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# APPLICATION OF THE ELZAKI TRANSFORM ITERATIVE METHOD FOR THE FOKKER-PLANCK EQUATION

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**ABSTRACT:** *In this Paper, I propose a New Integral Transform which is still not widely known, nor used. The aim of the present paper is to investigate the application of the Elzaki Transform with combination of simple iteration Method for solving the Fokker-Planck equation and some similar equations. The method can easily be applied to many linear and non-linear Partial Differential Equations and is capable to reduce the size of computational work. In this approach the solution is found in the form of a convergent series with easily computed components. To give overview of Methodology, I have presented several examples in one- and two-dimensional cases.*
**KEY WORDS***:* Elzaki Transform, Fokker-Planck Equation, Kolmogorov Equation, Analytical Solution.

## INTRODUCTION

In the last decade, many of new methods were used to solve linear and non-linear partial differential equations arising from the Mathematical Modelling of problems in Mathematics, Physics, Engineering and Other various branches of science which have imperative applications in a real life such as Adomain Decomposition Method (ADM)[1,2], Variational Iteration Method (VIM) [3], Homotopy Perturbation Method (HPM)[4,5], Homotopy Analysis Method (HAM) [6,7] , New Iterative Method (NIM) [8,9], Differential Transform Method (DTM) [10,11], Laplace Transform Method (LTM) [12], and Other Methods.

Elzaki Transform first proposed by Tarig M. Elazki and Salih M. Elzaki [13] in 2011. This Method is Highly recommended to solve linear and Non-linear differential Equation with shorten calculation part compare to other Methods. For example, Adomain decomposition Method [14].
In this Paper, I solve Fokker-Planck equation (FPE) which was first applied to investigate the Brownian motion of particles, is now largely employed in various generalized forms in Physics, Engineering biology and Chemistry [15].
The motivation of this work is to extend the application of Elzaki Transform for solving Linear and Non-linear FPEs.

**Fokker-Planck Equation (FPE):**

The general form of FPE for variables x and t is as follows, according to [2,7,9,11,12]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x)\right]u(x,t) \tag{2.1}$$

With the initial condition $u(x,0) = f(x),\ x \in \mathbb{R}$ (2.2)

$Where\ A(x)\ is\ the\ drift\ coefficient\ and\ B(x) > 0\ is\ the\ diffusion\ coefficient.$

The drift and diffusion coefficients can also be functions of x and t that is

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)\right]u(x,t) \tag{2.3}$$

Equation (2.1) is a linear second order partial differential equation of parabolic type called Kolmogorov equation, which is an equation for the motion of a concentration field $u(x,t)$.

The backward Kolmogorov equation can be written in the following form:

$$\frac{\partial u}{\partial t} = \left[-A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2}\right]u(x,t) \tag{2.4}$$

A generalized form of equation (2.1) for N variables $x_1, x_2, x_3, \dots\dots\dots, x_N$ can be written as follows:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x)\right]u(x,t) \tag{2.5}$$

With the initial condition $u(x,0) = f(x),\ x = (x_1, x_2, x_3, \dots\dots\dots, x_N) \in \mathbb{R}^N$ (2.6)

Where the drift vector $A_i$and the diffusion tensor $B_{i,j}$ in equation (2.5) depend on N variables $x_1, x_2, x_3, \dots\dots\dots, x_N$.

There is a more general form of FPE, which is non-linear FPE. Nonlinear FPE has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, Engineering, Neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern for motion, psychology and Marketing.

The nonlinear FPE for one variable is in the following form:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x}A(x,t,u) + \frac{\partial^2}{\partial x^2}B(x,t,u)\right]u(x,t) \tag{2.7}$$

Equation (2.7) for N variables $x_1, x_2, x_3, \dots\dots\dots, x_N$ is given by below equation.

