International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

ASYMPTOTICALLY AUTONOMOUS SYSTEM IN MATHEMATICAL MODEL OF SIS MODEL DETERMINE BY REPRODUCTIVE NUMBER

Briuk Dea, Asefa Erba, Tadesse Lamessa*
College of Natural and Computational science,
Department of Applied Mathematics
Woalita Sodo University,
sodo Ethiopia,P.O.BOX .138

ABSTRACT: *We describe and analyze a simple SIS model with treatment. In particular we give a completely qualitative analysis by means of the theory of asymptotically autonomous system. It is found that a backward bifurcation occurs if the adequate contact rate is small. It is also found that there is exists the bistable (if two steady state coexist) endemic equilibria. In the case of disease -induced death, it is shown that the backward bifurcation also occurs. Moreover, there is no limit cycle under some condition, and the subcritical Hopf bifurcation occurs under another condition.*

KEYWORD*:* asymptotically autonomous system, simple SIS model, backward bifurcation, endemic equilibria

Introduction

In compartmental model for the case transmission of communicable disease, there is standard, basic reproductive number R_0 representing the mean number of secondary infections caused by single infective introduced into susceptible population. If $R_0 < 1$ there is a disease –free equilibrium which is asymptotically stable and the infections dies-out. If $R_0 > 1$the usual situation is, there is an endemic equilibrium which is asymptotically stable and the infections persist. In this case the bifurcation leading from a disease –free equilibrium to an endemic equilibrium is forward. Even, if the endemic equilibrium is unstable, the instability commonly arises from a Hopf bifurcation and the infection still persists but in oscillatory manner. More precisely, as R_0 is increase through 1 there is an exchange of stability between the disease –free equilibrium and the endemic equilibrium (which is negative as well as unstable and the biologically meaningless if $R_0 < 1$). There is bifurcation or change in equilibrium behaviour at $R_0 = 1$, but the equilibrium infective population size depends on continuously on R_0 such a transition is a transcriptional bifurcation. In this case, reducing the basic reproductive number R_0 below one may fail to control the disease. Thus, it is important to identify the backward bifurcation to obtain threshold to control disease. We know that quarantine is an important method to decrease the spread of disease. In classical epidemic models the treatment rate is assumed to be proportional to the number of infective. In fact, this assumption is irrational because every community should have a suitable capacity for treatment. If it is too large, the community pays for unnecessary cost. If it is too small, the community has the risk of the outbreak of a disease. In this paper we modify it into

$$T(I) = \begin{cases} rI \text{ , if } 0 \le I \le I_0 \\ M \text{ , if } I \ge I_0 \end{cases}$$

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

where $M = rI_0$.This means that, the treatment rate ratio is proportional to the number of being infected when the capacity of treatment is not reached, and otherwise, take the maximal capacity.

The SIS model

Suppose there is a fixed population of size N which is initially entirely composed of susceptible becomes whose population is denoted by S. Then some number I_0 become infected at t= 0.If the population is mixed randomly, new infected denoted by I are predicted to appear at a rate proportional to both the number of infected and the number of susceptible. Infected individual recover at some rate and become susceptible again or transmission may occur from an infected person to another infected person which result is nothing happening, since the person is already infected. Or the potential transmission may occur from an infected person to recovered or immune person. In this case again nothing to change since only R_0 percent of the population is susceptible each infected person generates only βS_t is a new infection each period.

Let's as describe the susceptible population we begin period t with S_t individuals in susceptible population from this population we lose an average $\beta S_t I_t$ from the population. Thus in the period $t+1$ we have $S_{t+1} = S_1 - \beta\, S_t I_t$

Through similarly reasoning we see that

$I_{t+1} = I_t + \beta\, S_t I_t$ Where β is adequate contact rate.

In this paper, we will give a complete qualitative analysis of a simple SIS model with this treatment rate. Our model is given as follows

$$\begin{cases} \dfrac{ds}{dt} = A - dS - \beta SI + T(I) \\ \dfrac{dI}{dt} = \beta SI - (d+\varepsilon)I - T(I) \end{cases} \qquad (1.1)$$

Where A is the recruitment of population, d is the natural death rate of the population, ε is the disease –induced rate and $T(I)$ is treatment rate. For all parameters are positive constants.

SIS model without Disease –induced Death

In this section we can consider the case in which there is no disease –induced death.

That is when $\varepsilon = 0$, then we get

$$\begin{cases} \dfrac{ds}{dt} = A - dS - \beta SI + T(I) \\ \dfrac{dI}{dt} = \beta SI - dI - T(I) \end{cases} \qquad (2.1)$$

In (2.1) the total population size is given by $N = S + I$. This implies that $S + I = 1$

And $N' = (S+I)' = \dfrac{dS}{dt} + \dfrac{dI}{dt} = A - dN$. Then $\lim_{t\to\infty} N(t) = \dfrac{A}{d}$ for every choice of initial value and (2.1) is asymptotically autonomous. Now we let $K = \dfrac{A}{d}$, by the theory of asymptotically autonomous system, this implies if N has a constant limit then the system is equivalent to the system in which, we may replace N by this limit that is K and reduce the dimensional of system (2.1) by using $S = N - I$ to give the single differential equation

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

$$I' = \beta(K - I)I - dI - T(I) \qquad (2.2)$$

Equilibrium of the System under without Disease –Induced Death

Now we first consider the equilibrium of the system (2.1) and asymptotical stability. Let

$$R_0 = \frac{\beta K}{d + r}.$$

Then, R_0 is :

- ❖ One of the fundamental concepts in mathematical biology.
- ❖ Defined as "the average number of secondary infections caused by a single infectious individual during their entire infectious lifetime.
- ❖ It is expected number of secondary individuals produced by an individual in its lifetime.

