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AN APPLICATION OF HOMOTOPY ANALYSIS METHOD TO THE STUDY OF ROGUE WAVES

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ABSTRACT: *The mathematical study of rogue wave phenomenon had been on-going for years. Rogue waves are unusually large-amplitude surface waves that appear from nowhere in the open ocean. Collisions with such waves have caused catastrophic damage to ships and offshore structures. In this work, we apply an analytic technique, namely the Homotopy Analysis Method (HAM) to derive the rogue wave solution for the fully nonlinear wave equations with nonlinear boundary conditions. On the basis of the HAM, we obtain an analytic expression for the surface wave elevation* η *and velocity potential* ϕ *for a rogue wave. The expressions obtained for* η *are exact and depends on the computed values of the velocity potential* ϕ*. Due to the highly nonlinear nature of the problem (having nonlinear boundary conditions) the velocity potentials are obtained up to 5*th *order approximation. Surface plots of the wave profile are presented. These show high level agreement with the famous New Year wave at the Draupner platform. It is expected that this study will deepen and enrich our understanding of rogue waves.*

KEYWORDS; homotopy analysis method, rogue wave, surface wave elevation, velocity potential

INTRODUCTION

Rogue Waves

Rogue waves are large surface waves that appear infrequently in the ocean without warning. Rogue wave are also referred to in the literature as an extreme wave, giant wave, a monster in the ocean, abnormal wave or freak wave. The popular definition of rogue wave is a wave which the height is twice more than the average wave height present in the ocean surface (i.e. the average significant wave height). Such waves are usually accompanied by deep troughs (holes), between which is a high crest. These rogue waves are often described as vertical walls of water in the ocean. Occurrences of such large waves have been reported worldwide from ships, off shore platforms and radars (Lawton 2001; Kharif & Pelinovsky 2003; Forristall 2005). As stated by Lawton (2001), the rogue waves have been noticeable part of marine problem for centuries. The rogue wave event, have been extensively studied in the last years, in attempts to reveal the physics of this phenomenon. They have been reported as unusually high waves on the sea surface, appearing for a short time and causing eventually severe damages to the marine structures on their way (Touboul et al 2006; Kharif and Pelinovsky, 2003). The Rogue wave event that occurred on January 1st 1995 under the Draupner platform in the North Sea (Dysthe et al, 2008) provided evidence that such rogue waves can occur in the open ocean. During this rogue wave events, an extreme high wave crest with an amplitude of 18.5m occurred. The maximal wave height of 25.6m was much more than twice the significant wave height H_s of about 10.8m, which is usually the critical for rogue wave description. Liao (2011) investigated the steady condition for the nonlinear interaction of two trains of propagating wave in deep water and obtained the solution for both resonant and non-resonant cases. By means of the analytical method called homotopy analysis method (HAM) developed in (Liao 1997, 2004, 2011, 2012) a powerful analytic method for highly nonlinear problems. Tongkai (2005) derived the exact approximation analytical rogue wave solution in consideration of the Hirota equation with fractional and integer-order time using homotopy analysis method. Ejinkonye (2020a,b) obtain higher order surface displacement using homotopy analysis method (HAM).

In this work, we apply the homotopy analysis method (HAM) to the nonlinear boundary-value problem governed by the partial differential equations to derive the mechanism for rogue wave generation.

Basic equations and boundary conditions

Let us consider the nonlinear interactions of two trains of gravity waves with large amplitudes, propagating in water of finite depth. We assume that the fluid is inviscid and incompressible, the flow is irrotational and the surface tension is neglected.

Let z denote the vertical co-ordinate, x, y the horizontal co-ordinates, t the time, $z=\eta(x,y,t)$ the wave elevation, respectively.

Wang et al. (2016) Shows that governing equation of the velocity potential $\phi(x,y,z,t)$ is given by

$$\nabla^2\phi(x,y,z;t)=0 \quad -d<z<\eta(x,y,t) \tag{1}$$

Subject to the two boundary conditions on the unknown wave elevation $z=\eta(x,y,t)$

$$\frac{\partial\eta(x,y;t)}{\partial t}+\nabla\phi(x,y,z;t)\cdot\left(-\nabla\frac{\partial\phi(x,y,z;t)}{\partial t}-\frac{1}{2}\nabla\left(\nabla\phi(x,y,z;t)\cdot\nabla\phi(x,y,z;t)+\left(\frac{\partial\phi(x,y,z;t)}{\partial z}\right)^2\right)\right)-\frac{\partial\phi(x,y,z;t)}{\partial z}=0 \quad \text{on } z=\eta(x,y,t) \tag{2}$$

$$\frac{\partial\phi(x,y,z;t)}{\partial t}+\eta(x,y;t)+\frac{1}{2}\left(\nabla\phi(x,y,z;t)\cdot\nabla\phi(x,y,z;t)+\left(\frac{\partial\phi(x,y,z;t)}{\partial z}\right)^2\right)=0 \quad \text{on } z=\eta(x,y,t) \tag{3}$$

And the bottom boundary condition

$$\frac{\partial\phi(x,y,z;t)}{\partial z}=0 \qquad at\ z=-\infty \tag{4}$$

where del is define as

$$\nabla=\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z} \tag{5}$$

However, for simplicity, let us consider a wave system consisting of two trains of primary travelling gravity waves in deep water with wave numbers k_1 *and* k_2 and the corresponding angular frequencies w_1 *and* w_2 respectively, where $k_1\times k_2\neq 0$ (i.e. the two traveling waves are not collinear). In this work, the nonlinear boundary-value problem governed by the PDEs (1-4) is solved by means of the homotopy analysis method.

Let α_1, α_2 denote the angles between the positive x-axis and the wave number vectors $\vec{K}_1$ *and* $\vec{K}_2$ respectively, where $k_1 \cdot k = k_2 \cdot k = 0$, i.e. the z axis is perpendicular to the wave number $\vec{K}_1$, $\vec{K}_2$. Then

$$\vec{K}_1 = k_1(\cos\alpha_1 i + \sin\alpha_1 j) \text{ and } \vec{K}_2 = k_2(\cos\alpha_2 i + \sin\alpha_2 j) \tag{6}$$

We write $r = xi + yj$, *where* $k_1 = |\vec{K}_1|$ *and* $k_2 = |\vec{K}_2|$

$$\xi_1 = \vec{K}_1 \cdot \vec{r} - w_1 t, \quad \xi_2 = \vec{K}_2 \cdot \vec{r} - w_2 t \tag{7}$$

In other words, one can express the potential function $\phi(x, y, z; t) = \phi(\xi_1, \xi_2, z)$ and the wave elevation $\eta(x, y; t) = \eta(\xi_1, \xi_2)$ respectively, then we have $\nabla = \vec{K}_1 \frac{\partial}{\partial \xi_1} + \vec{K}_2 \frac{\partial}{\partial \xi_2} + \hat{k} \frac{\partial}{\partial z}$

So

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \vec{K}_1 \frac{\partial\phi}{\partial\xi_1} + \vec{K}_2\frac{\partial\phi}{\partial\xi_2} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{k}_1(\cos\alpha_1 i + \sin\alpha_1 j)\frac{\partial\phi}{\partial\xi_1} + \hat{k}_2(\cos\alpha_2 i + \sin\alpha_1 j)\frac{\partial\phi}{\partial\xi_2} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi\cdot\nabla\phi = k_1^2\left(\frac{\partial\phi}{\partial\xi_1}\right)^2 + 2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi}{\partial\xi_1}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right) + k_2^2\left(\frac{\partial\phi}{\partial\xi_2}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 \tag{8}$$

Where $\vec{K}_1 \cdot \hat{k}_1 = \vec{K}_2 \cdot \hat{k}_2 = 0$ is used, and $\vec{K}_1 \cdot \vec{K}_2 = 2k_1k_2\cos(\alpha_1 - \alpha_2)$

In general, it holds that, the original governing equation reads

$$\nabla^2\phi = k_1^2\frac{\partial^2\phi}{\partial\xi_1^2} + 2k_1k_2\cos(\alpha_1-\alpha_2)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2} + k_2^2\frac{\partial^2\phi}{\partial\xi_2^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{9}$$

which has the general solution

$$\phi = [A\cos(a\xi_1 + b\xi_2) + B\sin(a\xi_1 + b\xi_2)]e^{|\vec{K}+\vec{K}|z} \tag{10}$$

where a, b are integers, A, B are integral constants.