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}A_i(x,t,u) + \sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x,t,u)\right]u(x,t) \tag{2.8}$$

$Where\ x = (x_1, x_2, x_3, \dots \dots \dots ., x_N) \in \mathbb{R}^N$

**Elzaki Transform Iterative Method (ETIM):**
The Basic definition of Elzaki Transform is given below

A New integral transform called Elzaki transform [16–19] defined for functions of exponential order is proclaimed. We consider functions in the set A defined by

$$A = \left\{ f(t): \exists M, K_1, K_2 > 0, |f(t)| < Me^{\frac{|t|}{K_j}}, if\ t \in (-1)^j \times\ [0, \infty) \right\}$$

Definition: If $f(t)$ is function defined for all $t \geq 0$, its Elzaki transform is the integral of $f(t)$ times $e^{-\frac{t}{v}}\ from\ t = 0\ to\ \infty$. It is a function of $v$ and is defined as

$$E[f(t)] = T(v) = v \int_0^\infty f(t)\, e^{-\frac{t}{v}}\ dt \qquad v \in (K_1, K_2)$$

$$or\ equivalently\ T(v) = v^2 \int_0^\infty f(vt)\, e^{-t}\ dt \qquad K_1, K_2 > 0$$

Theorem 1: Elzaki transform amplifies the coefficients of the power series function,

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \tag{3.1}$$

On the new integral transform “Elzaki Transform” is

$$E\{f(t)] = T(v) = \sum_{n=0}^{\infty} n!\, a_n v^{n+2} \tag{3.2}$$

Theorem:2 Let $f(t)$ be in A and Let $T_n(v)$ denote Elzaki transform of nth derivative, $f^n(t)\ of\ f(t), then\ for\ n \geq 1,$

$$T_n(v) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k}\, f^{(k)}(0) \tag{3.3}$$

To obtain Elzaki transform of partial derivative we use integration by parts, and then we have

$$E\left(\frac{\partial f(x,t)}{\partial t}\right)=\frac{1}{v}T(x,v)-vf(x,0),$$

$$E\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right)=\frac{1}{v^2}T(x,v)-f(x,0)-v\frac{\partial f(x,0)}{\partial t} \qquad (3.4)$$

Properties of Elzaki transform

1. $E(1)=v^2$
2. $E(t^n)=n!\,v^{n+2}$
3. $E(t)=v^3$
4. $E^{-1}(v^{n+2})=\frac{t^n}{n!}$
5. $E\left(\frac{\partial f(x,t)}{\partial x}\right)=\frac{d}{dx}\big(f(x,t)\big)$
6. $E\left(\frac{\partial^2 f(x,t)}{\partial x^2}\right)=\frac{d^2}{dx^2}\big(f(x,t)\big)$

To illustrate the basic idea of the ETIM for linear and non-linear partial differential equation. The very first step is applying Elzaki transform on both side of the below equation

$$\frac{\partial^m u(x,t)}{\partial t^m}+R\,.u(x,t)+N.u(x,t)=g(x,t), Where\ m\ =\ 1,2,3,\ldots. \qquad (3.5)$$

$$With\ the\ initial\ conditions\quad \left.\frac{\partial^m u(x,t)}{\partial t^m}\right|_{t=0}=\ f_{m-1}(x), Where\ m\ =\ 1,2,3,\ldots.$$

Where $\frac{\partial^m u(x,t)}{\partial t^m}$ is the Partial derivative of the function $u(x,t)$ of order m, R is the linear differential operator, N represents the general nonlinear differential operator and g(x, t) is the source term. Applying the Elzaki Transform (denoted by in this paper is E) on both side of equation, we get

$$v^{-m}E[u(x,t)]=\sum_{k=0}^{m-1}v^{2-m+k}\frac{\partial^k u(x,0)}{\partial t^k}+E[g(x,t)]-E[R.u(x,t)+N.u(x,t)] \qquad (3.6)$$

$Where\ m\ =\ 1,2,3,\ldots.\ and\ thus\ we\ have$

$$E[u(x,t)]=\ \sum_{k=0}^{m-1}v^{2+k}\frac{\partial^k u(x,0)}{\partial t^k}+v^mE[g(x,t)]-v^mE[R.u(x,t)+N.u(x,t)] \qquad (3.7)$$

Now, by operating Inverse transform on both side of the equation we get

$$u(x,t)=G(x,t)-E^{-1}\big[v^mE[R.u(x,t)+N.u(x,t)]\big] \qquad (3.8)$$

Where $G(x,t)$ represent the term arising from the source term and the given initial condition.