However, "secondary" depends on context:-

- ✓ Mean lifetime reproductive success (demographics and ecology).
- ✓ It is the number of individuals infected within a single infected individual's entire infectious lifetime (epidemiology);
- ✓ Number of newly infected cells produced by a single infected cell (in-host dynamics).

A threshold criterion of R_0 is:

- If $R_0 < 1$, each individual produces, on average, less than one new infected individual and hence the disease dies out.
- If $R_0 > 1$, each individual produces more than one new infected individual and hence the disease is able to invade the susceptible population. This allows us to determine the effectiveness of control measures. Predicts whether a disease will become endemic or die out.

When, $0 \le I \le I_0$, we have two equilibrium of the system (2.1).

Thus $S = K - I, I = 0 \Rightarrow S = K$

$S = \frac{A}{d}, E_0(\frac{A}{d}, 0)$. And from system (2.1) we have $I' = \beta(K - I)I - dI - rI$ for, then we get

$\beta(K - I)I - dI - rI = 0$

$I = (K - \frac{d + r}{\beta})$ and $S = \frac{d + r}{\beta}$

Now by using Jacobin matrix and Taylor expansion we can see that $E_0(0, \frac{d + r}{\beta})$ is a disease-free equilibrium, and $E_1(k - \frac{d + r}{\beta}, \frac{d + r}{\beta})$ is an endemic equilibrium of system (2.1) if and only if

$$1 < R_0 \le \frac{\beta I_0}{d + r} \triangleq P_2$$

When $I > I_0$ an endemic equilibrium of (2.2) satisfies

$$\beta(K - I)I - dI - M = 0 \qquad (2.3)$$

It is clear that (2.3) does not have positive solution when $d \ge \beta K$. Let $\Delta = (d - \beta K)^2 - 4\beta M$

We know that equation (2.3) has positive solution is equivalent to

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

$$\begin{cases} d < \beta K \\ \Delta \geq 0 \end{cases} \Rightarrow R_0 \geq \frac{\sqrt{4\beta M} + d}{d + r} \triangleq P_0 \qquad (2.4)$$

Let us suppose that (2.4) holds. Then (2.3) has two positive solution I_* and I^* where

$$I_* = \frac{\beta K - d - \sqrt{\Delta}}{2\beta}, I^* = \frac{\beta K - d + \sqrt{\Delta}}{2\beta}$$

Clearly $E_*(I_*, S_*), E^*(I^*, S^*)$ is an endemic equilibrium of the system (2.2) if and only $I_* > I_0$ and $I^* > I_0$ By simple computation, we have

$$I_* > I_0 \Rightarrow \frac{\beta K - d - \sqrt{\Delta}}{2\beta} > I_0 \Rightarrow \beta K - d - \sqrt{\Delta} > 2\beta I_0$$
$$\Rightarrow \beta K - \sqrt{\Delta} > 2\beta I_0 + d$$
$$\Rightarrow 0 > -\sqrt{\Delta} > 2\beta I_0 + d - \beta K$$
$$\Rightarrow 2\beta I_0 + d - \beta K < 0$$
$$\frac{2\beta I_0 + d}{d + r} < \frac{\beta K}{d + r} < P_2$$

This implies that $I_* > I_0 \Rightarrow P_1 \triangleq \frac{2\beta I_0 + d}{d + r} < R_0 < P_2$.

And for $I^* > I_0 \Rightarrow \frac{\beta K - d + \sqrt{\Delta}}{2\beta} > I_0 \Rightarrow \beta K - d + \sqrt{\Delta} > 2\beta I_0$

Therefore, $R_0 > P_1$ or $P_2 < R_0 \leq P_1$

Summarizing the discussion above, we have the following conclusions.

Theorem 2.1

$E_0(0, \frac{d + r}{\beta})$ is locally stable when $R_0 < 1$ and unstable when $R_0 > 1$

$E_1(k - \frac{d + r}{\beta}, \frac{d + r}{\beta})$ Exist $1 < R_0 < P_2$ and locally stable if and only if $1 < R_0 \leq P_2$

$E_0(0, \frac{d + r}{\beta}), E_1(k - \frac{d + r}{\beta}, \frac{d + r}{\beta})$ is globally asymptotically stable if $1 < R_0 \leq P_2$ and one of the following conditions is satisfied:

$$i, R_0 < P_0$$
$$ii, R_0 < P_1$$

Proof: 1, from the equation (2.1), we have

$$\begin{cases} \frac{ds}{dt} = A - dS - \beta SI + T(I) \\ \frac{dI}{dt} = \beta SI - dI - T(I) \end{cases}$$

Then by Jacobin matrix we get

$$J = \begin{bmatrix} \frac{\partial S'}{\partial S} & \frac{\partial S'}{\partial I} \\ \frac{\partial I'}{\partial S} & \frac{\partial I'}{\partial I} \end{bmatrix}$$

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

Now at $(K,0)$, $\det(J-\lambda I)$. Then, we get the SIS Eigen values

$$J\big|_{(K,0)}=\begin{bmatrix}-d-\lambda & \beta I\\ 0 & \beta K-d-r-\lambda\end{bmatrix}$$

Then, the Eigen values are $\lambda_{1,2}=-d,-(\beta K-d-r)$. In this case the disease free equilibrium is stable if the second Eigen value is negative i.e. $\beta K<d+r$ this implies that $R_0<1$. Hence $E_0(0,\frac{d+r}{\beta})$ is locally stable, when $R_0<1$ and unstable when $R_0>1$.

Proof similarly we can that (2) for, since the endemic equilibrium point (S,I) is actually only available to the system if $\beta K<d+r$ since I cannot be negative equilibrium point.