Therefore, the boundary conditions in equations (2) and (3) will become

$$\eta = w_1\frac{\partial\phi}{\partial\xi_1} + w_2\frac{\partial\phi}{\partial\xi_2} - \left(k_1^2\left(\frac{\partial\phi}{\partial\xi_1}\right)^2 + 2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi}{\partial\xi_1}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right) + k_2^2\left(\frac{\partial\phi}{\partial\xi_2}\right)^2 + 2\left(\frac{\partial\phi}{\partial z}\right)^2\right) \tag{11}$$

and

$$w_1^2\frac{\partial^2\phi}{\partial\xi_1^2} + 2w_1w_2\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2} + w_2^2\frac{\partial^2\phi}{\partial\xi_2^2} + 2w_1k_1^2\left(\frac{\partial\phi}{\partial\xi_1}\right)\frac{\partial^2\phi}{\partial\xi_1^2} + 2w_2k_1^2\left(\frac{\partial\phi}{\partial\xi_1}\right)\frac{\partial^2\phi}{\partial\xi_1^2} + 2w_1k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1^2} +$$

$$+ 2w_1k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi}{\partial\xi_1}\right)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2} + 2w_2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2} + 2w_2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_2^2} +$$

$$2w_1k_2^2\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_2^2} + 2w_2k_1^2\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_2^2} + \left(\frac{\partial\phi}{\partial z}\right)^2 + \nabla\phi\cdot\left(\nabla\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\nabla\phi\cdot\nabla\phi + \left(\frac{\partial\phi}{\partial z}\right)^2\right)\right) + \frac{\partial\phi}{\partial z} = 0 \tag{12}$$

and the bottom boundary condition

$$\frac{\partial\phi(\xi_1, \xi_2, z)}{\partial z} = 0 \qquad as \;\; z = -\infty \tag{13}$$

where

$$\nabla\phi\cdot\left(\nabla\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\nabla\phi\cdot\nabla\phi+\left(\frac{\partial\phi}{\partial z}\right)^2\right)\right)=\rho \tag{14}$$

$$\begin{aligned}
\rho = (\; & k_1^4\left(\frac{\partial^2\phi}{\partial\xi_1^2}\right)\left(\frac{\partial\phi}{\partial\xi_1}\right)^2+2k_1k_2\cos(\alpha_1-\alpha_2)k_1^2\left(\frac{\partial\phi}{\partial\xi_1}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1^2}-k_1^2w_1\left(\frac{\partial\phi}{\partial\xi_1}\right)\frac{\partial^2\phi}{\partial\xi_1^2}-k_1^2w_2\left(\frac{\partial\phi}{\partial\xi_1}\right)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}\\
& -2k_1k_2\cos(\alpha_1-\alpha_2)w_1\left(\frac{\partial\phi}{\partial\xi_1}\right)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}-2k_1k_2\cos(\alpha_1-\alpha_2)w_2\left(\frac{\partial\phi}{\partial\xi_1}\right)\frac{\partial^2\phi}{\partial\xi_2^2}-2k_1k_2\cos(\alpha_1-\alpha_2)w_1\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1^2}+\\
& -2k_1k_2\cos(\alpha_1-\alpha_2)w_2\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}-k_2^2w_1\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}-k_2^2w_2\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_2^2}+2k_1^2k_2^2\left(\frac{\partial\phi}{\partial\xi_1}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}\\
& +2k_1k_2\cos(\alpha_1-\alpha_2)k_1^2\left(\frac{\partial\phi}{\partial\xi_1}\right)^2\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}+k_1^2k_2^2\left(\frac{\partial\phi}{\partial\xi_1}\right)^2\left(\frac{\partial^2\phi}{\partial\xi_2^2}\right)+2k_1k_2\cos(\alpha_1-\alpha_2)k_1^2\left(\frac{\partial\phi}{\partial\xi_1}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_2^2}+\\
& 2k_1k_2\cos(\alpha_1-\alpha_2)k_1^2\left(\frac{\partial\phi}{\partial\xi_1}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1^2}+k_1^2k_2^2\left(\frac{\partial^2\phi}{\partial\xi_1^2}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right)^2+2k_1k_2\cos(\alpha_1-\alpha_2)k_2^2\left(\frac{\partial\phi}{\partial\xi_1}\right)^2\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}+\\
& 2k_1^2k_2^2\left(\frac{\partial\phi}{\partial\xi_1}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}+2k_1k_2\cos(\alpha_1-\alpha_2)k_2^2\left(\frac{\partial\phi}{\partial\xi_1}\right)\left(\frac{\partial\phi}{\partial\xi_2}\right)\frac{\partial^2\phi}{\partial\xi_2^2}+k_2^4\left(\frac{\partial\phi}{\partial\xi_2}\right)^2\frac{\partial^2\phi}{\partial\xi_2^2}+2\left(\frac{\partial\phi}{\partial z}\right)^2\left(\frac{\partial^2\phi}{\partial z^2}\right)\;)
\end{aligned} \tag{15}$$

Our aim is to find out the corresponding unknown potential function $\phi(\xi_1,\xi_2,z)$ and the unknown wave elevation $\eta(\xi_1,\xi_2)$, which are governed by the linear partial differential Eq. (9) subject to two nonlinear boundary conditions (11) and (12) on the unknown free surface elevation $z=\eta(\xi_1,\xi_2)$, and the linear boundary condition (13) on the bottom.

METHODOLOGY

Analytic approach based on the homotopy analysis method

The nonlinear boundary-value problem governed by the PDEs in equations (9, 11, 12, 13) will be solved by means of the homotopy analysis method.

Let $\phi_0(\xi_1,\xi_2,z)$, $\eta_0(\xi_1,\xi_2)$ denote the initial guesses of the velocity potential $\phi(\xi_1,\xi_2,z)$ and wave elevation $\eta(\xi_1,\xi_2)$ respectively. Let $p\in[0,1]$ denote an embedding parameter and let $\hbar\neq 0$ be the so-called convergence-control parameter. Here, both p and $\hbar$ are auxiliary parameters without physical meaning. Instead of solving the nonlinear PDEs (9, 11, 12, 13) directly, we first construct a family (with respect to p) of PDEs $\eta(\xi_1,\xi_2,p)$ and $\phi(\xi_1,\xi_2,z,p)$, governed by the so-called zeroth-order deformation equations,

$$\nabla^2\phi(\xi_1,\xi_2,z;p)=0 \tag{16}$$

subject to the two boundary conditions on the unknown wave elevation $z=\eta(\xi_1,\xi_2;p)$,

$$(1-p)[\eta(\xi_1,\xi_2;p)-\eta_0(\xi_1,\xi_2)]=p\hbar N_1[\eta(\xi_1,\xi_2;p)] \tag{17}$$

$$(1-p)L[\phi(\xi_1,\xi_2,z;p)-\phi_0(\xi_1,\xi_2,z)]=\hbar pN_2[\phi(\xi_1,\xi_2,z;p)] \tag{18}$$

and the bottom condition

$$\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial z}=0 \qquad as\ z\to-\infty \tag{19}$$

