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A STUDY ON THE MIXTURE OF EXPONENTIATED-WEIBULL DISTRIBUTION PART I (THE METHOD OF MAXIMUM LIKELIHOOD ESTIMATION)

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ABSTRACT: *Mixtures of measures or distributions occur frequently in the theory and applications of probability and statistics. In the simplest case it may, for example, be reasonable to assume that one is dealing with the mixture in given proportions of a finite number of normal populations with different means or variances. The mixture parameter may also be denumerable infinite, as in the theory of sums of a random number of random variables, or continuous, as in the compound Poisson distribution. The use of finite mixture distributions, to control for unobserved heterogeneity, has become increasingly popular among those estimating dynamic discrete choice models. One of the barriers to using mixture models is that parameters that could previously be estimated in stages must now be estimated jointly: using mixture distributions destroys any additive reparability of the log likelihood function. In this thesis, the maximum likelihood estimators have been obtained for the parameters of the mixture of exponentiated Weibull distribution when sample is available from censoring scheme.The maximum likelihood estimators of the parameters and the asymptotic variance covariance matrix have been obtained. A numerical illustration for these new results is given*.

KEYWORDS: Mixture distribution, Exponentiated Weibull Distributiom (EW), Mixture of two Exponentiated Weibull Distribution(MTEW), Maximum Likelihood Estimation, Moment Estimation, Monte-Carlo Simulation

INTRODUCTION

In probability and statistics, a mixture distribution is the probability distribution of a random variable whose values can be interpreted as being derived in a simple way from an underlying set of other random variables. In particular, the final outcome value is selected at random from among the underlying values, with a certain probability of selection being associated with each. Here the underlying random variables may be random vectors, each having the same dimension, in which case the mixture distribution is a multivariate distribution.

In Many applications, the available data can be considered as data coming from a mixture population of two or more distributions. This idea enables us to mix statistical distributions to get a new distribution carrying the properties of its components. In cases where each of the underlying random variables are continuous, the outcome variable will also be continuous and its probability density function is sometimes referred to as a mixture density. The c.d.f. of a mixture is convex combination of the c.d.f's of its components. Similarly, the p.d.f. of the mixture can also express as a convex combination of the p.d.f's of its components. The number of components in mixture distribution is often restricted to being finite, although in some cases the components may be countable. More general cases (i.e., an uncountable set of component distributions), as well as the countable case, are treated under the title of compound distributions

A mixture is a weighted average of probability distribution with positive weights that add up to one. The distributions thus mixed are called the components of the mixture. The weights themselves comprise a probability distribution called the mixing distribution. Because of these weights, a mixture is in particular again a probability distribution. Probability distributions of this type arise when observed phenomena can be the consequence of two or more related, but usually unobserved phenomena, each of which leads to a different probability distribution. Mixtures and related structures often arise in the construction of probabilistic models. Pearson (1894) was the first researcher in the field of mixture distributions who considered the mixture of two normal distributions. After the study of Pearson (1894) there was long gap in the field of mixture distributions. Decay (1964) has improved the results of Pearson (1894), Hasselblad (1968) studied in greater detail about the finite mixture of distributions.

Life testing is an important method for evaluating component's reliability by assuming a suitable lifetime distribution. Once the test is carried out by subjecting a sample of items of interest to stresses and environmental conditions that typify the intended operating conditions, the lifetimes of the failed items are recorded. Due to time and cost constraints, often the test is stopped at a predetermined time (Type I censoring) or at a predetermined number of failures (Type II censoring).

If each item in the tested sample has the same chance of being selected, then the equal probability sampling scheme is appropriate, and this has lead theoretically to the use of standard distributions to fit the obtained data. If the proper sampling frame is absent and items are sampled according to certain measurements such as their length, size, age or any other characteristic (for example, observing in a given sample of lifetimes that large values are more likely to be observed than small ones). In such a case the standard distributions cannot be used due to the presence of certain bias (toward large value in our example), and must be corrected using weighted distributions.

In lifetesting reliability and quality control problems, mixed failure populations are sometimes encountered. Mixture distributions comprise a finite or infinite number of components, possibly of different distributional types, that can describe different features of data. Some of the most important references that discussed different types of mixtures of distributions are Jaheen (2005) and AL-Hussaini and Hussien (2012).

Mixture of distributions can be treated from two points of view. The first one is that the experimenter knows in advance the population of origin of each tested item before placed on test or after the test has been terminated by failure analysis. This kind of data can be named classified data or post-mortem data (in the case of post failure analysis). This idea was adopted by Mendenhall and Hader (1958) who derived the likelihood function adequate for this situation in the case of two –component mixture under type I censored data and generalized it to the case of k – components (k >2).