Remark 2.2: To the global stability of the equilibrium, we can know the system (2.1) is bounded and $E_0(E_1)$ is the unique locally stable equilibrium under the condition of theorem (2.1) in the one -dimensional system (2.2).So we can easily obtain that $E_1(E_0)$ is globally asymptotically stable under condition in Theorem (2.1)

Theorem 2.3

Endemic equilibria E_* and E^* do not exist if $R_0<P_0$. Further, if $R_0\geq P_0$ we have the following:

Both E_* and E^* exist when $\beta I_0<r$ and $R_0<P_2$.

E_*, does not exist but E^* exists when $\beta I_0<r$ and $R_0\geq P_2$.

Let $\beta I_0\geq r$, then E_* does not exist. Further, E^* exists when $R_0>P_0$ and E^* does not exist when $R_0\leq P_2$.

Notice that $\beta I_0<r$ is equivalent to $P_1<P_2$ or $P_1<P_0$ and $\beta I_0\neq r$ is equivalent to $P_0<P_2$. So the Theorem 2.3 can be got from the above analysis.

Theorem 2.4

E_* is unstable whenever it exists and E^* is locally stable whenever it exists.

Further, E^* is globally asymptotically stable when E^*exists and $R_0>P_2$.

Notice that the local stability of equilibria is easy to get in one-dimensional system. The global stability of equilibria is similar to the Theorem 2.1. We know that$E^*$is the uniquelocally stable equilibrium under the conditions of Theorem 2.4, so we can obtain that E^* is globally asymptotically stable.

We consider $P_0>1$. *if* $r\leq\beta I_0$, a typical bifurcation diagram is illustrated in Fig. 1, where the bifurcation at $R_o=1$ is forward and (2.2) has one unique endemic equilibrium for all $R_o>1$. According to Theorem 2.1 (3), Theorem 2.3 (iii) and Theorem 2.4, we know that E_0, E_1 and E^* are all globally asymptotically stable under this case. Further, if $r>\beta I_0$ a typical bifurcation diagram is illustrated in Fig. 2, where the bifurcation from the disease-free equilibrium at $R_0=1$ is forward and there is a backward bifurcation from an endemic equilibrium at $R_0=P_2=1+\frac{\beta I_0}{d+r}$, which give rise to the existence of multiple endemic equilibria.

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

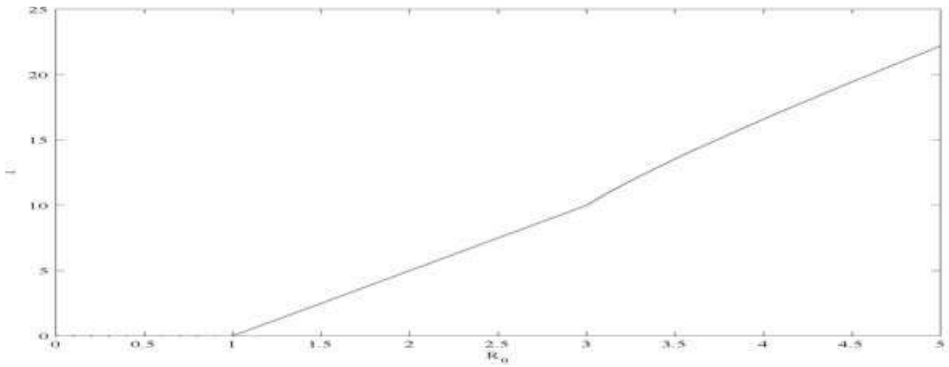


Figure1. The diagram of $I_0, I_1, and\ I^*$ versus R_0 when I_0 =10, β = 0.2, d= 0.2, r =0.8, where (iii) of Theorem 2.3 holds. The bifurcation at $R_0 = 1$ is forward and (2.2) has one unique endemic equilibrium for all $R_0 > 1$

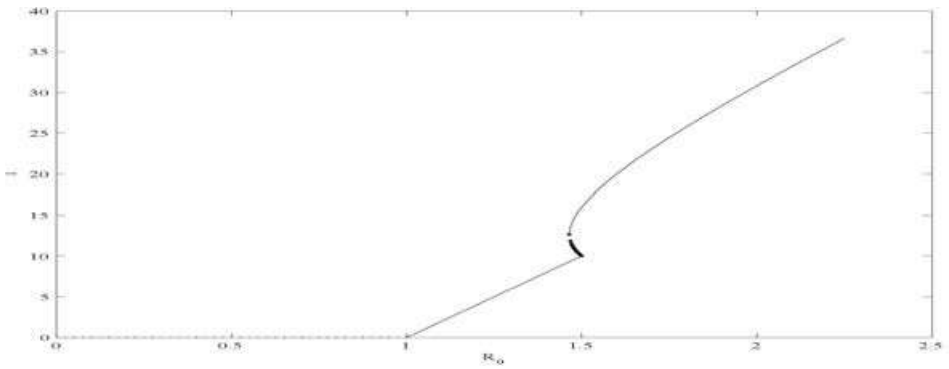


Figure2. The figure of infective sizes at equilibria *versus* R_0, *when* I_0 =10, $\beta = 0{,}05, d = 0.2$,r=0,8, where (i) of Theorem 2.3 holds. The bifurcation from the disease-free equilibrium at R_0= 1 is forwardand there is a backward bifurcation from an endemic equilibrium at $R_0 = p_2 = 1.5$.

We know that a backward bifurcation with endemic equilibria when $R_0 > 1$ is very interesting in applications. So we present the following proposition to give conditions for such aback ward bifurcation to occur.

Proposition2.5. If $P_0 < 1$ there is a backward bifurcation with endemic equilibria when $R_0 > 1$ in the system (2.1).