The next step in Elzaki Transform Iterative Method is that we represent the solution as an infinite series given below and linear and non-linear term given by simple recurrence relation

$u_0 = u(x,0)$

$$u_{r+1} = [R(u_r) + N(u_r)], Where\ r = 0,1,2,3 \ldots \ldots \tag{3.9}$$

Finally, we approximate the analytical solution $u(x,t)$ as follows

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \ldots \ldots ..$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.10}$$

The above series solution generally converges very rapidly.

**Application of ETIM to FPEs:**

**<u>Example: 1</u>** Consider Equation (2.1) with the initial condition $u(x,0) = x,\ x \in \mathbb{R}$

Let $A(x) = -1\ and\ B(x) = 1$ in equation (2.1).

By applying Elzaki Transform on equation by putting these values as below

$$E\left(\frac{\partial u}{\partial t}\right) = E\left[\left[-\frac{\partial}{\partial x}(-1) + \frac{\partial^2}{\partial x^2}(1)\right] u(x,t)\right]$$

We get solution according to equation (3.6) as follow

$$v^{-1}E[u(x,t)] = v.u(x,0) + E\left[\frac{\partial}{\partial x}u(x,t) + \frac{\partial^2}{\partial x^2}u(x,t)\right]$$

$$E[u(x,t)] = v^2.x + vE\left[\frac{\partial}{\partial x}u(x,t) + \frac{\partial^2}{\partial x^2}u(x,t)\right]$$

Now by Applying inverse Elzaki Transform on above equation we get

$$E^{-1}\big[E[u(x,t)]\big] = E^{-1}\left[v^2.x + vE\left[\frac{\partial}{\partial x}u(x,t) + \frac{\partial^2}{\partial x^2}u(x,t)\right]\right]$$

$$u = \left[x + E^{-1}\left[vE\left[\frac{\partial}{\partial x}u(x,t) + \frac{\partial^2}{\partial x^2}u(x,t)\right]\right]\right]$$

Now, the recursive relation is given as below according to equation (3.9)

$$u_0 = x$$

$$u_1 = E^{-1}\left[vE\left[\frac{\partial}{\partial x}u_0 + \frac{\partial^2}{\partial x^2}u_0\right]\right] = t$$

$$u_2 = E^{-1}\left[vE\left[\frac{\partial}{\partial x}u_1 + \frac{\partial^2}{\partial x^2}u_1\right]\right] = 0$$

$$u_3 = E^{-1}\left[vE\left[\frac{\partial}{\partial x}u_2 + \frac{\partial^2}{\partial x^2}u_2\right]\right] = 0$$

$$\dots\dots\dots\dots$$

Finally, we approximate the analytical solution $u(x,t)$ as follows

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \dots\dots$$
$$= x + t + 0 + 0 + \cdots$$
$$u(x,t) = x + t$$

**Example: 2** Consider Equation (2.3) with the initial condition $u(x,0) = \sinh x\,,\ x \in \mathbb{R}$
Let $A(x,t) = e^t \coth x \cosh x + e^t \sinh x - \coth x$ *and* $B(x,t) = e^t \cosh x$ in equation (3).

By applying Elzaki Transform on equation by putting these values as below

$$E\left(\frac{\partial u}{\partial t}\right) = E\left[\left[-\frac{\partial}{\partial x}(e^t \coth x \cosh x + e^t \sinh x - \coth x) + \frac{\partial^2}{\partial x^2}(e^t \cosh x)\right]u(x,t)\right]$$

We get solution according to equation (3.6) as follow

$$v^{-1}E[u(x,t)] = v.u(x,0) + E\left[-\frac{\partial}{\partial x}\{(e^t \coth x \cosh x + e^t \sinh x - \coth x).u(x,t)\} + \frac{\partial^2}{\partial x^2}\{(e^t \cosh x).u(x,t)\}\right]$$

$$E[u(x,t)] = v^2.\sinh x + vE\left[-\frac{\partial}{\partial x}A(x,t)u(x,t) + \frac{\partial^2}{\partial x^2}B(x,t)u(x,t)\right]$$