Finite mixture models have been used for more years, but have seen a real boost in popularity over the last decade due to the tremendous increase in available computing power. The areas of application of mixture models range from biology and medicine to physics, economics and marketing. On the one hand these models can be applied to data where observations originate from various groups and the group affiliations are not known, and on the other hand to provide approximations for multi-modal distributions [see Everitt and Hand (1981); Titterington et al. (1985); Maclachlan and Peel (2000), Shawky and Bakoban (2009) and Hanna Abu-Zinadah (2010)].We shall consider the exponentiated Weibull model, which includes as special case the Weibull and exponential models. The Exponentiated Weibull family EW [introduced by Mudholkar and Srivastava (1993) as extention of the Weibull family] Contains distributions with bathtub shaped and unimodal failure rates besides a broader class of monoton failure rates. Applications of the exponentiated models have been carried out by some authors as Bain (1974); Gore et al. (1986); and Mudholkar and Hutson (1996).

Some statistical properties of this distribution (EW) are discussed by Singh et al. (2002). Ashour and Afifiy (2008) derived maximum likelihood estimators of the parameters for EW with type II progressive interval censoring with random removals and their asymptotic variances.

The aim of this research is to introduce a study of a mixture of two Exponentiated Weibull distribution, study of the behavior of the failure rate function of this mixture and handle the problems of estimation.

Research Outline

1. Derivation of statistical properties of the model.
2. Obtain maximum likelihood estimators of the parameters, reliability and hazard functions from type II censored samples..
3. Monte Carlo simulation study will be done to compare between these estimators and the maximum likelihood.

Additional to this introductory chapter, this thesis contains four chapters:
In chapter (2) some properties of the mixture of two exponentiated Weibull distribution will be studying .Chapter (3) is concerned with the estimation of the Mixture of the exponentiated Weibull distribution parameters has been drived via maximum likelihood estimation method. Chapter(4) a nenumerical data will be illustrated using real data and Simulation technique has been used to study the behaviour of the estimators using the Mathcad (2011) packages.

THE MIXTURE OF TWO EXPONENTIATED WEIBULL DISTRIBUTION

In this chapter, we consider the mixture of two – component Exponentiated Weibull (MTEW) distribution. Some properties of the model with some grahps of the density and hazard functions are discussed. The maximum likelihood estimation is used for estimating the parameters, reliability, and hazard functions of the model under type II censored samples.

Mixture Models

Mixtures of life distributions occur when two different causes of failure are present, each with the same parametric form of life distributions. In recent years, the finite mixtures of life distributions have proved to be of considerable interest both in terms of their methodological development and practical applications [see Titterington et al. (1985), Mclachlan and Basford (1988), Lindsay (1995), Mclachlan and Peel (2000) and Demidenko (2004)].

Mixture model is a model in which independent variables are fractions of a total. One of the types of mixture of the distribution functions which has its practical uses in a variety of disciplines. Finite mixture distributions go back to end of the last century when Everitt and Hand (1981) published a paper on estimating the five parameters in a mixture of two normal distributions. Finite mixtures involve a finite number of components. It results from the fact that different causes of failure of a system could lead to different failure distributions, this means that the population under study is non-homogenous.

Suppose that T is a continuous random variable having a probability density function of the form:

$$f(t)=\sum_{j=1}^{k} p_j f_j(t), \qquad t>0,\ \ k>1, \tag{1}$$

where $0 \le p_j \le 1, \quad j=1,2,...,k$ and $\sum_{i=1}^{k} p_i = 1$. The corresponding c.d.f. is given by:

$$F(t)=\sum_{j=1}^{k} p_j F_j(t), \qquad t>0,\ \ k>1, \tag{2}$$

where k is the number of components, the parameters $p_1, p_2, \dots, p_k$ are called mixing parameters, where p_i represent the probability that a given observation comes from population "i" with density $f_i(.)$, and $f_1(.), f_2(.), \dots, f_k(.)$ are the component densities of the mixture. When the number of components k=2, a two component mixture and can be written as:

$$f(t)=p\ \ f_1(t)+(1-p)\, f_2(t),$$

When the mixing proportion 'p' is closed to zero, the two component mixture is said to be not well separated.

Definition (1): Suppose that T and Y be two random variables. Let $F(t|y)$ be the distribution function of T given Y and G(y) be the distribution function of Y. The marginal distribution function $F(t)$, defined by:

$$F(t)=\int_{-\infty}^{\infty} F(t\mid y).dG(y), \tag{3}$$

is called a mixture of the distribution function $F(t|y)$ and $G(y)$ where $F(t|y)$ is known as the kernel of the integral and $G(y)$ as the mixing distribution .