Proof: Suppose that $P_2 > P_0$ when $\beta I_0 \neq r$, and $\beta I_0 < r$ is equivalent to $P_1 < P_0$ since $P_0 < 1$ when $R_0 < 1$.Therefore by using theorem 2.3, the Proposition holds. Moreover, we can get the thresholds for control of the disease under this case. The threshold is P_0. By the definition of P_0, we know that P_0 is increasing with β increasing. When β is large Such that $P_0 > 1$, it follows from Theorem 2.3 that there is no backward bifurcation with endemic equilibria when $R_0 < 1$, see Fig. 2 (β = 0.05). When β is larger such that $\beta I_0 \geq r$, there does not have a backward bifurcation because the endemic equilibria $E_*\ and\ E^*$ do not exit according to Theorem 2.3, see Fig. 1(β = 0.2). Moreover, when, β is small such- that $P_0 < 1$, there is a backward bifurcation with endemic equilibria when $R_0 < 1$, see Fig. 3($\beta = 0.01$). This means that a small adequate contact rate is a source of the backward bifurcation. Similarly, we also find that an insufficient capacity for the treatment is a source of the backward bifurcation.

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

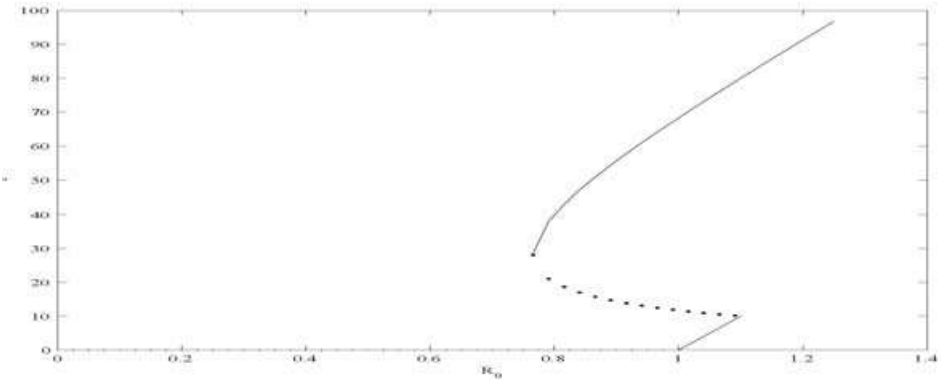


Figure 3. The figure of infective sizes at equilibria *versus* R_0, *when* $I_0 = 10, \beta = 0{,}01, d = 0.2$,r=0.8 that shows a backward bifurcation with endemic equilibria when $R_0 < 1$ where the Proposition 2.5 holds.

SIS Model with Disease-induced Death

Now we consider the system (1.1). The total population size

$N = S + I$ *and* $N' = (S + I)' = A - dN - \varepsilon I < A - dN$

It follows that $\lim_{t\to\infty} N(t) < \frac{A}{d}$ that means N is bounded. Let

$\Gamma = \left\{ (S, I) \in R^2 : S + I < \frac{A}{d}; \text{ such that } S \geq 0, I \geq 0 \right\}$, for

$S = 0 \Rightarrow S'(t) = A > 0,\ I = 0 \Rightarrow I'(t) = 0,\ S + I = 1 \Rightarrow (S + I)' = A - dN - \varepsilon I < A - dN$

Then Γ is positive invariant with respect to system (1.1). Since the total population is not constant when $\varepsilon \neq 0$, we do not reduce the dimension of the system to simple the computation.

Equilibrium of system under Disease-induced Death

Equilibrium of system (1.1) satisfies;

$$\begin{cases} A - dS - \beta SI + T(I) = 0 \\ \beta SI - (d + \varepsilon)I - T(I) = 0 \end{cases} \qquad (3.1)$$

Let $R_0 = \frac{A\beta}{d(d + r + \varepsilon)}$ then R_0 are a basic reproductive number of systems (1.1). When $0 \leq I \leq I_0$, we have two equilibrium points. Thus $E_0(\frac{A}{d}, 0)$ is a disease-free equilibrium, and E_0 is locally stable when $R_0 < 1$ *and* E_0 is a saddle, when $R_0 < 1$ by analyzing the Eigen values of the Jacobean matrix of system (1.1) at E_0. And

$E_1(\frac{d + r + \varepsilon}{\beta}, \frac{(R_0 - 1)d(d + r + \varepsilon)}{\beta(d + \varepsilon)})$ is an endemic equilibrium of system (1.1) if and only if

$1 < R_0 \leq \frac{I_0\beta(d + \varepsilon)}{d(d + r + \varepsilon)} + 1 \underline{\Delta} P_2$

when $I > I_0$, we solve S from the second equation (3.1) to obtain $S = \frac{M + (d + \varepsilon)I}{\beta I}$ Substituting into the first equation (3.1), we have

$\beta(d + \varepsilon)I^2 + (d^2 + d\varepsilon - A\beta)I + dM = 0$ 3.2

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

If $d^2+d\varepsilon-A\beta\geq 0,$, it is clear that equation (3.2) does not have a positive solution. Let $\Delta=(d^2+d\varepsilon-A\beta)^2-4\beta(d+\varepsilon)dM$, we know that equation (3.2) have positive solutions is equivalent to

$$\begin{cases} d^2+d\varepsilon-A\beta<0 \\ \Delta\geq 0 \end{cases} \Rightarrow R_0\geq 1-\frac{dr}{d(d+r+\varepsilon)}+\frac{2\sqrt{\beta(d+\varepsilon)dM}}{d(d+r+\varepsilon)}\underline{\underline{\Delta}}P_0 \qquad 3.3$$

Let us suppose (3.3) holds, then (3.2) has two positive solution I_* *and* I^* where

$$I_*=\frac{-(d^2+d\varepsilon-A\beta)-\sqrt{\Delta}}{2\beta(d+\varepsilon)},\ I^*=\frac{-(d^2+d\varepsilon-A\beta)+\sqrt{\Delta}}{2\beta(d+\varepsilon)}\ then,\ E_*(\frac{M+(d+\varepsilon)I_*}{\beta I_*},I_*),E^*(\frac{M+(d+\varepsilon)I^*}{\beta I^*},I^*)$$

is an endemic equilibrium of system (1.1) if $I_*>I_0, or\ I^*>I_0$. By the simple computation, we have $I_*>I_0\Rightarrow P_1\underline{\underline{\Delta}}1-\frac{dr}{d(d+r+\varepsilon)}+\frac{2\beta(d+\varepsilon)I_0}{d(d+r+\varepsilon)}<R_0<P_2$

$$I_*>I_0\Rightarrow R_o>P_1\ \ or\ P_2<R_o\leq P_1.$$

Summarizing the above discussions, we have the following conclusions.