where L is an auxiliary linear operator with the property $L(0)=0$, N_1 and N_2 are nonlinear differential operators defined respectively as

$$L(\phi(\xi_1,\xi_2,z;p))=\left(w_1^2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1^2}+2w_1w_2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1\partial\xi_2}+w_2^2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2^2}+\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial z}\right) \tag{20}$$

$$N_1[\eta,\phi]=\eta(\xi_1,\xi_2;p)-\left(w_1\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1}+w_2\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2}\right)-k_1^2\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1}\right)^2$$
$$-2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1}\right)\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2}\right)-k_2^2\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2}\right)^2-2\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial z}\right)^2 \tag{21}$$

and

$$N_2[\phi]=w_1^2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1^2}+2w_1w_2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1\partial\xi_2}+w_2^2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2^2}+$$
$$+2w_1k_1^2\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1}\right)\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1^2}+2w_2k_1^2\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1}\right)\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1^2}+$$
$$+2w_1k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2}\right)\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1^2}+2w_1k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1}\right)\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1\partial\xi_2}+$$
$$+2w_2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2}\right)\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1\partial\xi_2}+2w_2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2}\right)\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2^2}+$$
$$2w_1k_2^2\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2}\right)\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2^2}+2w_2k_1^2\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2}\right)\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2^2}+\left(\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial z}\right)^2+\rho+\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial z}=0$$

(22)

where the definition of ρ is in equation (15)

When $p=0,$ the zeroth-order deformation equations (16)–(19) have the solution

$$\eta(\xi_1,\xi_2;0)=\eta_0(\xi_1,\xi_2) \tag{23}$$

$$\phi(\xi_1,\xi_2,z;0)=\phi_0(\xi_1,\xi_2,z) \tag{24}$$

When $p=1$, the zeroth-order deformation equations (16)–(19) are equivalent to the original PDEs (9, 11, 12,13), so we have the solution

$$\eta(\xi_1,\xi_2;1)=\eta(\xi_1,\xi_2) \tag{25}$$

$$\phi(\xi_1,\xi_2,z;1)=\phi(\xi_1,\xi_2,z) \tag{26}$$

Thus, as the embedding parameter $p\in[0,1]$ increases from 0 to 1, $\eta(\xi_1,\xi_2;p)$ and $\phi(\xi_1,\xi_2,z;p)$ vary continuously from their initial guess $\eta_0(\xi_1,\xi_2)$ and $\phi_0(\xi_1,\xi_2,z)$ respectively to the exact solution $\eta(\xi_1,\xi_2)$ and $\phi(\xi_1,\xi_2,z)$. Thus, the zeroth-order deformation equations (16)–(19) indeed construct two continuous deformations $\eta(\xi_1,\xi_2;p)$ and $\phi(\xi_1,\xi_2,z;p)$. Such continuous deformations are called homotopies in topology, expressed by

$$\eta(\xi_1,\xi_2;p):\eta_0(\xi_1,\xi_2)\simeq\eta(\xi_1,\xi_2) \tag{27}$$

$$\phi(\xi_1,\xi_2,z;p):\phi_0(\xi_1,\xi_2,z)\simeq\phi(\xi_1,\xi_2,z) \tag{28}$$

assuming that $\hbar$ is properly chosen that the Maclaurin series

$$\eta(\xi_1,\xi_2;p)=\sum_{n=0}^{\infty}\eta_n(\xi_1,\xi_2)p^n \tag{29}$$

and $$\phi(\xi_1,\xi_2,z;p)=\sum_{n=0}^{\infty}\phi_n(\xi_1,\xi_2,z)p^n \tag{30}$$

exist and convergent at $p=1$, we have due to equation (26) the so-called homotopy-series solution ,

$$\eta(\xi_1,\xi_2)=\eta_0(\xi_1,\xi_2)+\sum_{n=1}^{\infty}\eta_n(\xi_1,\xi_2) \tag{31}$$

and $$\phi(\xi_1,\xi_2,z)=\phi_0(\xi_1,\xi_2,z)+\sum_{n=1}^{\infty}\phi_n(\xi_1,\xi_2,z) \tag{32}$$

where $$\eta_n(\xi_1,\xi_2)=\frac{1}{n!}\frac{\partial^n\eta(\xi_1,\xi_2;p)}{\partial p^n}\bigg|_{p=0} \tag{33}$$

$$\phi_n(\xi_1,\xi_2,z)=\frac{1}{n!}\frac{\partial^n\phi(\xi_1,\xi_2,z;p)}{\partial p^n}\bigg|_{p=0} \tag{34}$$

are called homotopy derivatives.

We can obtain wave elevation $\eta_n(\xi_1,\xi_2)$ and velocity potential $\phi_n(\xi_1,\xi_2,z)$ which is governed by a linear PDE, as long as $\phi_{n-1}(\xi_1,\xi_2,z)$ is known.

Since the HAM provides freedom in the choice of auxiliary linear operator, and considering the linear part of (17), we choose

$$L(\phi(\xi_1,\xi_2,z;p))=\left(w_1^2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1^2}+2w_1w_2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_1\partial\xi_2}+w_2^2\frac{\partial^2\phi(\xi_1,\xi_2,z;p)}{\partial\xi_2^2}+\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial z}\right) \tag{35}$$

Higher-order deformation equation

Differentiating the zeroth-order deformation equations (16)–(19) m times with respect to p , then dividing them by $m!$ and setting $p=1$, we have the m^{th}-order deformation equation

$$\nabla^2\phi_m(\xi_1,\xi_2,z;p)=0 \tag{36}$$

subject to the two boundary conditions at $z=\eta(\xi_1,\xi_2;p)$

$$\frac{\partial^m}{\partial p}\frac{1}{m!}[(1-p)\eta(\xi_1,\xi_2;p)=\chi_m p\hbar N_1[\eta(\xi_1,\xi_2;p),\phi(\xi_1,\xi_2,z;p)]] \tag{37}$$

and $$\frac{\partial^m}{\partial p}\frac{1}{m!}[(1-p)L[\phi(\xi_1,\xi_2,z;p)-\phi_0(\xi_1,\xi_2,z)]=\chi_m p\hbar N_2[\phi(\xi_1,\xi_2,z;p)]] \tag{38}$$

and the bottom condition

$$\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial z}=0 \qquad as\ z\to-\infty \tag{39}$$

We obtain the respective wave elevation and velocity potential at the boundary conditions defined as

$$\eta_m = m\left[(\chi_m + \hbar)\eta^{m-1} + \hbar \frac{D^{m-1}N_1[\phi(\xi_1,\xi_2,z;p)]}{Dp^{m-1}}\right]\Bigg|_{p=0} \tag{40}$$

and

$$\frac{D^m L[\phi(\xi_1,\xi_2,z;p)]}{Dp^m}\Bigg| = m\left[\chi_m \frac{D^{m-1}L[\phi(\xi_1,\xi_2,z;p)]}{Dp^{m-1}} + \hbar \frac{D^{m-1}N_2[\phi(\xi_1,\xi_2,z;p)]}{Dp^{m-1}}\right]\Bigg|_{p=0} \tag{41}$$

and the bottom condition

$$\frac{\partial\phi(\xi_1,\xi_2,z;p)}{\partial z} = 0 \tag{42}$$

Where $\chi_m = \begin{cases} 0 & m \le 0 \\ 1 & m > 1 \end{cases}$ (43)

$$\eta_m = m(\chi_m + \hbar)\eta^{m-1} + m\hbar\left(w_1 \frac{\partial\phi(\xi_1,\xi_2,z)^{m-1}}{\partial\xi_1} + w_2 \frac{\partial\phi(\xi_1,\xi_2,z)^{m-1}}{\partial\xi_2}\right) - \tag{44}$$