A special case from definition (1) when the random variable Y is a discrete number of points $\{y_j, j = 1, 2, 3, \dots, k\}$ and G is discrete and assigns positive probabilities to only those values of Y; the integral (2.16) can be replaced by a sum to give a countable mixture:

$$F(t)=\sum_{j=1}^{\infty} g(y_j).F(t\mid y_j),$$

where $g(y_j)$ is the probability of y_j . If the random variable Y assumes only a finite number of distributions $\{y_j, j = 1, 2, 3, \dots, k\}$, Ahmed et al. (2013) have been used the finite mixture:

$$F(t)=\sum_{j=1}^{k} w_i F_j(t), \tag{4}$$

By differentiating (4) with respect to T, the finite mixture of probability density functions can be obtained as follows

$$f(t)=\sum_{j=1}^{k} w_i f_j(t), \tag{5}$$

where

$$f_j(t) = \frac{dF_j(t)}{dt} = \frac{dF(t|y_j)}{dt} = f\big(t|y_j\big),$$

In (5), the masses w_j called the mixing proportions, they satisfy the conditions:

$w_j \geq 0$ and $\sum_{j=1}^{k} w_j = 1,$

$F_j(.)$ *and* $f_j(.)$ are called the j^{th} component in the finite mixture of distributions (4) and probability density functions (5), respectively. Thus, the mixture of the distribution functions can be defined as a distribution function that is a linear combination of other distribution functions where all coefficients are non-negative and add up to 1.
The parameters in number of expressions (4) or (5) can be divided into three types. The first consists solely of k, the components of the finite mixture. The second consists of the mi- xing proportions w. The third consists of the component parameters (the parameters of $F_j(.)$ or $f_j(.)$).

Reliability of finite mixture of distributions
An important topic in the field of lifetime data analysis is to select and specify the most appropriate life distribution that describes the times to failure of a component, subassembly, assembly or system. This requires the collection and analysis of the failure data obtained by measurements or simulations in order to fit the model empirically to the observed failure process.

There are two general approaches to fitting reliability distributions to failure data. The first approach is to derive an empirical reliability function directly from data, since no parameters exist. The second and usually preferred approach is to identify an appropriate parametric distribution, such as exponential, Weibull, normal, lognormal or gamma, and to estimate the unknown parameters. There are several reasons to prefer the later approach, for instance, binning the data does not provide information beyond the range of the sample data, whereas with a parametric distribution this is possible. Continuous reliability distribution can be applied also in performing more complex analysis of the failure process, [see Ebeling (1997)].

The two and three-parameter Weibull distribution are one of the most commonly used distributions in reliability engineering because of the many shapes they attain for various values of shape and scale parameters. It can therefore model a great variety of data and life characteristics. Since the shape of the life distribution is often composed of more than one basic shape, a natural alternative is to introduce the mixture distribution as the genuine distribution for times to failure modeling. A significant difficulty common to all mixed distributions is the estimation of unknown parameters.

There is anumber of papers dealing with 2-fold mixture models for times to failure modeling. For example, Jiang and Murthy (1995) characterized the 2-fold Weibull mixture models in terms of the Weibull probability plotting, and examined the graphical plotting approach to determine if a given data set can be modeled by such models. Ling and pan (1998) proposed the method to estimate the parameters for the sum of two three- parameter Weibull distributions. Based on these findings, a new procedure for the selection of population distribution and parameter estimation was presented.
The reliability of the mixture distributions is given by:

$$\begin{aligned} R(t) &= \sum_{j=1}^{k} p_j R_j(t), \qquad t > 0, \;\; k > 1 \\ &= p_1[1 - F_1(t)] + p_2[1 - F_2(t)], \end{aligned} \tag{6}$$

Exponentiated Weibull Distribution (EW)

Salem and Abo-Kasem (2011) drived EW distribution in the following details; the "Exponentiated Weibull family" introduced by Mudholkar and Srivastava (1993) as extension of the Weibull family, contains distribution with bathtub shabed and unimodale failure rates besides a broader class of monotone failure rates. The applications of the exponentiated Weibull (EW) distribution in reliability and survival studies were illustrated by Mudholkar et al. (1995). Its' properties have been studied in more detail by Mudholkar and Hutson (1996) and Nassar and Eissa (2003). The probability density function (p.d.f.), the cumulative distribution function (c.d.f.) and the reliability function of the exponentiated Weibull are given respectively by;

$$f(t) = \alpha\theta(1-e^{-t^{\alpha}})^{\theta-1}e^{-t^{\alpha}}t^{\alpha-1}, \qquad t,\alpha,\theta > 0, \tag{7}$$

$$F(t,\alpha,\theta) = [1-e^{-t^{\alpha}}]^{\theta}, \tag{8}$$

and

$$R(t) = [1-(1-e^{-t^{\alpha}})^{\theta}], \qquad t > 0, \tag{9}$$

Where α and θ are the shape parameters of the model (7). The distinguished feature of EW distribution from other life time distribution is that it accommodates nearly all types of failure rates both monotone and non-monotone (unimodal and bathtub). The EW distribution includes a number of distributions as particular cases: if the shape parameter θ = 1, then the p.d.f. is that Weibull distribution, when α = 1 then the p.d.f is that Exponentiated Exponential distribution, if α = 1 and θ = 1 then the pdf is that Exponential distribution and if α = 2 then the p.d.f is that one parameters Burr-X distribution. Mudholkar and Hutson (1996) showed that the density function of the EW distribution is decreasing when αθ ≤ 1 and unimodal when αθ ≥ 1.