Theorem 3.1.

$E_0(\frac{A}{d},0)$ is locally stable when $R_0<1$ and is a saddle when $R_0>1$

$E_1(\frac{d+r+\varepsilon}{\beta},\frac{(R_0-1)d(d+r+\varepsilon)}{\beta(d+\varepsilon)})$ exists and is locally stable if and only $1<R_0\leq P_2$.

The locally stability of E_1 is easy to obtain by analyzing the Jacobean matrices of system (1.1) at E_1 .

Theorem3.2 Endemic equilibria E_* and E^* do not exist if $R_0<P_0$. Further, if $R_0\geq P_0$ we have the following:

Both E_* and E^* exist when $\beta(d+\varepsilon)I_0<dr\ and\ R_0<P_2$

E_* does not exist but E^* exists when $\beta(d+\varepsilon)I_0<dr\ and\ R_0\geq P_2$

Let $\beta(d+\varepsilon)I_0\geq dr$. The E_* does not exist. Further, E^* is exists when $R_0>P_2$ and E^* does not exist when $R_0\leq P_2$

Theorem 3.2 can be easily obtained from the above analysis. We notice that $\beta(d+\varepsilon)I_0<dr$ is equivalent $P_1<P_2\ or\ P_1<P_0\ and\ \beta(d+\varepsilon)I_0\neq dr\ is\ equivalent\ to\ P_1<P_2$.

So we also can get the following Proposition.

Proposition3.3. If $P_1<1$, there is a backward bifurcation with endemic equilibria when $R_0<1$ in System (1.1).

The above conclusions are similar to the results in section 2. Moreover, P_0 increases with $\beta\ or\ I_0$ increasing by the definition. So we also can obtain that a small adequate contact rate or an insufficient capacity for the treatment is a source of the backward bifurcation. That is to say, when the adequate contact rate or capacity is large, there is no backward bifurcation (see Fig. 4).

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

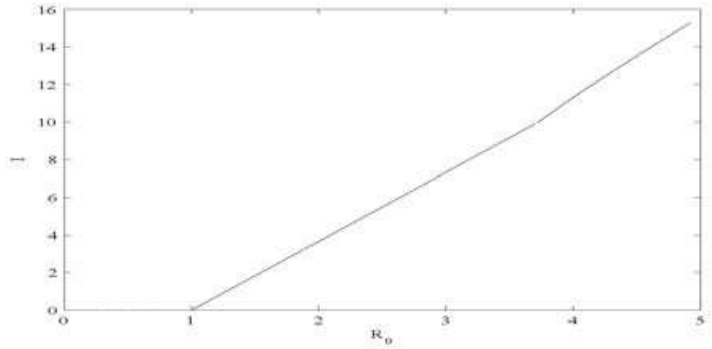


Figure4. The diagram of $I_0, I_1 and\ I^*$ versus R_0 when $R_0 = 10, \beta = 0.2, d = 0.2, \varepsilon = 0.1, r = 0.8,$ Where (iii) of Theorem3.2 holds. The bifurcation at $R_0 = 1$ is forward and (2.2) has one unique endemic equilibrium for all $R_0 > 1$.

When the adequate contact rate or capacity is small, there is a backward bifurcation with endemic equilibria when $R_0 > 1$ (see Fig. 5).With the adequate contact rate or capacity becomes more smaller, there will be a backward bifurcation with endemic equilibria when $R_0 < 1$ (see Fig. 6)

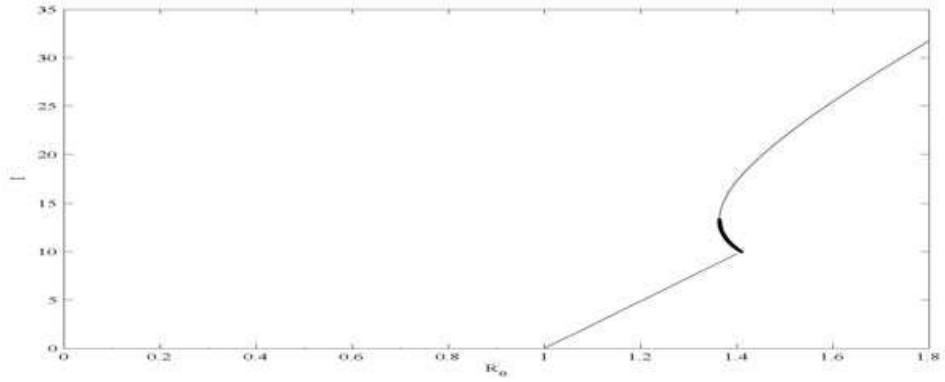


*figure*5 *The figureof inictivesize at equilibrum* R_0 *when* $I_0 = 10, \beta = 0.01, d = 0.2, r = 0.8, \varepsilon = 0.1.$ Where (i)of Theorem3.2 holds. The bifurcation from the disease-free equilibrium at $R_0 = 1$ is forward and there is a backward bifurcation from an endemic equilibrium at $R_0 = P_2 = 1.40909$.