Now by Applying inverse Elzaki Transform on above equation we get

$$E^{-1}\big[E[u(x,t)]\big] = E^{-1}\left[v^2.\sinh x + vE\left[-\frac{\partial}{\partial x}A(x,t).u(x,t) + \frac{\partial^2}{\partial x^2}B(x,t).u(x,t)\right]\right]$$

$$u = \left[\sinh x + E^{-1}\left[vE\left[-\frac{\partial}{\partial x}A(x,t).u(x,t) + \frac{\partial^2}{\partial x^2}B(x,t).u(x,t)\right]\right]\right]$$

Now, the recursive relation is given as below according to equation (3.9)

$$u_0 = \sinh x$$

$$u_1 = E^{-1}\left[vE\left[-\frac{\partial}{\partial x}A(x,t).u_0 + \frac{\partial^2}{\partial x^2}B(x,t).u_0\right]\right] = \sinh x\,.\,t$$

$$u_2 = E^{-1}\left[vE\left[-\frac{\partial}{\partial x}A(x,t).u_1 + \frac{\partial^2}{\partial x^2}B(x,t).u_1\right]\right] = \sinh x\,.\frac{t^2}{2!}$$

$$u_3 = E^{-1}\left[vE\left[-\frac{\partial}{\partial x}A(x,t).u_2 + \frac{\partial^2}{\partial x^2}B(x,t).u_2\right]\right] = \sinh x\,.\frac{t^3}{3!}$$

$$\dots\dots\dots\dots$$

Finally, we approximate the analytical solution $u(x,t)$ as follows

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots\dots\dots..$$

$$= \sinh x + \sinh x\,.\,t + \sinh x\,.\frac{t^2}{2!} + \sinh x\,.\frac{t^3}{3!} + \sinh x\,.\frac{t^4}{4!} + \cdots$$

$$= \sinh x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\dots\dots.\right)$$

$$u(x,t) = e^t \sinh x$$

**Example: 3** Consider the backward Kolmogorov Equation (2.4) with the drift and diffusion coefficients given by below respectively

$A(x,t) = -(x+1)\ and\ B(x,t) = x^2e^t$

the initial condition $u(x,0) = x+1,\ x \in \mathbb{R}$

By applying Elzaki Transform on equation by putting these values as below

$$E\left(\frac{\partial u}{\partial t}\right) = E\left[\left[(x+1)\frac{\partial}{\partial x} + (x^2e^t)\frac{\partial^2}{\partial x^2}\right]u(x,t)\right]$$

We get solution according to equation (3.6) as follow

$$v^{-1}E[u(x,t)] = v.u(x,0) + E\left[(x+1)\frac{\partial}{\partial x}\{u(x,t)\} + (x^2e^t)\frac{\partial^2}{\partial x^2}\{u(x,t)\}\right]$$

$$E[u(x,t)] = v^2.(x+1) + vE\left[(x+1)\frac{\partial}{\partial x}u(x,t) + (x^2e^t)\frac{\partial^2}{\partial x^2}u(x,t)\right]$$

Now by Applying inverse Elzaki Transform on above equation we get

$$E^{-1}\big[E[u(x,t)]\big] = E^{-1}\left[v^2.(x+1) + vE\left[(x+1)\frac{\partial}{\partial x}u(x,t) + (x^2e^t)\frac{\partial^2}{\partial x^2}u(x,t)\right]\right]$$

$$u = \left[(x+1) + E^{-1}\left[vE\left[(x+1)\frac{\partial}{\partial x}u(x,t) + (x^2e^t)\frac{\partial^2}{\partial x^2}u(x,t)\right]\right]\right]$$

Now, the recursive relation is given as below according to equation (3.9)

$$u_0 = \sinh x$$

$$u_1 = E^{-1}\left[vE\left[(x+1)\frac{\partial}{\partial x}u_0 + (x^2e^t)\frac{\partial^2}{\partial x^2}u_0\right]\right] = (x+1).t$$

$$u_2 = E^{-1}\left[vE\left[(x+1)\frac{\partial}{\partial x}u_1 + (x^2e^t)\frac{\partial^2}{\partial x^2}u_1\right]\right] = (x+1).\frac{t^2}{2!}$$

