp(2h_i)$$

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$$\vartheta_3''(h_i) = 4{h_i}^2\exp(2h_i) + 8h_i\exp(2h_i) + 2\exp(2h_i)$$

$$\vartheta_3'''(h_i) = 8{h_i}^2\exp(2h_i) + 24h_i\exp(2h_i) + 12\exp(2h_i)$$

Clearly the condition of continuity and differentiability as contained in lemma 3.1 has been satisfied. It remain to verify the second condition which entails showing that

$$\vartheta_1'(h)\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\left(\left(\frac{\vartheta_3'(h)}{\vartheta_1'(h)}\right)'\Big/\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\right)' > 0 \; for \; h \in [A, B)$$

Consider,

$$\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)' = \left[\frac{\exp(2h_i) + 2h_i\exp(2h_i)}{2\exp(2h_i)}\right]'$$

$$= \frac{2\exp(2h_i)\,[4\exp(2h_i) + 4\,h_i\exp(2h_i)] - 4\exp(2h_i)[\exp(2h_i) + 2\,h_i\exp(2h_i)]}{4\exp(4h_i)}$$

$$\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)' = \frac{4\exp(4h_i)}{4\exp(4h_i)} = 1$$

$$\left(\frac{\vartheta_3'(h)}{\vartheta_1'(h)}\right)' = \left(\frac{2{h_i}^2\exp(2h_i) + 2h_i\exp(2h_i)}{2\exp(2h_i)}\right)' = \frac{8h_i\exp(4h_i) + 4\exp(4h_i)}{4\exp(4h_i)} = 1 + 2h_i$$

$$\left(\left(\frac{\vartheta_3'(h)}{\vartheta_1'(h)}\right)'\Big/\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\right)' = (1 + 2h_i)' = 2$$

$$\Rightarrow \vartheta_1'(h)\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\left(\left(\frac{\vartheta_3'(h)}{\vartheta_1'(h)}\right)'\Big/\left(\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\right)'\right)' = 2\exp(2h_i) * 1 * 2 = 4\exp(2h_i) > 0$$

Having verified the second condition, next is to check for the last condition which is given as

$$lim_{h\downarrow B}\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\big(\vartheta_1(B) - \vartheta_1(h)\big) = 0$$

$$\frac{\vartheta_2'(h)}{\vartheta_1'(h)}\big(\vartheta_1(B) - \vartheta_1(h)\big) = \frac{\exp(2h_i)[1 + 2h_i]}{2\exp(2h_i)}\big(\vartheta_1(B) - \vartheta_1(h)\big)$$

$$lim_{h\downarrow B}\frac{\exp(2h_i)[1 + 2h_i]}{2\exp(2h_i)}\big(\vartheta_1(B) - \vartheta_1(B)\big) = lim_{h\downarrow B}\frac{\exp(2h_i)[1 + 2h_i]}{2\exp(2h_i)}(0) = 0$$

Obviously, all the three conditions of lemma 3.1 are satisfied and this indicates that the function is a type II function; hence condition (ii) of theorem 3.1 applies.

Application of theorem 3.1 implies having an optimal design with two support points and one of which is the upper boundary point, x_2. The next important task is to show the dependence of support point on the unknown parameter value and determine the lower support point using recursive algorithm.

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4.3.2 Dependence of Support Point on the Unknown Parameter Value

Given that

$$f(x_i,\theta)=\left(\frac{\partial\eta(x_i,\theta)}{\partial\theta_1},\frac{\partial\eta(x_i,\theta)}{\partial\theta_r}\right)'=[\exp(a+bx_i)\quad x_i\exp(a+bx_i)]$$

Then for a saturated design,

$$M(\xi,\theta)=\frac{1}{2}\begin{bmatrix}\exp[2(a+bx_1)]+\exp[2(a+bx_2)] & x_1\exp[2(a+bx_1)]+x_2\exp[2(a+bx_2)]\\ x_1\exp[2(a+bx_1)]+x_2\exp[2(a+bx_2)] & {x_1}^2\exp[2(a+bx_1)]+{x_2}^2\exp[2(a+bx_2)]\end{bmatrix}$$

and

$$|M(\xi,\theta)|=\tfrac{1}{4}(x_1^2+x_2^2-2x_1x_2)\exp[2(a+bx_1)+2(a+bx_2)]$$

$$=\tfrac{1}{4}(x_2-x_1)^2\exp[2(a+bx_1)+2(a+bx_2)]$$

Remember that the function is a type II function and hence x_2 is fixed at the upper boundary point, 1. This implies $|M(\xi,\theta)|=\frac{1}{4}(1-x_1)^2\exp[2(a+bx_1)+2(a+b)]$

Taking the log of both sides we get

$$\log(|M(\xi,\theta)|)=log\frac{1}{4}+2log(1-x_1)+4a+2b(1+x_1)$$

Taking the partial derivatives with respect to x_1 we have

$$x_1=1+\frac{1}{b};b<0 \qquad 3.11$$

Evidently, equation (3.11) shows clearly the dependence of design support point on the unknown parameter value. Now the design in (3.2) becomes

$$\hat{\xi}=\begin{bmatrix}1+\frac{1}{b} & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

4.3.3 Determination of Lower Support Point Using Recursive Algorithm

The function $g(x_1,b)=1+b(1-x_1)=0$ is called the gradient of the objective function. $for\ b_0=-2,\qquad x_0=1+\frac{1}{-2}=0.5$

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since we are able to established/found an optimal design x_0 with respect to b_0 then we can use recursive algorithm as in (Dette et al (2001)) to find an approximate optimal experimental design $x_1(b)$ for different values of b provided the Jacobian of the function exist.