v

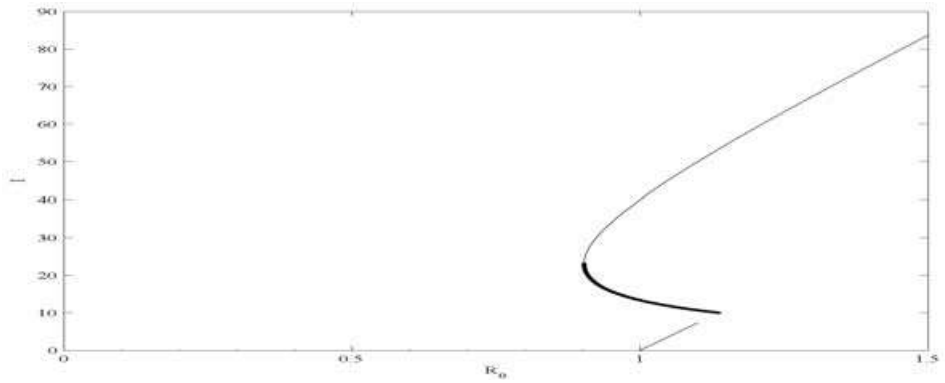


Figure.6.The figure of infective sizes at equilibria I_0 =10, β = 0.2, *d*= 0.2, *r* =*0*.8,ε = 0.1 .It shows a backward bifurcation with endemic equilibria when $R_0 < 1$ when Proposition

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

Moreover, we can get the thresholds to the control of disease in this case. The threshold is p_0 **Theorem3.4.**

$E_*(S_*, I_*)$ is a saddle whenever it exists.

When $E^*(S^*, I^*)$ exists, then E* is stable $A\beta + \sqrt{\Delta} < (d+\varepsilon)\sqrt{4\beta M + d^2}$ and is unstable.

If $A\beta + \sqrt{\Delta} > (d+\varepsilon)\sqrt{4\beta M + d^2}$

Proof (1): Now we begin by analyzing the stability of these two endemic equilibria. The Jacobin matrices of system (1.1) at E_* is

$$J_* = \begin{bmatrix} -d - \beta I_* & -\beta S_* \\ \beta I_* & \beta S_* - (d+\varepsilon) \end{bmatrix}$$

We have

$$\det(J_*) = (d+\beta I_*)(d+\varepsilon-\beta S_*) + \beta^2 S^* I^*$$

$$= d(d+\varepsilon) - (A\beta - \beta^2 S^* I^* + \beta M) + \beta^2 S_* I_* - \beta M$$

$= d^2 + d\varepsilon - A\beta + 2(d+\varepsilon)\beta I_* = -\sqrt{\Delta} < 0$, This implies eigen value is negative. It follows that (S_*, I_*) is a saddle point. By the same argument, we obtain that det $(J_*) = \sqrt{\Delta} > 0$ Moreover, we see that the trace of (J^*) is given by.

$$\mathrm{tr}(J^*) = -d - \beta I^* + \beta S^* - (d+\varepsilon) = \frac{-\beta I^2 - dI^* + M}{I^*}.$$

Let us verity that existence of a Hopf bifurcation in (1.1) and determine its direction. Set A_0 is the positive solution of the following equation.

$$A\beta + \sqrt{\Delta} = (d+\varepsilon)\sqrt{4\beta M + d^2}$$

Theorem3.5. Suppose the endemic equilibria of System (1.1) exist. Then there is a family of unstable limit cycles if A is greater than and close to A_0 , i.e., a subcritical Hopf bifurcation occurs when A passes through the critical value A_0

Proof. Suppose $A = A_0$ .Then $\mathrm{tr}(J^*) = 0$ we can obtain that

$$I^* = \frac{-d + \sqrt{4\beta M + d^2}}{2\beta}, S^* = \frac{\sqrt{4\beta M + d^2} + 3d + 2\varepsilon}{2\beta}$$

The eigenvalues of (J^*) are $\lambda_{1,2} = \pm \omega i$, where $\omega = \sqrt{\Delta} = (d^2 + d\varepsilon - A\beta)^2 - 4\beta(d+\varepsilon)dM$, Let $x = S - S^*, y = I - I^*,$ then System (1.1) becomes.

$$\begin{cases} \dfrac{dx}{dt} = -(d+\beta I^*)x - \beta S^* I - \beta xy \\ \dfrac{dy}{dt} = \beta I^* x + \beta S^* - (d+\varepsilon)y + \beta xy \end{cases}$$

Setting $x = -\beta S^* V, y = \omega U + (d+\beta^*)V$ we can obtain

$$\begin{cases} \dfrac{dU}{dt} = -\omega V + F(U,V), \\ \dfrac{dV}{dt} = \omega U + G(U,V) \end{cases}$$

Where $F(U,V) = \dfrac{\beta V(-\beta S^* + \beta I^* + d)(\omega U + dV + \beta I^* + V)}{\omega}$

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)

$$G(U,V) = -\beta V(\omega U + dV + \beta I^* V)$$

We know that tr $(J^*) = 0$, so we get $G(U,V) = \dfrac{\omega F(U,V)}{d+\varepsilon}$

Let $\mu = \dfrac{1}{16}\left[F_{UUU} + F_{UVV} + G_{UUV}\right] + \dfrac{1}{16\omega}\left[F_{UV}(F_{UU} + F_{VV}) - G_{UV}(G_{UU} + G_{VV}) - F_{UU}G_{UU} + F_{vv}G_{VV}\right]$.

By some calculations, we obtain

$$\mu = \frac{\beta^2(d+\beta I^*)(2d^2 + 3\varepsilon d + \varepsilon^2 + 2\beta S^* d + \varepsilon\beta I^*)}{8\omega^2} > 0$$

As an example, we fix I_0= 10, β= 0.04, d = 0.2, r = o.8, ε = 0.1. Then we obtain that A_0 = 8.430703310. We know that there is an unstable limit cycle when A is greater than and near A_0 from Theorem 3.5, which is shown the following figure (see Fig. 7)

At this stage, the local stability of the endemic equilibria of System (1.1) is clear. Next, we begin to study the global stability of the equilibria.