Statistical Properties of EW distribution

The statistical properties are very important to identify the distributions. Once a life time distribution representation for a particular item is known, it may be of interest to compute a moment or fractile of the distribution. Although moments and fractiles contain less information than a life time distribution representation, they are often useful ways to summarize the distribution of a random life time. Mudholkar and Hutson (1996) discussed som statistical measure for the EW distribution in the following detailes.

Moments: the r^th central moment $\mu_r' = E(t^r)$ of the EW distributin with density given by equation (7) is given by:

$$\mu_r^{\backslash} = E(T^r) = \theta\int_0^{\infty} t^{\frac{r}{\alpha}}(1-e^{-t})^{\theta-1}\, e^{-t}\, dt, \tag{10}$$

In general the moments are analytically intractable, but can be studied numerically. Also an examination of (10) shows that for α > 0 the moments of all orders exist but it is not always so when the family is extended to α < 0.

Skewness and Kurtosis: The coefficient of skewness v_3 the coefficient of kurtosis v_4 can be used to understand the nature of the exponentiated Weibulle distribution.

Quantile function: The exponentiated Weibull family introduced by Mudholkar and Srivastava (1993) is defined by the quantil function

$$Q(U) = [-\ln(1-U^{1/\theta})]^{1/\alpha}, \qquad \alpha,\theta > 0, \tag{11}$$

At $\theta = 1$, (11) corresponds to the Weibull family which includes the exponential distribution the exponentiated Weibull may be extended to negative values, continuously at $\alpha = 0$, by modifying $Q(U)$ at (11) to $[Q(U)-\alpha]/\alpha$. This extended family includes the reciprocal Weibull family, and at $\alpha=0$ consists of the extreme value distributions.

Currently, there are little studies for the use of the EW in reliability estimation. Ashour and Afify (2007) considered the analysis of EW family distributed lifetime data observed under type I progressive interval censoring with random removals, maximum likelihood estimators of the parameters and their asymptotic variances are derived. Ashour and Afify (2008) derived maximum likelihood estimators for the parameters of EW with type II progressive interval censoring with random removals and their asymptotic variances. Kim et al. (2009) derived the maximum likelihood and Bayes estimators for EW lifetime model using symmetric and asymmetric loss functions [see Salem and Abo-Kasem (2011)].

Statistical properties for The Mixture of Exponentiated Weibull Distribution.

The failure of an item or a system can be caused by one or more than one cause of failure; it results that the density of time to failure can have one mode or multimodal shape and in that case, finite mixtures represent a good tool to model such phenomena. Suppose that two populations of the exponentiated Weibull (EW) distribution with two shapes parameters α and θ [see Mudholkar and Hutson (1996)] mixed in unknown proportions p and (1-p) respectively.

A random variable T is said to follow a finite mixture distribution with k components, if the p.d.f. of T can be written in the form (1) [see Titterington et al. (1985)]. Where $j=1,\dots,k$, $f_j(t)$ the j^{th} p.d.f. component (7) and the mixing proportions ,p_j , satisfy the conditions $0<p_j<1$ and $\sum_{j=1}^{k} p_j=1$, the corresponding c.d.f., is given by (2), where $F_j(t)$ is the j^{th} c.d.f., component (8) , the reliability function (RF) of the mixture is given by (6), where $R_j(t)$ is the j^{th} reliability component (9) . The hazard function (HF) of the mixture is given by

$$H(t)=\frac{\sum_{j=1}^{k} p_j f_j(t)}{\sum_{j=1}^{k} p_j R_j(t)},$$

where $f(t)$ and $R(t)$ are defined in (1) and (6) respectively.

Mixture of K EW components: Substituting (7) and (8) in (1) and (2), the p.d.f. and c.d.f. of MTEW components are given respectively, by:

$$f(t)=\sum_{j=1}^{k} p_j\alpha_j\theta_j t^{\alpha-1}e^{-t^{\alpha_j}}(1-e^{-t^{\alpha_j}})^{\theta_j-1},\quad t>0,\ \ \alpha_j,\theta_j>0 \tag{12}$$

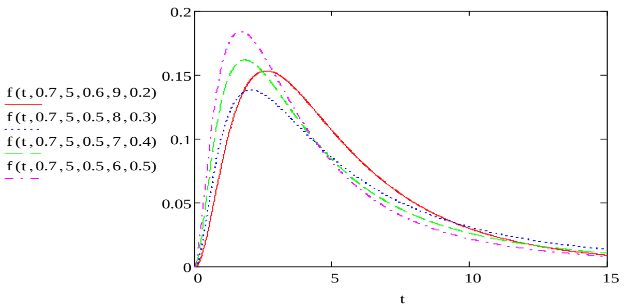

Fig (1) Shapes of MTEW distribution with $(P,\alpha_1,\theta_1,\alpha_2,\theta_2)$

Figure (1) shows some densities of MTEW distribution.

$$F(t) = \sum_{j=1}^{k} p_j (1 - e^{-t^{\alpha_j}})^{\theta_j}, \quad t > 0, \ \ \alpha_j, \theta_j > 0 \tag{13}$$

where, for $j = 1,\ldots,k$, $0 < p_j < 1$ and $\sum_{j=1}^{k} p_j = 1$.