$$u_3 = E^{-1}\left[vE\left[(x+1)\frac{\partial}{\partial x}u_2 + (x^2e^t)\frac{\partial^2}{\partial x^2}u_2\right]\right] = (x+1).\frac{t^3}{3!}$$

$$\dots\dots\dots\dots$$

Finally, we approximate the analytical solution $u(x,t)$ as follows

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots\dots\dots..$$

$$= (x+1) + (x+1).t + (x+1).\frac{t^2}{2!} + (x+1).\frac{t^3}{3!} + (x+1).\frac{t^4}{4!} + \cdots$$

$$= (x+1)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\dots\dots.\right)$$

$$u(x,t) = (x+1).e^t$$

**Example: 4** Consider the Equation (2.5) with:

$A_1(x_1,x_2) = x_1$ *and* $A_2(x_1,x_2) = 5x_2$

$B_{1,1}(x_1,x_2) = {x_1}^2$

$B_{1,2}(x_1,x_2) = 1$

$B_{2,1}(x_1,x_2) = 1$

$B_{2,2}(x_1,x_2) = {x_2}^2$

With initial condition $u(x,0) = x_1, where\ x = (x_1,x_2) \in \mathbb{R}^2$

By applying Elzaki Transform on equation by putting these values as below

$$E\left(\frac{\partial u}{\partial t}\right) = E\left[\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x) + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x)\right]u(x,t)\right]$$

We get solution according to equation (3.6) as follow

$$v^{-1}E[u(x,t)] = v.u(x,0) + E\begin{bmatrix} -\frac{\partial}{\partial x_1}\{x_1.u(x,t)\} - \frac{\partial}{\partial x_2}\{5x_2.u(x,t)\} + \frac{\partial^2}{\partial x_1 \partial x_1}\{x_1{}^2.u(x,t)\} \\ +\frac{\partial^2}{\partial x_1 \partial x_2}\{1.u(x,t)\} + \frac{\partial^2}{\partial x_2 \partial x_1}\{1.u(x,t)\} + \frac{\partial^2}{\partial x_2 \partial x_2}\{x_2{}^2.u(x,t)\}\end{bmatrix}$$

$$E[u(x,t)] = v^2.x_1 + vE\left[\left(-\frac{\partial}{\partial x_1}\{x_1\} - \frac{\partial}{\partial x_2}\{5x_2\}\right)u(x,t) + \left(\frac{\partial^2}{\partial x_1 \partial x_1}\{x_1{}^2\} + \frac{\partial^2}{\partial x_1 \partial x_2}\{1\} + \frac{\partial^2}{\partial x_2 \partial x_1}\{1\} + \frac{\partial^2}{\partial x_2 \partial x_2}\{x_2{}^2\}\right)u(x,t)\right]$$

Now by Applying inverse Elzaki Transform on above equation we get

$$E^{-1}\big[E[u(x,t)]\big] = E^{-1}\left[v^2.x_1 + vE\left[\left(-\frac{\partial}{\partial x_1}\{x_1\} - \frac{\partial}{\partial x_2}\{5x_2\}\right)u(x,t) + \left(\frac{\partial^2}{\partial x_1 \partial x_1}\{x_1{}^2\} + \frac{\partial^2}{\partial x_1 \partial x_2}\{1\} + \frac{\partial^2}{\partial x_2 \partial x_1}\{1\} + \frac{\partial^2}{\partial x_2 \partial x_2}\{x_2{}^2\}\right)u(x,t)\right]\right]$$

$$u = \left[x_1 + E^{-1}\left[vE\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x).u(x,t) + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x).u(x,t)\right]\right]\right]$$

Now, the recursive relation is given as below according to equation (3.9)

$$u_0 = x_1$$

$$u_1 = E^{-1}\left[vE\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x).u_0 + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x).u_0\right]\right] = x_1.t$$

$$u_2 = E^{-1}\left[vE\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x)\,.\,u_1 + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x)\,.\,u_1\right]\right] = x_1.\frac{t^2}{2!}$$

$$u_3 = E^{-1}\left[vE\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x)\,.\,u_2 + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x)\,.\,u_2\right]\right] = x_1.\frac{t^3}{3!}$$

$$\dots\dots\dots\dots$$

Finally, we approximate the analytical solution $u(x,t)$ as follows

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \dots \dots$$
$$= x_1 + x_1.t + x_1.\frac{t^2}{2!} + x_1.\frac{t^3}{3!} + x_1.\frac{t^4}{4!} + \cdots$$
$$= x_1\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \dots \dots.\right)$$
$$u(x,t) = x_1.e^t$$