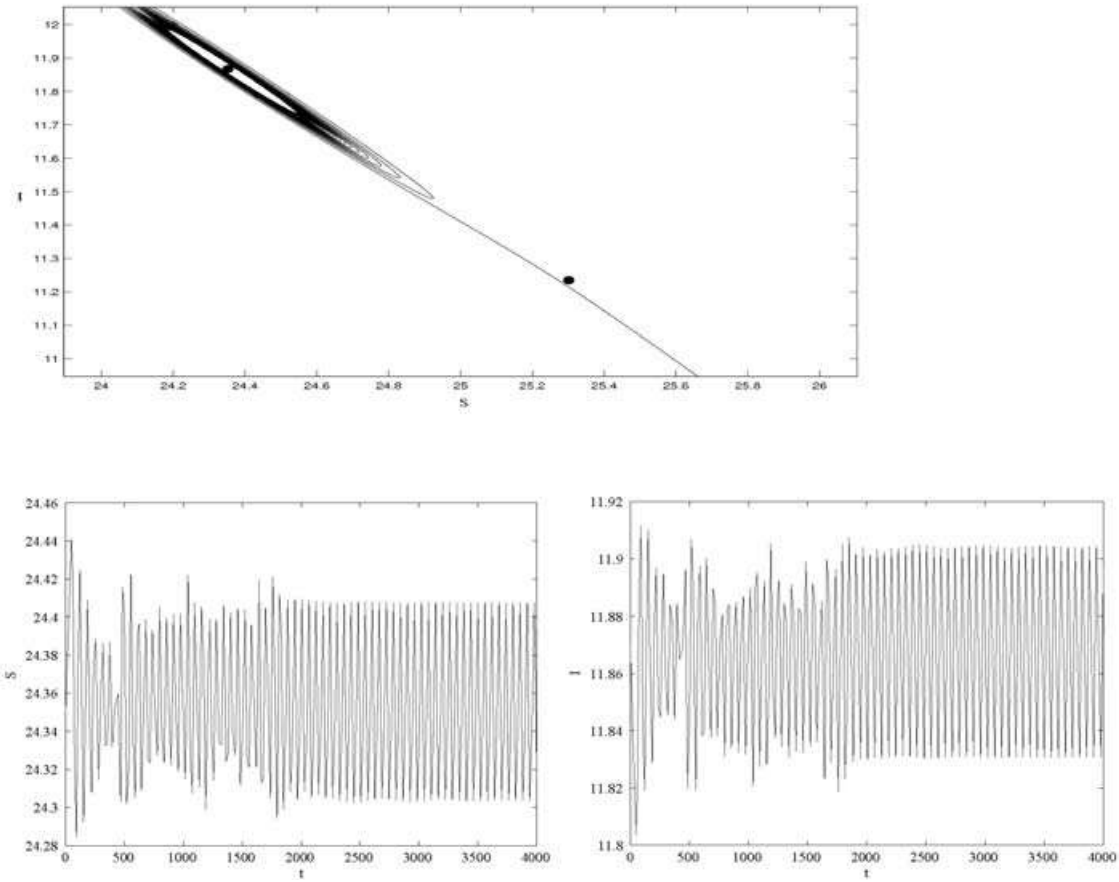
Theorem3.6. The disease-free equilibrium E_0 is globally asymptotically stable if $R_0 < 1$ and one of the following conditions is satisfied:

$$i, R_0 < p_0, ii, R_0 < p_1$$

Proof. Let $R_0 < 1$ implies E_0 that does not exist. Suppose $R_0 < p_0$. It follows from the discussions for the **Theorem 3.2** at exists E_* and E^* only if $R_0 < p_0$, which is impossible since we have $R_0 < 1$.Let us now suppose $p_0 < 1$ and $p_0 \geq 1$. If $I_0\beta(d+\varepsilon) < dr$, since $p_1 < p_2$, it follows from the discussions for (i),(ii),of that E_* or E^*.**Theorem 3.2** exists only if $R_0 < p_0$, which is impossible since we have R_0 1.If $I_0\beta(d+\varepsilon) < dr$, since $1 < p_2$, it follows from , (iii) of **Theorem 3.2** that E_* and E^* do not exist

It is easy to verify that positive solutions of are ultimately bounded. Note that the nonnegative S-axis is positively invariant and that the nonnegative I-axis repels positive solutions of (1.1). Since E_0 is asymptotically stable, it follows from the Bendixson Theorem that every positive solution of (1.1) approaches E_0 as t approaches infinity. The limit cycles of (1.1) play crucial roles on the structure of dynamical behaviours of the model. For example; if there is no limit cycle and its endemic equilibrium is unique, the unique endemic equilibrium is globally stable. For this reason, we adopt Dulac functions to obtain conditions for the nonexistence of a limit cycle in (1.1). Generally speaking, the approach of Dulac functions applies to smooth vector fields. However, it is applicable to (1.1).Where the vector field defined by (1.1) is not smooth at the line $I = I_0$. Indeed, denote the right-hand sides of (1.1) by f1 andf2.

International Research Journal of Natural Sciences
Vol.8, No.1, pp.7-21, March 2020
Published by ECRTD-UK
Print ISSN: ISSN 2053-4108(Print)
Online ISSN: ISSN 2053-4116(Online)



Figur7. An unstable limit cycle when $I_0 = 10$, $\beta = 0.04$, d =0.2, r = 0.8, ε = 0.1,A = 8.4308.