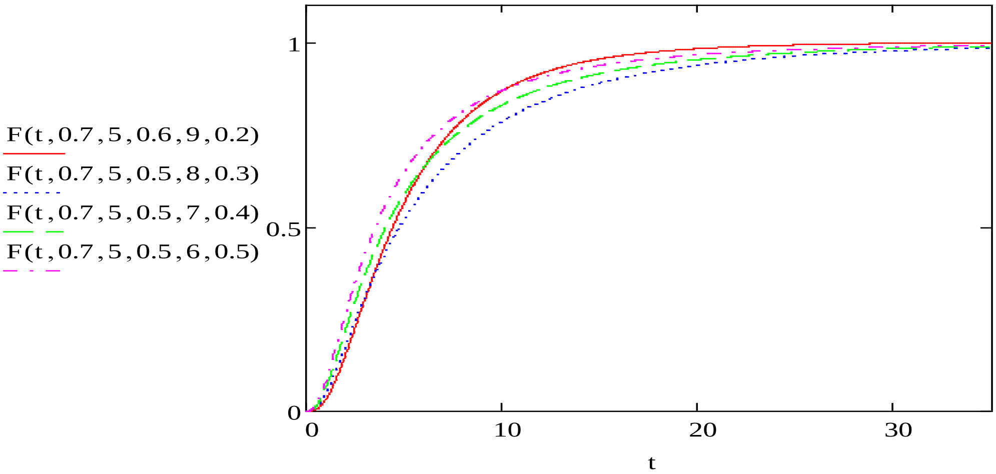

Fig (2) Shapes of (MTEW) distribution c.d.f.

Figure (2) shows some cumulative distribution functions of MTEW distribution.

By observing that R(t) = 1- F(t) and $\sum_{j=1}^{k} p_j = 1$, the RF of MTEW distribution , $j = 1, 2, \ldots, k$ components can be obtained from (6) and (9) as :

$$R(t) = \sum_{j=1}^{k} p_j [1 - (1 - e^{-t^{\alpha_j}})^{\theta_j}], \quad t > 0, \ \ \alpha_j, \theta_j > 0 \tag{14}$$

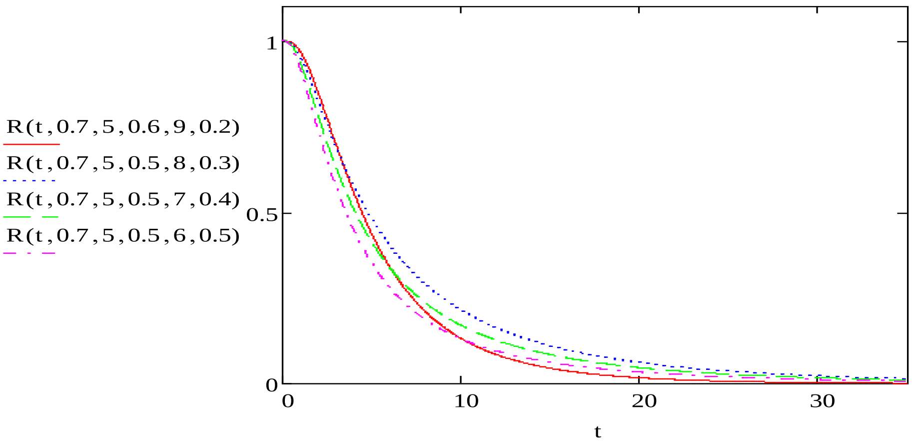

Fig (3) Shapes of (MTEW) distribution RF

Figure (3) shows some reliability functions (RF) of MTEW distributions.

dividing (12) by (14), we obtain the HF of MTEW distribution as:

$$H(t)=\frac{\sum_{j=1}^{k} p_j\alpha_j\theta_j t^{\alpha-1}e^{-t^{\alpha_j}}(1-e^{-t^{\alpha_j}})^{\theta_j-1}}{\sum_{j=1}^{k} p_j[1-(1-e^{-t^{\alpha_j}})^{\theta_j}]}\quad t>0,\ \ \alpha_j,\theta_j>0, \tag{15}$$