**Example: 5** Consider the non-linear Fokker-Planck Equation (2.7) with
Initial condition $u(x,0) = x^2, x \in \mathbb{R}$

$$A(x,t,u) = 4\frac{u}{x} - \frac{x}{3} \text{ and } B(x,t,u) = u$$

By applying Elzaki Transform on equation by putting these values as below

$$E\left(\frac{\partial u}{\partial t}\right) = E\left[\left[-\frac{\partial}{\partial x}\left(4\frac{u}{x} - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}(u)\right]u(x,t)\right]$$

We get solution according to equation (3.6) as follow

$$v^{-1}E[u(x,t)] = v.u(x,0) + E\left[-\frac{\partial}{\partial x}\left\{\left(4\frac{u}{x} - \frac{x}{3}\right)u(x,t)\right\} + \frac{\partial^2}{\partial x^2}\{u.u(x,t)\}\right]$$

$$E[u(x,t)] = v^2.x^2 + vE\left[-\frac{\partial}{\partial x}\left\{\left(4\frac{u}{x} - \frac{x}{3}\right)u(x,t)\right\} + \frac{\partial^2}{\partial x^2}\{u.u(x,t)\}\right]$$

Now by Applying inverse Elzaki Transform on above equation we get

$$E^{-1}\big[E[u(x,t)]\big] = E^{-1}\left[v^2.x^2 + vE\left[-\frac{\partial}{\partial x}\left\{\left(4\frac{u}{x} - \frac{x}{3}\right)u(x,t)\right\} + \frac{\partial^2}{\partial x^2}\{u.u(x,t)\}\right]\right]$$

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$$u = \left[x^2 + E^{-1}\left[vE\left[-\frac{\partial}{\partial x}\left\{\left(4\frac{u}{x} - \frac{x}{3}\right)u(x,t)\right\} + \frac{\partial^2}{\partial x^2}\{u.u(x,t)\}\right]\right]\right]$$

Now, the recursive relation is given as below according to equation (3.9)

$$u_0 = x^2$$

$$u_1 = E^{-1}\left[vE\left[-\frac{\partial}{\partial x}\left\{\left(4\frac{u_0}{x} - \frac{x}{3}\right)u_0\right\} + \frac{\partial^2}{\partial x^2}\{u_0.u_0\}\right]\right] = x^2.t$$

$$u_2 = E^{-1}\left[vE\left[-\frac{\partial}{\partial x}\left\{\left(4\frac{u_1}{x} - \frac{x}{3}\right)u_1\right\} + \frac{\partial^2}{\partial x^2}\{u_1.u_1\}\right]\right] = x^2.\frac{t^2}{2!}$$

$$u_3 = E^{-1}\left[vE\left[-\frac{\partial}{\partial x}\left\{\left(4\frac{u_2}{x} - \frac{x}{3}\right)u_2\right\} + \frac{\partial^2}{\partial x^2}\{u_2.u_2\}\right]\right] = x^2.\frac{t^3}{3!}$$

$$\dots\dots\dots\dots$$

Finally, we approximate the analytical solution $u(x,t)$ as follows

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \dots\dots..$$

$$= x^2 + x^2.t + x^2.\frac{t^2}{2!} + x^2.\frac{t^3}{3!} + x^2.\frac{t^4}{4!} + \cdots$$

$$= x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \dots\dots.\right)$$

$$u(x,t) = x^2.e^t$$

**Example: 6** Consider the non-linear Fokker-Planck Equation (2.8) with
Initial condition $u(x,0) = {x_1}^2, x = (x_1, x_2) \in \mathbb{R}^2$

$A_1(x,t,u) = \dfrac{4}{x_1}u$ *and* $A_2(x,t,u) = x_2$

$B_{1,1}(x_1,x_2) = u$

$B_{1,2}(x_1,x_2) = 1$

$B_{2,1}(x_1,x_2) = 1$

$B_{2,2}(x_1,x_2) = u$

With initial condition $u(x,0) = {x_1}^2, where\ x = (x_1, x_2) \in \mathbb{R}^2$