For this problem, the Jacobian of the function is given by

$$J = \frac{\partial\big(1 + b(1 - x_1)\big)}{\partial x_1} \Big|_{b=b_0} = -b \Big|_{b=b_0} = 2$$

Using the relation

$$\xi^*(\mathrm{s}+1, \tau_0) = -J^{-1}(\tau_0)\frac{1}{(s+1)!}\left(\frac{d}{d\tau}\right)^{s+1} g\left(\xi^*{}_{(s)}(\tau), \tau\right) \text{, for s} = 0,1,2, \ldots$$

$When\ s = 0$

$$\frac{d\big(1 + b(1 - x_1)\big)}{db} = 1 - x_1 \Big|_{x_1 = x_0} = 1 - 0.5 = 0.5$$

$$\Rightarrow \xi^*(1, -2) = -\frac{0.5}{2} = -0.25$$

and $\xi^*{}_{(s)}(\tau) = \xi^*(\tau_0) + \sum_{j=1}^{k} \xi^*(\mathrm{k}, \tau_0)(\tau - \tau_0)^j$ is obtained as

$$\xi^*{}_{(0)}(\mathrm{b}) = 0.5 - 0.25(b + 2)$$

$when\ s = 1,$

$$\left(\frac{d}{d\tau}\right)^2 g\left(\xi^*{}_{(s)}(\tau), \tau\right) = \left(\frac{d}{d\mathrm{b}}\right)^2 g(0.5 - 0.25(b + 2), b)$$

now,

$$g(0.5 - 0.25(b + 2), b) = 1 + b\big(0.5 - 0.25(b + 2)\big) = 1 + b + 0.25b^2$$

$$\left(\frac{d}{d\mathrm{b}}\right) g(0.5 - 0.25(b + 2), b) = 1 + 0.5b \Rightarrow \left(\frac{d}{d\mathrm{b}}\right)^2 g(0.5 - 0.25(b + 2), b) = 0.5$$

$$\xi^*(2, -2) = -\frac{0.5}{2(2)!} = -0.125$$

$$\Rightarrow \xi^*{}_{(1)}(\mathrm{b}) = 0.5 - 0.25(b + 2) - 0.125(b + 2)^2$$

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$when\ s = 2,$

$$\left(\frac{d}{d\tau}\right)^3 g\left(\xi^*{}_{(s)}(\tau), \tau\right) = \left(\frac{d}{d\mathrm{b}}\right)^3 g(0.5 - 0.25(b+2) - 0.125(b+2)^2, b)$$

but, $g(0.5 - 0.25(b+2) - 0.125(b+2)^2, b) = 1 + b\big(1 - (0.5 - 0.25(b+2) - 0.125(b+2)^2)\big)$

$$\left(\frac{d}{d\mathrm{b}}\right)^3 \Big(1 + b\big(1 - (0.5 - 0.25(b+2) - 0.125(b+2)^2)\big)\Big) = 0.125$$

$$\xi^*(3, -2) = -\frac{1}{2(3)!}(0.125) = -0.0104$$

$$\Rightarrow \xi^*{}_{(2)}(\mathrm{b}) = 0.5 - 0.25(b+2) - 0.125(b+2)^2 - 0.0104(b+2)^3$$

$when\ s = 3,$

$$\left(\frac{d}{d\tau}\right)^4 g\left(\xi^*{}_{(s)}(\tau), \tau\right) = \left(\frac{d}{d\mathrm{b}}\right)^4 g(0.5 - 0.25(b+2) - 0.125(b+2)^2 - 0.0104(b+2)^3, b)$$

but, $g(0.5 - 0.25(b+2) - 0.125(b+2)^2 - 0.0104(b+2)^3, b) = 1 + b\big(1 - (0.5 - 0.25(b+2) - 0.125(b+2)^2 - 0.0104(b+2)^3)\big)$

$$\left(\frac{d}{d\mathrm{b}}\right)^4 \Big(1 + b\big(1 - (0.5 - 0.25(b+2) - 0.125(b+2)^2 - 0.0104(b+2)^3)\big)\Big) = 0.0104$$

$$\xi^*(4, -2) = -\frac{1}{2(4)!}(0.0104) = -2.17 * 10^{-4}$$

$$\Rightarrow \xi^*{}_{(3)}(\mathrm{b}) = 0.5 - 0.25(b+2) - 0.125(b+2)^2 - 0.0104(b+2)^3 - 2.17 * 10^{-4}(b+2)^4 \qquad (4.5)$$

From the above computation we can deduce that for a given value of b, the lower support point can be approximated by the polynomial in equation 4.5 above.

CONCLUSION

In this paper, we have proved that the support points of locally D-optimal designs for Poisson regression model are dependent on the guess parameter value, in addition the design is shown to be minimally supported (saturated) at two design points including B, the upper boundary point. We also proposed the use of Taylor series to approximate the lower support points for Poisson model.

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