$$m\hbar\left(k_1^2\left(\frac{\partial\phi(\xi_1,\xi_2,z)^{m-1}}{\partial\xi_1}\right)^2 + 2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi(\xi_1,\xi_2,z)^{m-1}}{\partial\xi_1}\right)\left(\frac{\partial\phi(\xi_1,\xi_2,z)^{m-1}}{\partial\xi_2}\right) + k_2^2\left(\frac{\partial\phi(\xi_1,\xi_2,z)^{m-1}}{\partial\xi_2}\right)^2 + 2\left(\frac{\partial\phi(\xi_1,\xi_2,z)^{m-1}}{\partial z}\right)^2\right)$$

and

$$w_1^2\frac{\partial^2\phi^m}{\partial\xi_1^2} + 2w_1w_2\frac{\partial^2\phi^m}{\partial\xi_1\partial\xi_2} + w_2^2\frac{\partial^2\phi^m}{\partial\xi_2^2} + \frac{\partial^2\phi^m}{\partial z^2} = m\chi_m\left(w_1^2\frac{\partial^2\phi^{m-1}}{\partial\xi_1^2} + 2w_1w_2\frac{\partial^2\phi^{m-1}}{\partial\xi_1\partial\xi_2} + w_2^2\frac{\partial^2\phi^{m-1}}{\partial\xi_2^2} + \frac{\partial^2\phi^{m-1}}{\partial z^2}\right) -$$

$$-\hbar m\left(+2w_1k_1^2\frac{\partial^2\phi^{m-1}}{\partial\xi_1^2} + 2w_2k_1^2\frac{\partial^2\phi^{m-1}}{\partial\xi_1\partial\xi_2} + 2w_1k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi^{m-1}}{\partial\xi_2}\right)\frac{\partial^2\phi^{m-1}}{\partial\xi_1^2} + +2w_1k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi^{m-1}}{\partial\xi_1}\right)\frac{\partial^2}{\partial\xi_1\partial\xi_2}\right) - \tag{45}$$

$$-\hbar m\left(2w_2k_1k_2\cos(\alpha_1-\alpha_2)\left(\frac{\partial\phi^{m-1}}{\partial\xi_2}\right)\frac{\partial^2\phi^{m-1}}{\partial\xi_1\partial\xi_2} + 2w_1k_2^2\left(\frac{\partial\phi^{m-1}}{\partial\xi_2}\right)\frac{\partial^2\phi^{m-1}}{\partial\xi_2^2} + 2w_2k_1^2\left(\frac{\partial\phi^{m-1}}{\partial\xi_2}\right)\frac{\partial^2\phi^{m-1}}{\partial\xi_2^2} + \left(\frac{\partial\phi^{m-1}}{\partial z}\right)^2 + \rho + \frac{\partial\phi^{m-1}}{\partial z}\right)$$

Therefore, the sub-problems for $\eta_m(\xi_1,\xi_2)$ and $\phi_m(\xi_1,\xi_2,z)$ are not only linear but also decoupled: given $\eta_{m-1}(\xi_1,\xi_2)$ and $\phi_{m-1}(\xi_1,\xi_2,z)$, we can get $\eta_m(\xi_1,\xi_2)$directly, and then $\phi_m(\xi_1,\xi_2,z)$is obtained by solving the linear Laplace equation (36) with two linear boundary conditions (42) and(44). Thus, the higher-order deformation equations can be easily solved by means of the symbolic computation software such as mathematica.

The higher order expansion

Our interest in this investigation is obtained to higher order surface wave elevation equations and observes the behaviour of surface wave elevation. If we choose the initial approximation of the velocity potential as

$$\phi_0 = A_0\frac{w_1}{k_1}\psi_{1,0} + B_0\frac{w_2}{k_2}\psi_{0,1} \tag{46}$$

where A_0 and B_0. $A_0 = \pm\frac{\varepsilon}{k_1^2}\sqrt{[1+2(k_1\cos(\alpha_1-\alpha_2))]\left(\frac{w_1^2}{k_2^2}-1\right)\left(\frac{w_1^2}{k_1^2}-1\right)}$ (47)

$$B_0 = \pm\frac{\varepsilon}{k_2^2}\sqrt{[1+2(k_2\cos(\alpha_1-\alpha_2))]\left(\frac{w_1^2}{k_1^2}-1\right)-\left(\frac{w_1^2}{k_2^2}-1\right)} \tag{48}$$

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We can see that A_0 *and* B_0 have multiple values; it can be neither positive nor negative.

$$\psi_{1,0} = \left[A\cos\xi_1 + B\sin\xi_1\right]\ell^{k_1^2 z} \tag{49}$$

$$\psi_{0,1} = \left[A\cos\xi_2 + B\sin\xi_2\right]\ell^{k_2^2 z} \tag{50}$$

$$\phi_1(\xi_1,\xi_2,z) = A_1\frac{w_1}{k_1}\psi_{1,0} + B_1\frac{w_2}{k_2}\psi_{0,1} + \gamma_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + d_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + \gamma_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + d_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + \gamma_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) + d_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) \tag{51}$$

$$\phi_2(\xi_1,\xi_2,z) = A_2\frac{w_1}{k_1}\psi_{1,0} + B_2\frac{w_2}{k_2}\psi_{0,1} + \gamma_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + d_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + \gamma_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + d_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + \gamma_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) + d_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) \tag{52}$$

$$\phi_3(\xi_1,\xi_2,z) = A_3\frac{w_1}{k_1}\psi_{1,0} + B_3\frac{w_2}{k_2}\psi_{0,1} + \gamma_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + d_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + \gamma_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + d_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + \gamma_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) + d_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) \tag{53}$$

$$\phi_4(\xi_1,\xi_2,z) = A_4\frac{w_1}{k_1}\psi_{1,0} + B_4\frac{w_2}{k_2}\psi_{0,1} + \gamma_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + d_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + \gamma_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + d_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + \gamma_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) + d_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) \tag{54}$$

$$\phi_5(\xi_1,\xi_2,z) = A_5\frac{w_1}{k_1}\psi_{1,0} + B_5\frac{w_2}{k_2}\psi_{0,1} + \gamma_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + d_1^{1,1}\left(\frac{\psi_{1,1}}{\lambda_{1,1}}\right) + \gamma_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + d_1^{2,1}\left(\frac{\psi_{2,1}}{\lambda_{2,1}}\right) + \gamma_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) + d_1^{1,2}\left(\frac{\psi_{1,2}}{\lambda_{1,2}}\right) \tag{55}$$

But from equation (44) we obtain the solutions of wave elevation. Taken our initial guess of wave elevation as $\eta_0 = 0$ and the initial guess of the velocity potential in equation (46) we have

$$\begin{aligned}
\eta_1(\xi_1,\xi_2) = {} & \frac{h}{k_1k_2}\ell^{2zk_1^2}\left(\sin(2\xi_1)-1\right)A_1k_1k_2w_1^2 + \\
& \frac{h}{k_1k_2}\ell^{zk_1^2}A_0k_2w_2\left(-\left(\sin(\xi_1)+\cos(\xi_1)\right)k_1^2 + \left(\cos(\xi_1)-\sin(\xi_1)\right)w_1\right) + \\
& -\frac{2h}{k_1k_2}\ell^{zk_2^2}B_0k_1w_2\cos(\alpha_1-\alpha_2)\left(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2)\right) + \\
& +\frac{h}{k_1k_2}\ell^{zk_2^2}B_1k_1w_2\left(-\left(\cos(\xi_2)+\sin(\xi_2)\right)k_2^2 + \left(\cos(\xi_2)-\sin(\xi_2)\right)w_2\right) + \\
& +\frac{h}{k_1k_2}\ell^{zk_2^2}B_0k_2w_2\left(\sin(2\xi_2)-1\right)
\end{aligned} \tag{56}$$