Theorem3.7. System (1.1) does not have a limit cycle if $rA < dA + rdI_0$ Proof.We know Γ is a positive invariant set, so we have $S < \frac{A}{d}$ .Take a Dulac function $D = \frac{1}{SI}$.*If* $0 < I < I_0$ we have $\frac{\partial(Df_1)}{\partial S} + \frac{\partial(Df_2)}{\partial I} = \frac{A}{S^2 I} - \frac{r}{S^2} < 0$. from equation (1,1)

If $I > I_0$ it is easy to see that

$$\frac{\partial(Df_1)}{\partial S} + \frac{\partial(Df_2)}{\partial I} = \frac{A}{S^2 I} - \frac{M}{SI^*} \le \frac{I_0}{S^2 I^2}(rS - rI_0) < \frac{I_0}{S^2 I^2}\left(r\frac{A}{d} - rI_0 - A\right) < 0.$$

Hence System (1.1) does not have a limit cycle.

Theorem 3.7 implies that there is no limit cycle in System (1.1) if $rA < dA + rdI_0$ is to say, if the treatment rate is less than the death rate, there will no limit cycle.

Remark3.8. In the above example $rA = 6.74464 > dA + rdI_0 = 3.28616$ and the condition of Theorem 3.7 is violated . So there is an unstable limit cycle, Theorem 3.5 is not in contradiction to Theorem 3.7.

CONCLUSSION

We have examined a simple SIS model with treatment. Firstly, we do not consider the disease-induced death. By the theory of asymptotically autonomous system, we can easily obtain the global behaviour. In the case of limited resources, that is to say we use the function $T(I)$ as the

treatment rate, we find that the small adequate contact rate or insufficient capacity can induce the backward bifurcation. We also show that the bistable endemic equilibria occur under this case. Secondly, when we consider the case of disease-induced death, we obtain more complex dynamical behaviour. It is shown that there is a backward bifurcation in some conditions, and the subcritical Hopf bifurcation occurs in some conditions, and there is no the control of disease when backward bifurcation occurs. The threshold is $P_{0.}$ This means that driving the basic reproductive number below one is not enough to eradicate the disease. Although the small adequate contact rate or insufficient capacity for the treatment may lead to backward bifurcations, we emphasize that it always decreases the spread of disease and the infective population size.

Bibliography

[1] Acevedo, H.G., Li, Michael Y. Backward bifurcation in a model for HTLV-I infection of CD4+ T cells. Bulletin of Mathematical Biology, 67: 101–114 (2005
[2] Brauer, F. Backward bifurcation in simple vaccination models. J. Math. Anal. Appl., 289: 418–431 (2004)
[3] Dushoff, J., Huang, W., C-Chavez, C. Backward bifurcation and catastrophe in simple models of fatal diseases. J. Math. Biol., 36: 227–248 (1998)
[4] Hui, J., Zhu, D.M. Global stability and periodicity on SIS epidemic models with backward bifurcation.Comp. Math. Appl., 50: 1271–1290 (2005)
[5] Hadeler, K.P., C-Chavez, C. A core group model for disease transmission. Math. Biosci., 128: 41–55(1995)
[6] Hadeler, K.P., Van den driessche, P. Backward bifurcation in epidemic control. Math.Biosci., 146: 1533(1997)
[7] K-Zaleta, C.M., V-Hern´andez, Jorge X. A simple vaccination model with multiple endemistates. Math.Biosci., 164: 183–201 (2000)
[8] Li, J.Q., Ma, Z.E., Zhou, Y.C. Global analysis of SIS epidemic model with a simple vaccination andmultipleendemic equilibria. Acta.Math.Scientia, 300: 273–284 (2004)
[9] Perko, L. Differential Equations and Dynamical Systems. Springer-Verlag, New York, 1996
[10]] Thieme,H.R. Asymptotically autonomous differential equations in the plane. Rocky Mountain J.Math.,24: 351–380 (1994)
[11] Van den Driesch, P., Watmough, James. A simple SIS epidemic model with a backward bifurcation.J.Math. Biol., 40: 525–540 (2000)
[12] Wang, W.D. Backward bifurcation of an epidemic model with treatment. Math.Biosci., 201: 58–71 (2006)
[13] H. W. Hethcote, The mathematics of infectious diseases, SIAM Rev., 42 (2000), 599-653
[14] W. Wang and S. Ruan, “Bifurcation in an epidemic model with constant removal rate of the infectives,”Journal of Mathematical Analysis and Applications, vol. 291, no. 2, pp. 775–793, 2004
[15] Hethcote H.W, Tudor D.W (1980). Integral equation models for epidemic infectious diseases, J. Math. Biol, 9, pp.37-47.
[16] W.M. Liu, H.W. Hethcote, S.A. Levin, Dynamical behavior of epidemiological models with nonlinear incidence rates, J. Math. Biol. 25 (1987) 359–380.

[17] Hethcote H.W, Driessche P.van den (1991). Some Epidemiological model with nonlinear incidence,J. Math.Biol, 29, pp. 271-287.
[18] Derrick W.R, Driessche P. van den (1993). A disease transmission model in non-constant population, J. Math. Biol. 31, pp. 495-512.
[19] Beretta E, Takeuchi Y (1995). Global stability of a SIR epidemic model with time delay, J. Math. Biol. 33, pp. 250-260.
[20] Beretta E, Takeuchi Y (1997). Convergence results in SIR epidemic model with varying population sizes, Nonlinear Anal. 28, pp. 1909-1921.
[21] Beretta E, Hara T, Ma W, Takeuchi Y (2001). Global asymptotic stability of a SIR epidemic model with distributed time delay, Nonlinear Anal. 47, pp. 4107-4115
[22] [22] Ma W, Hara T, Beretta E, Takeuchi Y (2002). Permanence of an SIR epidemic model with distributed time delays, Tohoku Math. J. 54, pp. 581-591.
[23] [23] Runa S, Wang W (2003). Dynamical behavior of an epidemic model with nonlinear incidence rate, J. Differ. Equations, 188, pp. 135-163.
[24] [24] Ma W, Song M, Takeuchi Y (2004). Global stability of an SIR epidemic model with time delay, Appl. Math.Lett. 17, pp.1141-1145.