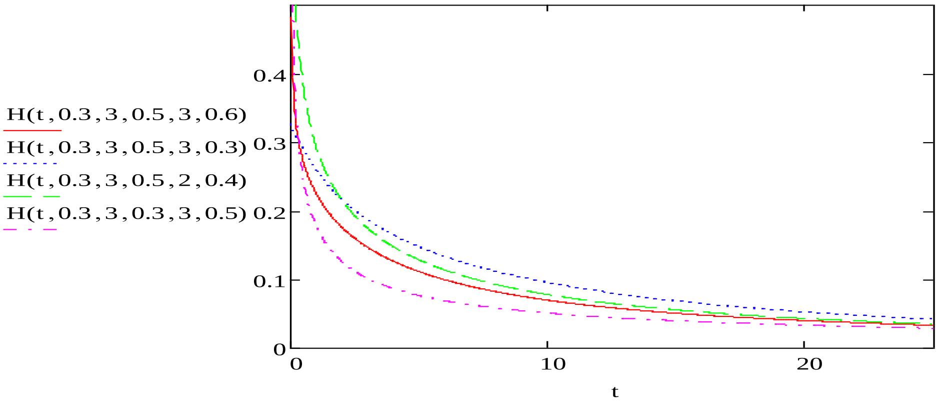

Fig (4) (MTEW) distributions HF

Figure (4) shows some reliability functions (HF) of MTEW distribution.
If k = 2, the p.d.f., c.d.f. RF and HF of MTEW distribution are then given, respectively by
$f(t)=p\,\alpha_1\theta_1 t^{\alpha_1-1}e^{-t^{\alpha_1}}(1-e^{-t^{\alpha_1}})^{\theta_1-1}+\ (1-p)\alpha_2\theta_2 t^{\alpha_2-1}e^{-t^{\alpha_2}}(1-e^{-t^{\alpha_2}})^{\theta_2-1}$,
$F(t)=p(1-e^{-t^{\alpha_1}})^{\theta_1}+\ (1-p)\ (1-e^{-t^{\alpha_2}})^{\theta_2}$,
$R(t)=p(1-\left(1-e^{-t^{\alpha_1}}\right)^{\theta_1})+\ (1-p)\ (1-\left(1-e^{-t^{\alpha_2}}\right)^{\theta_2})$,
and

$$H(t)=\frac{f(t)}{R(t)},$$

- **Moments and some measures**: The r^th moment about the origin, $\mu_r^{\backslash}=E(T^r)$ of MTEW distribution, with p.d.f (12) in the non – closed form is:

$$\mu_r^{\backslash}=\sum_{j=1}^{K} p_j\alpha_j\theta_j\int_0^{\infty} t^{\alpha_j+r-1}\,e^{-t^{\alpha_j}}(1-e^{-t^{\alpha_j}})^{\theta_j-1}\,dt\,,r=1,2,\ldots$$

which is a non-closed form. To evaluate $\mu_r^{\backslash}$ using this form, we resort to numerical integration for all positive values of α_j and θ_j. For positive integer values of θ_j, $\mu_r^{\backslash}$ takes the form

$$\mu_r^{|}=\sum_{j=1}^{k}\sum_{j_1=1}^{\theta_j-1} p_j\theta_j(-1)^{j_1}\binom{\theta_j-1}{j_1}\int_0^{\infty} t^{\frac{r}{\alpha_j}}e^{-t(j_1+1)}dt,$$

$$=\sum_{j=1}^{K}\sum_{j_1=1}^{\theta_j-1} P_j\theta_j(-1)^{j_1}\binom{\theta_j-1}{j_1}\frac{\Gamma(\frac{r}{\alpha_j}+1)}{(j_1+1)^{\frac{r}{\alpha_j}+1}}$$

$$=\sum_{j=1}^{k} p_j\theta_j[\Gamma(\frac{r}{\alpha_j}+1)A_r(\theta_j)],$$

where

$$A_r(\theta_j)=\sum_{j_1=1}^{\theta_j-1}(-1)^{j_1}\binom{\theta_j-1}{j_1}\frac{1}{(j_1+1)^{\frac{r}{\alpha_j}+1}},$$

when k=2, the rth moment about the origin , $\mu_r^{\backslash} = E(T^r)$ of MTEW distribution with p.d.f (12) is given by

$$\mu_r^{\backslash} = P\,\theta_1\left[\Gamma\left(\frac{r}{\alpha_1}+1\right)\ A_r(\theta_1)\right] + (1-P)\theta_2\left[\Gamma\left(\frac{r}{\alpha_2}+1\right)\ A_r(\theta_2)\right] \tag{16}$$

The closed form (16) of $\mu_r^{\backslash}$ allows us to derive the following forms of statistical measures for MTEW distribution:

1. **Coefficient of variation:**

$$c.v=\frac{\sigma}{\mu}=\frac{\sqrt{\sum_{j=1}^{k}p_j\theta_j\left[\Gamma(\frac{2}{\alpha_j}+1)A_2(\theta_j)\right]-\left(\sum_{j=1}^{k}p_j\theta_j\left[\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]\right)^2}}{\sum_{j=1}^{k}p_j\theta_j\left[\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]},$$

where

$$\mu=\mu_1^{\backslash}=\sum_{j=1}^{k}p_j\theta_j[\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)],$$

and

$$\sigma^2=\sum_{j=1}^{k}p_j\theta_j\left[\Gamma(\frac{2}{\alpha_j}+1)A_2(\theta_j)\right]-\left(\sum_{j=1}^{k}p_j\theta_j\left[\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]\right)^2,$$