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$$
\begin{aligned}
\eta_2(\xi_1,\xi_2) &= \frac{2h}{k_1k_2}\ell^{2zk_1^2}(\sin(2\xi_1)-1)A_1k_1k_2w_1^2 + \frac{2h}{k_1k_2}\ell^{zk_1^2}A_0k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right) \\
&-\frac{4h}{k_1k_2}\ell^{zk_2^2}B_0k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{2h}{k_1k_2}\ell^{zk_2^2}B_1k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+ \\
&+\frac{2h}{k_1k_2}\ell^{zk_2^2}B_0k_2w_2(\sin(2\xi_2)-1)+\frac{4h}{k_1k_2}\ell^{zk_1^2}A_0A_1k_1w_18\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+ \\
&-\frac{12\hbar}{k_1k_2}\ell^{zk_2^2}B_0B_1k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{2h}{k_1k_2}\ell^{zk_2^2}B_1B_2k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+ \\
&+\frac{2h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_2(\sin(2\xi_2)-1)+\frac{2h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+ \\
&-\frac{12h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{12h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right) \\
&+\frac{2h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_24(\sin(2\xi_2)-1)-\frac{4h}{k_1k_2}\ell^{zk_2^2}B_0k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+ \\
&+\frac{3h}{k_1k_2}\ell^{zk_2^2}B_1k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{3h}{k_1k_2}\ell^{zk_2^2}B_0k_2w_2(\sin(2\xi_2)-1)+ \\
&+\frac{6h}{k_1k_2}\ell^{zk_1^2}A_0A_1k_2w_28\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+ \\
&-\frac{18h}{k_1k_2}\ell^{zk_2^2}B_0B_1k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))
\end{aligned} \tag{57}
$$

$$\eta_3(\xi_1,\xi_2)=\frac{2h}{k_1k_2}\ell^{2zk_1^2}(\sin(2\xi_1)-1)A_1k_1k_2w_1^2+\frac{18h}{k_1k_2}\ell^{zk_1^2}A_0A_2A_3k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{12h}{k_1k_2}\ell^{zk_2^2}B_0k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_1B_2k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_2(\sin(2\xi_2)-1)+\frac{12h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2k_1w_18\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{36\hbar}{k_1k_2}\ell^{zk_2^2}B_0B_1k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{18h}{k_1k_2}\ell^{zk_2^2}B_1B_2k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_2(\sin(2\xi_2)-1)+\frac{6h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{24h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2A_3k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_24(\sin(2\xi_2)-1)-\frac{12h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{9h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{9h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_2(\sin(2\xi_2)-1)+$$
$$+\frac{18h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2k_2w_28\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{18h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_3k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{6h}{k_1k_2}\ell^{2zk_1^2}(\sin(2\xi_1)-1)A_1k_1k_2w_1^2+\frac{6h}{k_1k_2}\ell^{zk_1^2}A_0A_1k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$\frac{18h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3k_1w_18\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)-\frac{24\hbar}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_2(\sin(2\xi_2)-1)+$$
$$+\frac{8h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)-\frac{36h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))$$
$$+\frac{36h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2A_3k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_24(\sin(2\xi_2)-1)$$
$$-\frac{12h}{k_1k_2}\ell^{zk_2^2}B_0B_1k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{9h}{k_1k_2}\ell^{zk_2^2}B_1B_2k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$+\frac{9h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_2(\sin(2\xi_2)-1)+\frac{18h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2k_2w_28\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{54h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2)) \qquad (58)$$

$$\eta_4(\xi_1,\xi_2)=\frac{8h}{k_1k_2}\ell^{2zk_1^2}(\sin(2\xi_1)-1)A_1A_2A_3k_1k_2w_1^2+\frac{108h}{k_1k_2}\ell^{zk_1^2}A_0A_2A_3A_4k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)$$
$$-\frac{48h}{k_1k_2}\ell^{zk_2^2}B_0B_2B_3k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{24h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_2(\sin(2\xi_2)-1)+$$
$$+\frac{36h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4k_1w_18\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{144h}{k_1k_2}\ell^{zk_2^2}B_0B_1k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{72h}{k_1k_2}\ell^{zk_2^2}B_1B_2k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$-\frac{248h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{96h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2A_3k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right)$$
$$+\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_24(\sin(2\xi_2)-1)+\frac{36h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$+\frac{212h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_2w_2(\sin(2\xi_2)-1)+\frac{72h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4k_2w_28\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$+\frac{24h}{k_1k_2}\ell^{2zk_1^2}A_1A_2A_3k_1k_2w_1^2(\sin(2\xi_1)-1)+\frac{24h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{56h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))$$
$$+\frac{48h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$-\frac{12h}{k_1k_2}\ell^{zk_2^2}B_0k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_1B_2k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_2(\sin(2\xi_2)-1)+\frac{12h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2k_1w_18\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{36h}{k_1k_2}\ell^{zk_2^2}B_0B_1k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{18h}{k_1k_2}\ell^{zk_2^2}B_1B_2k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_2(\sin(2\xi_2)-1)+\frac{6h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{24h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2A_3k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$
$$+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_24(\sin(2\xi_2)-1)-\frac{12h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{9h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{9h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_2w_2(\sin(2\xi_2)-1)+$$
$$+\frac{18h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2k_2w_28\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)-\frac{18h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_3k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$
$$+\frac{6h}{k_1k_2}\ell^{2zk_1^2}(\sin(2\xi_1)-1)A_1k_1k_2w_1^2+\frac{6h}{k_1k_2}\ell^{zk_1^2}A_0A_1k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$
$$-\frac{6h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{6h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3\,k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)\qquad(59)$$

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$$\eta_5(\xi_1,\xi_2)=\frac{40h}{k_1k_2}\ell^{2zk_1^2}(\sin(2\xi_1)-1)A_1A_2A_3A_4k_1k_2w_1^2+$$

$$\frac{540h}{k_1k_2}\ell^{zk_1^2}A_0A_2A_3A_4A_5k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$

$$-\frac{224h}{k_1k_2}\ell^{zk_2^2}B_0B_2B_3B_4k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$

$$+\frac{120h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$

$$+\frac{120h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_2w_2(\sin(2\xi_2)-1)+$$

$$\frac{180h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4A_5k_1w_1 8\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$

$$-\frac{720\hbar}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$

$$+\frac{360h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$

$$-\frac{1240h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$

$$+\frac{480h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2A_3A_4A_5k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right)$$

$$+\frac{120h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_2w_2 4(\sin(2\xi_2)-1)+$$

$$+\frac{180h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$

$$+\frac{1060h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4B_5k_2w_2(\sin(2\xi_2)-1)+$$

$$\frac{360h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4A_5k_2w_2 8\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$

$$+\frac{8h}{k_1k_2}\ell^{2zk_1^2}(\sin(2\xi_1)-1)A_1A_2A_3k_1k_2w_1^2+\frac{108h}{k_1k_2}\ell^{zk_1^2}A_0A_2A_3A_4k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$

$$-\frac{48h}{k_1k_2}\ell^{zk_2^2}B_0B_2B_3k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$

$$+\frac{24h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$

$$+\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_2(\sin(2\xi_2)-1)+$$

$$\frac{36h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4k_1w_1 8\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$

$$-\frac{144\hbar}{k_1k_2}\ell^{zk_2^2}B_0B_1k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$

$$+\frac{72h}{k_1k_2}\ell^{zk_2^2}B_1B_2k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$

$$-\frac{248h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$

$$+\frac{96h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2A_3k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right)$$

$$+\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_2 4(\sin(2\xi_2)-1)+\frac{36h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$