By applying Elzaki Transform on equation by putting these values as below

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$$E\left(\frac{\partial u}{\partial t}\right) = E\left[\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x,t,u) + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x,t,u)\right]u(x,t)\right]$$

We get solution according to equation (3.6) as follow

$$v^{-1}E[u(x,t)] = v.u(x,0) + E\left[\begin{array}{l} -\frac{\partial}{\partial x_1}\left\{\frac{4u}{x_1}.u(x,t)\right\} - \frac{\partial}{\partial x_2}\{x_2.u(x,t)\} + \frac{\partial^2}{\partial x_1 \partial x_1}\{u.u(x,t)\} \\ + \frac{\partial^2}{\partial x_1 \partial x_2}\{1.u(x,t)\} + \frac{\partial^2}{\partial x_2 \partial x_1}\{1.u(x,t)\} + \frac{\partial^2}{\partial x_2 \partial x_2}\{u.u(x,t)\} \end{array}\right]$$

$$\begin{aligned} E[u(x,t)] = v^2.{x_1}^2 & \\ &+ vE\left[\left(-\frac{\partial}{\partial x_1}\{x_1\} - \frac{\partial}{\partial x_2}\{x_2\}\right)u(x,t)\right. \\ &\left. + \left(\frac{\partial^2}{\partial x_1 \partial x_1}\{u\} + \frac{\partial^2}{\partial x_1 \partial x_2}\{1\} + \frac{\partial^2}{\partial x_2 \partial x_1}\{1\} + \frac{\partial^2}{\partial x_2 \partial x_2}\{u\}\right)u(x,t)\right] \end{aligned}$$

Now by Applying inverse Elzaki Transform on above equation we get

$$\begin{aligned} E^{-1}&\big[E[u(x,t)]\big] \\ &= E^{-1}\left[v^2.{x_1}^2\right. \\ &+ vE\left[\left(-\frac{\partial}{\partial x_1}\left\{\frac{4u}{x_1}\right\} - \frac{\partial}{\partial x_2}\{x_2\}\right)u(x,t)\right. \\ &\left.\left. + \left(\frac{\partial^2}{\partial x_1 \partial x_1}\{u\} + \frac{\partial^2}{\partial x_1 \partial x_2}\{1\} + \frac{\partial^2}{\partial x_2 \partial x_1}\{1\} + \frac{\partial^2}{\partial x_2 \partial x_2}\{u\}\right)u(x,t)\right]\right] \end{aligned}$$

$$u = \left[{x_1}^2 + E^{-1}\left[vE\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x,t,u).u(x,t) + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x,t,u).u(x,t)\right]\right]\right]$$

Now, the recursive relation is given as below according to equation (3.9)

$$u_0 = {x_1}^2$$

$$u_1 = E^{-1}\left[vE\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x,t,u_0).u_0 + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x,t,u_0).u_0\right]\right] = -{x_1}^2.t$$

$$u_2 = E^{-1}\left[vE\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x,t,u_1)\,.\,u_1 + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x,t,u_1)\,.\,u_1\right]\right] = {x_1}^2.\frac{t^2}{2!}$$

$$u_3 = E^{-1}\left[vE\left[-\sum_{i=1}^{2}\frac{\partial}{\partial x_i}A_i(x,t,u_2)\,.\,u_2 + \sum_{i,j=1}^{2}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(x,t,u_2)\,.\,u_2\right]\right] = -{x_1}^2.\frac{t^3}{3!}$$

$$\dots\dots\dots\dots$$

Finally, we approximate the analytical solution $u(x,t)$ as follows

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \dots \dots$$

$$= {x_1}^2 - {x_1}^2.t + {x_1}^2.\frac{t^2}{2!} - {x_1}^2.\frac{t^3}{3!} + {x_1}^2.\frac{t^4}{4!} + \cdots$$

$$= {x_1}^2\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \dots \dots.\right)$$

$$u(x,t) = {x_1}^2.e^{-t}$$

## Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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