2. **Skewness:**

$$v_3=\frac{\sum_{j=1}^{k}p_j\theta_j\left[\Gamma(\frac{3}{\alpha_j}+1)A_3(\theta_j)\right]-3\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{2}{\alpha_j}+1)A_2(\theta_j)\right]\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]+w_1}{\left[\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{2}{\alpha_j}+1)A_2(\theta_j)\right]-\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]^2\right]^{\frac{3}{2}}},$$

where

$$w_1=2\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]^3$$

3. Kurtosis:

$$v_4=\frac{\sum_{j=1}^{k}p_j\theta_j\left[\Gamma(\frac{4}{\alpha_j}+1)A_4(\theta_j)\right]-4[\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{3}{\alpha_j}+1)A_3(\theta_j)\right]\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]+w_2}{\left(\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{2}{\alpha_j}+1)A_2(\theta_j)\right]-\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]^2\right)^2},$$

where

$$w_2=6\left[\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{2}{\alpha_j}+1)A_2(\theta_j)\right]\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]^2\right]$$

$$-3\left[\sum_{j=1}^{k}p_j\theta_j\Gamma(\frac{1}{\alpha_j}+1)A_1(\theta_j)\right]^4,$$

and abbreviation A_r is for $A_r(\theta_j), r=1,2,...,$

The result developed (16) can be derived from the moment generating function (m.g.f) of MTEW distribution as follows

$$\mu_t(y)=\sum_{j=1}^{k} p_j\alpha_j\theta_j\int_0^{\infty} t^{\alpha_j-1}e^{-t^{\alpha_j}}e^{yt}(1-e^{-t^{\alpha_j}})^{\theta_j-1}dt, \quad y>0, \tag{17}$$

when $p_1=1$, (17) gives the (m.g.f) of MTEW distribution (12).

Median and Mode: If T is continuous, then t_p is a solution to the equation

$$F(t_p)=p,$$

The median of T is the 50th percentile, denote by $t_{0.5}, m, m_t$. Half of the population values are above the median, and half are below it, so it is sometimes used instead of the mean as acentral measure [see Ahmed et al (2013)]. From equation (13), the median *m* is defined as the numerical solution of the fillow equation

$$\sum_{j=1}^{k} p_j(1-e^{-m^{\alpha_j}})^{\theta_j}=0.5, \quad t>0, \quad \alpha_j,\theta_j>0, \quad 0>p>1,$$

Next, to find the mode for MTEW distribution, we differentiate $f(t)$ with respect to t so (12) gives.

$$f^{\backslash}(t)=\sum_{j=1}^{k} p_j[\frac{(\alpha_j-1)-\alpha_j t^{\alpha_j}}{t}-\alpha_j(\theta_j-1)t^{\alpha_j-1}e^{-t^{\alpha_j}}(1-e^{-t^{\alpha_j}})^{-1}]f_j(t),$$

Then by equating $f^{\backslash}(t)$ with zero. Since $f(t)$ is always positive for $t>0,$ then we have

$$\sum_{j=i}^{k} p_j[\frac{(\alpha_j-1)-\alpha_j t^{\alpha_j}}{t}-\alpha_j(\theta_j-1)t^{\alpha_j-1}e^{-t^{\alpha_j}}(1-e^{-t^{\alpha_j}})^{-1}]=0,$$

MAXIMUM LIKELIHOOD ESTIMATION FOR THE UNKNOWN PARAMETERS of MTEW under Type II Censored Sample

Suppose a type-II censored sample $\underline{t}=(t_{1;n},t_{2;n},...,t_{r;n})$ where t_i the time of the ith component to fail. This sample of failure times are obtained and recorded from a life test of n items whose life time have MTEW distribution, with p.d.f, and c.d.f given, respectively, by (12) and (13). The likelihood function in this case [see lawless (1982)] can be written as

$$L(\underline{\theta}\,|\,\underline{t})=\frac{n!}{(n-r)!}\prod_{i=1}^{r} f(t_{(i)})[1-F(\Delta)]^{n-r}, \tag{18}$$

where $f(.)$ and $F(.)$ are the density and distribution functions, respectively.

when $\Delta=T$,then (18) reduces to the likelihood function of type-I censored, and when $\Delta=t_{(r)}$, then (18) reduces to the likelihood function of type–II censored. Type-I and type–II censoring corresponding to complete sampling when $n=r$.

where $f(t_{i;n})$ and $R(t_{i;n})$ are given, respectively, by (12) and (14).