$$+\frac{120h}{k_1k_2}\ell^{2zk_1^2}A_1A_2A_3A_4k_1k_2w_1^2(\sin(2\xi_1)-1)$$

$$+\frac{120h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4A_5k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+$$

$$-\frac{280h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+$$

$$+\frac{240h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4B_5k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+$$

$$
\begin{aligned}
&+\frac{72h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3k_2w_2 8\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)-\frac{54h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))\\
&+\frac{72h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4k_1w_1 8\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)-\frac{24\hbar}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\\
&+\frac{24h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{32h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+\\
&-\frac{144h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{144h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2A_3A_4k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right)\\
&+\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_2w_2 4(\sin(2\xi_2)-1)-\frac{48h}{k_1k_2}\ell^{zk_2^2}B_0B_1k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\\
&+\frac{36h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{102h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3k_2w_2(\sin(2\xi_2)-1)+\\
&+\frac{72h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4k_2w_2 8\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)-\frac{216h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))\\
&+\frac{68h}{k_1k_2}\ell^{2zk_1^2}(\sin(2\xi_1)-1)A_1A_2A_3A_4k_1k_2w_1^2+\frac{48h}{k_1k_2}\ell^{zk_1^2}A_0A_2A_3A_4A_5k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+\\
&-\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{28h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4B_5k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\\
&+\frac{128h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_2w_2(\sin(2\xi_2)-1)+\frac{48h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2k_1w_1 8\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+\\
&-\frac{150\hbar}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4B_5k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))++\frac{180h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\\
&+\frac{30h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4A_5k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+-\frac{72h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\\
&+\frac{72h}{k_1k_2}\ell^{zk_2^2}A_0A_1A_2A_3A_4A_5k_1w_1\left(-8(\cos(\xi_2)+\sin(\xi_2))k_2^2+8(\cos(\xi_2)-\sin(\xi_2))w_2\right)-\frac{48h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4B_5k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\\
&+\frac{120h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)+\frac{54h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3A_4A_5k_2w_2 8\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+8(\cos(\xi_1)-\sin(\xi_1))w_1\right)+\\
&-\frac{32h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_1w_2\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))+\frac{18h}{k_1k_2}\ell^{zk_1^2}A_0A_1A_2A_3k_2w_2\left(-(\sin(\xi_1)+\cos(\xi_1))k_1^2+(\cos(\xi_1)-\sin(\xi_1))w_1\right)+\\
&-\frac{24h}{k_1k_2}\ell^{zk_2^2}B_0B_1B_2B_3B_4k_1w_1\cos(\alpha_1-\alpha_2)(\cos(\xi_1-\xi_2)-\sin(\xi_1+\xi_2))++\frac{30h}{k_1k_2}\ell^{zk_2^2}B_1B_2B_3B_4B_5k_1w_2\left(-(\cos(\xi_2)+\sin(\xi_2))k_2^2+(\cos(\xi_2)-\sin(\xi_2))w_2\right)
\end{aligned}
\tag{60}
$$

Note that automatically ϕ_1 satisfies the Laplace equation (9) and the bottom condition (13) for any constants A_1 *and* B_1. On the other hand, given the initial guess ϕ_0 , we can calculate $\eta_1(\xi_1,\xi_2)$ directly by means of the formula (44).

The above approach has general meaning. In a similar way, we can obtain $\eta_m(\xi_1,\xi_2)$ *and* $\phi_m(\xi_1,\xi_2,z)$, successively, in the order m = 1,2,3, and so on.

$$k_1 = 0.70399,\ \ k_2 = \frac{\pi}{5} = 0.62831835,\ \ \alpha_1 = \frac{2\pi}{3},\ \ \alpha_2 = \frac{\pi}{6},\ \ w_1 = 0.83904,\ \ w_2 = 0.79266535$$

$$A_1 = \pm\frac{\varepsilon}{k_1^2}\sqrt{[1+2(k_1\cos(\alpha_1-\alpha_2))]\left(\frac{w_1^2}{k_2^2}-1\right)-\left(\frac{w_1^2}{k_1^2}-1\right)+[1+2(k_1\cos(\alpha_1-\alpha_2))]\left(\frac{w_1^2}{k_2^2}-1\right)} \tag{61}$$

$$B_1 = \pm\frac{\varepsilon}{k_2^2}\sqrt{[1+2(k_2\cos(\alpha_1-\alpha_2))]\left(\frac{w_1^2}{k_1^2}-1\right)-\left(\frac{w_1^2}{k_2^2}-1\right)+[1+2(k_2\cos(\alpha_1-\alpha_2))]\left(\frac{w_1^2}{k_1^2}-1\right)} \tag{62}$$

$$\varepsilon = \left(\sqrt{(1+2(k_2\cos(\alpha_1-\alpha_2)))}+1+2(k_2\cos(\alpha_1-\alpha_2))-1\right)^{-1} \tag{63}$$

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APPLICATIONS

Using these values, the higher-order deformation equations that was solved by means of the symbolic computation software mathematica
The higher order deformation of the surface wave's elevation is given by

$$\eta(\xi_1,\xi_2)=\eta_1(\xi_1,\xi_2)+\eta_2(\xi_1,\xi_2)+\eta_3(\xi_1,\xi_2)+\eta_4(\xi_1,\xi_2)+\eta_5(\xi_1,\xi_2) \tag{64}$$

and the higher order deformation of the velocity potential is given by

$$\phi(\xi_1,\xi_2,z)=\phi_0(\xi_1,\xi_2,z)+\phi_1(\xi_1,\xi_2,z)+\phi_2(\xi_1,\xi_2,z)+\phi_3(\xi_1,\xi_2,z)+\phi_4(\xi_1,\xi_2,z)+\phi_5(\xi_1,\xi_2,z) \tag{65}$$

where $\eta(\xi_1,\xi_2)$ is the higher order surface wave elevation. $\phi(\xi_1,\xi_2,z)$ is the higher order potential function.
The analytical analysis of the rogue waves equations have been carried out using Homotopy Analysis Method. Higher order deformation equation of the waves elevation and velocity potential have been derived in equations (64) and (65).

THE HIGHER ORDER DEFORMATION EQUATION TO GENERATE ROGUE WAVE

The Higher order deformation equation of wave elevation to generate Rogue wave in random sea state is given by equation (64). At different values convergence-control parameter h using equation (64) we have these plots as

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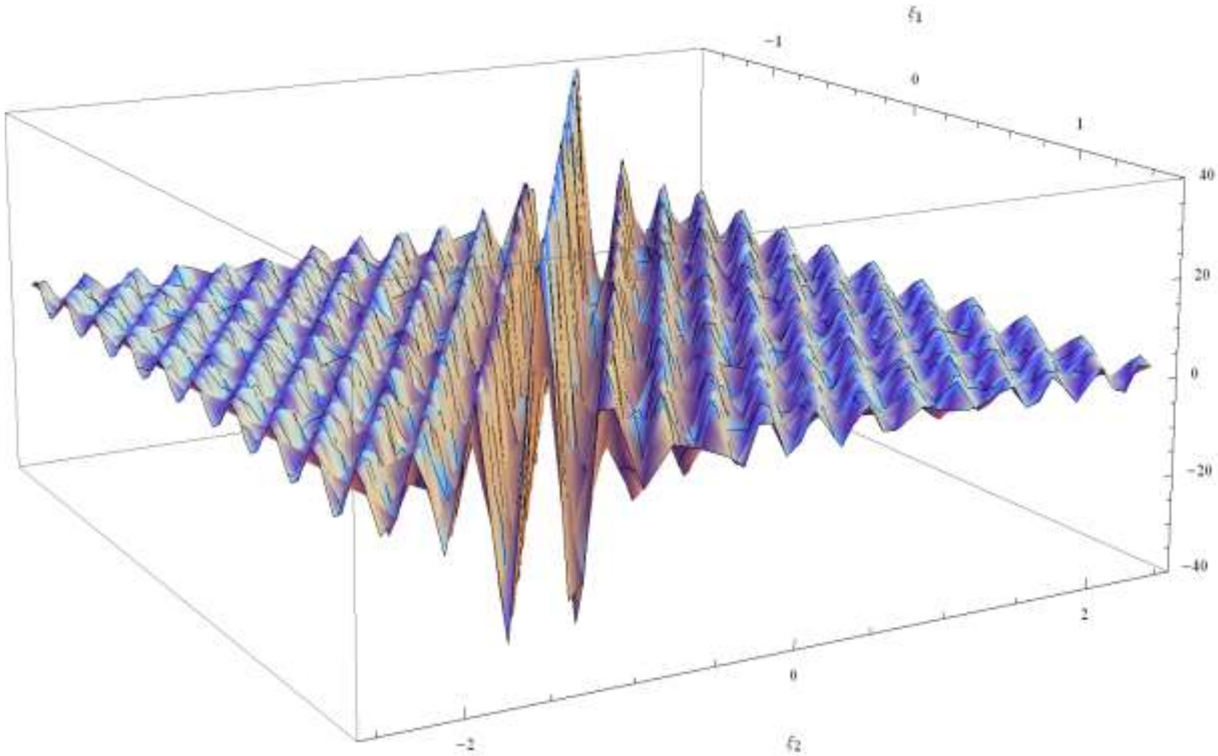

Fig. 1
The plot of wave elevation at the convergence-control parameter h = -1 using equation (64).

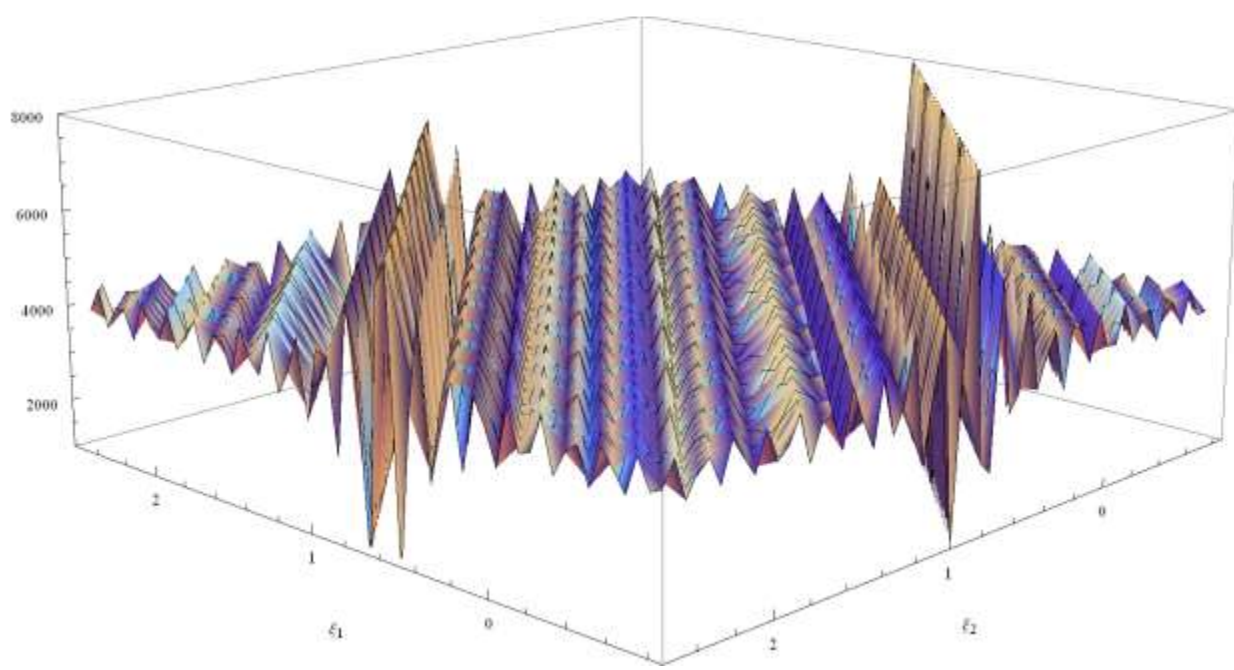

Fig. 2

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The plot of wave elevation at the convergence-control parameter h = -0.8 using equation (64)

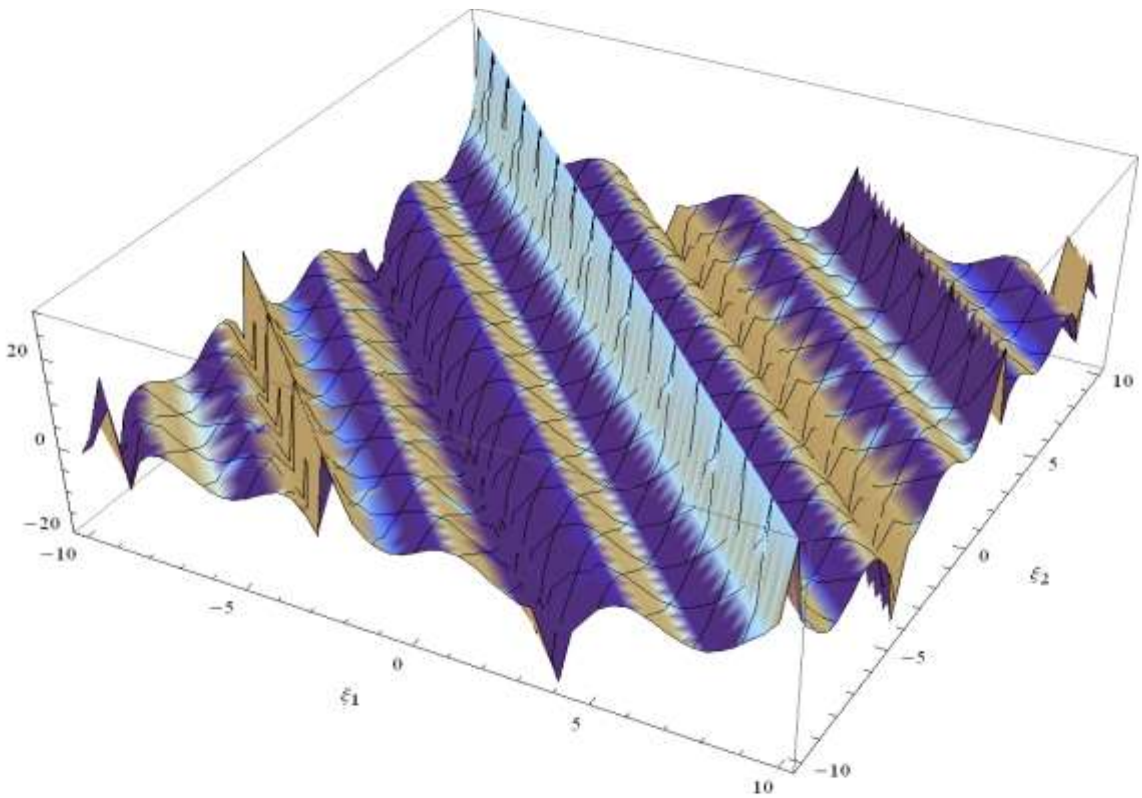

Fig. 3 The plot of wave elevation at the convergence-control parameter h = -0.6 using equation (64).

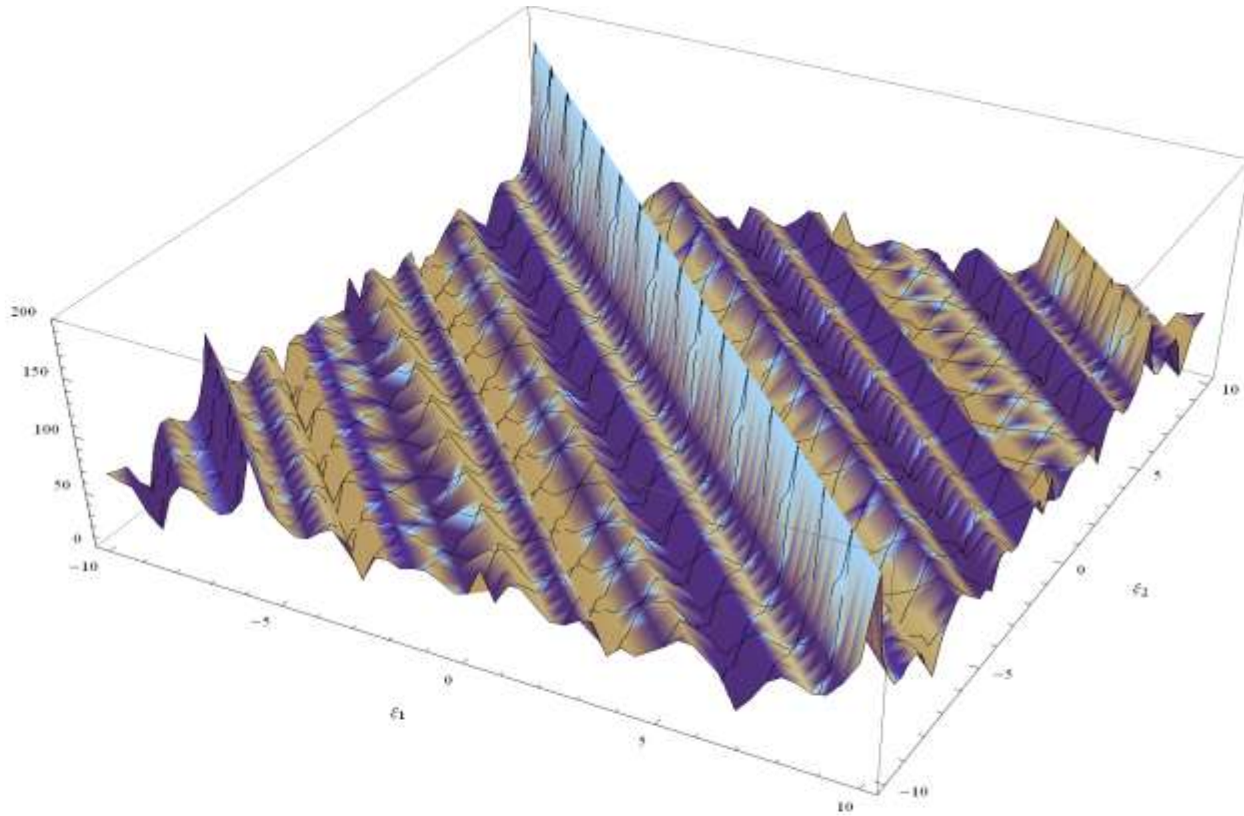

Fig. 4
The plot of wave elevation at the convergence-control parameter h = -0.4 using equation (64).

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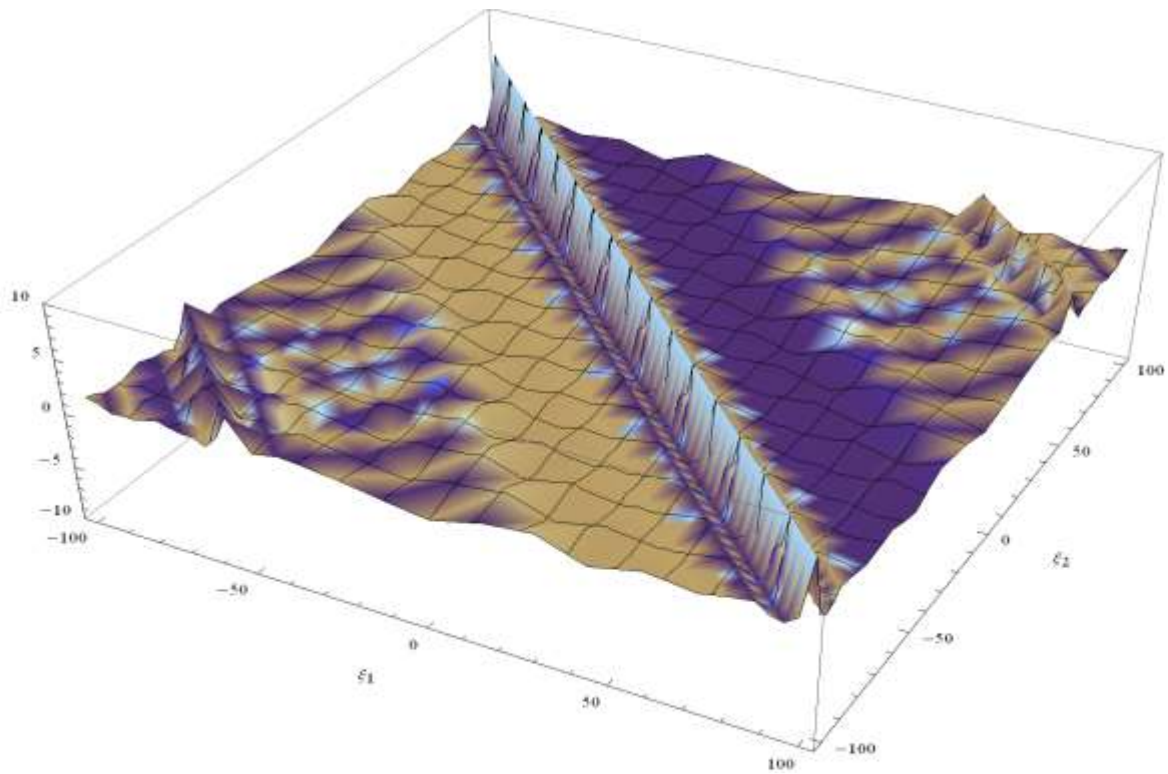

Fig. 5 The plot of wave elevation at the convergence-control parameter $h = -0.2$ using equation (64).

RESULTS AND CONCLUSION

The analytical analysis of the Rogue wave equations have been carried out using Homotopy Analysis Method. Higher order deformation equation of the wave elevation and velocity potential have been derived in equations (64) and (65). Using differenent values of the convergence- control parameter h say $-1 \le h < 0.2$ we observe the behaviour of the waves; and check if the optimal value of h we chose provdies the waves that satisfied the definition and characteristic of Rogue wave.

We plot the graph using equation (64) and different values of h against ε_1 *and* ε_2. Fig 1 is at the values of $h = -1$, shows a very clear graph of a Rogue wave that is showing giant wave (a monster) in the middle of an ocean. Such waves are usually accompanied by deep troughs (holes), between which is a high crest. Fig 2 is at the values of $h = -0.8$ shows that Rogue wave can appear concurrently, in the plot we can see two giant waves appearing in different location in the middle of an ocean, these waves are very dangerous to on coming ships and the marine structures. Figs 3-5 are at the values of $h = -0.6, -0.4, -0.2$ show the plot Rogue as a natural phenomenon, we have different shapes of Rogue wave. These waves satisfied the definition and characteristic of Rogue wave. The popular definition of Rogue wave is a wave whose height is twice more than the average wave height present in the ocean surface (i.e. the average significant wave height). These Rogue waves are often described as vertical walls of water in the ocean.

In fact, as we can see h the convergence-control parameter shows that the optimal value of the convergence-control parameter $h=-1$. It does not only make the series to converge fast but descries a Rogue waves.

Physically, we found that, the higher order deformation equations of wave elevation satisfied the definition and characteristic of Rogue wave, for a fully developed Rogue wave system.

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