Assuming that the parameters, θ_1 and θ_2 are unknown, we differentiate the natural logarithm of the likelihood function

$$l = \ln \quad L(\underline{\theta}) = \ln\left(\frac{n!}{(n-r)!}\right) + \sum_{i=1}^{r} \ln f(t_{i;n}) + (n-r)\ln R(t_{r:n}),$$

with respect to θ_j so the likelihood equations are given by

$$l_j = \frac{\partial l}{\partial \theta_j} = \sum_{i=1}^{r}\left[\frac{1}{f(t_{i:n})} \cdot \frac{\partial f(t_{i:n})}{\partial \theta_j}\right] + \frac{n-r}{R(t_{r:n})} \cdot \frac{\partial R(t_{r:n})}{\partial \theta_j}, \quad j = 1,2, \tag{19}$$

where l_j is the first derivatives of the natural logarithm of the likelihood function with respect to θ_j, from (12) and (14) respectively .we have:

$$\frac{\partial f(t_{i:n})}{\partial \theta_j} = p_j f_j(t_{i:n}) \mathrm{k}_j^*(t_{i:n}), \tag{20}$$

and

$$\frac{\partial R(t_{r:n})}{\partial \theta_j} = -p_j F_j(t_{r:n}) \omega^*(t_{r:n}), \tag{21}$$

where $p_1 = p$, $p_2 = 1 - p$,

$$k_j^*(t_{i:n}) = \theta_j^{-1} + \omega^*(t_{i:n}), \tag{22}$$

and

$$\omega^*(t_{i:n}) = \ln\left(1 - e^{-t_{i:n}{}^{\alpha_j}}\right), \tag{23}$$

Substituting (20) and (21) in (19), we obtain

$$l_j = p_j\left\{\sum_{i=1}^{r} \zeta_j^{**}(t_{i:n}) k_j^*(t_{i:n}) - (n-r)\, \zeta_j^{***}(t_{r:n}) \omega^*(t_{r:n})\right\} = 0, \tag{24}$$

where l_j is the first derivatives of the natural logarithm of the likelihood function with respect to θ_j for j = 1,2, and i = 1,2, ..., r

$$\zeta_j^{**}(t_{i:n}) = \frac{f_j(t_{i:n})}{f(t_{i:n})}, \ \zeta_j^{***}(t_{r:n}) = \frac{F_j(t_{r:n})}{R(t_{r:n})}, \tag{25}$$

and $k_j^*(t_{i:n})$, $\omega^*(t_{i:n})$ are given by (22) and (23) respectively. Assuming that the parameters, α_1 and α_2 are unknown, we differentiate the natural logarithm of the likelihood function with respect to α_j so the likelihood equations are given by:

$$\frac{\partial l}{\partial \alpha_j} = \sum_{i=1}^{r}\left[\left(\frac{1}{f(t_{i:n})} \ \frac{\partial f(t_{i:n})}{\partial \alpha_j}\right)\right] + \frac{n-r}{R(t_{r:n})} \ \frac{\partial R(t_{r:n})}{\partial \alpha_j}, \tag{26}$$

where $\frac{\partial l}{\partial \alpha_j}$ is the first derivatives of the natural logarithm of the likelihood function with respect to α_j, from (12) and (14) respectively, we have

$$\frac{\partial f(t_{i:n})}{\partial \alpha_j} = p_j f_j(t_{i:n}) S_j(t_{i:n}), \tag{27}$$

and

$$\frac{\partial R(t_{r:n})}{\partial \alpha_j} = -p_j F_j(t_{r:n}) O_j(t_{r:n}), \tag{28}$$

where $p_1 = p$, $p_2 = 1 - p$,

$$S_j(t_{i:n}) = \alpha_j^{-1} + \ln(t_{i:n}) - t_{i:n}^{\alpha_j} \ln(t_{i:n}) + (\theta_j - 1)(1 - e^{-t^{\alpha_j}})^{-1}$$

$$\times e^{-t_{(i:n)}^{\alpha_j}} t_{(i:n)}^{\alpha_j} \ln(t_{i:n}), \tag{29}$$

and

$$O_j(t_{r:n}) = \theta_j t_{(r:n)}^{\alpha_j} e^{-t_{(r:n)}^{\alpha_j}} (1 - e^{-t_{r:n}^{\alpha_j}})^{-1} \ln(t_{r:n}), \tag{30}$$

Substituting (4.1.15) and (4.1.16) in (4.1.14), we obtain

$$\frac{\partial l}{\partial \alpha_j} = p_j \left\{ \sum_{i=1}^{r} \zeta_j^{**}(t_{i:n}) S_j(t_{i:n}) - (n-r)\, \zeta_j^{***}(t_{r:n}) O_j(t_{r:n}) \right\} = 0, \tag{31}$$

where $\frac{\partial l}{\partial \alpha_j}$ is the first derivatives of the natural logarithm of the likelihood function with respect to α_j for $j = 1,2$, and $i = 1,2,\ldots, r$,

$\zeta_j^{**}(t_{i:n})$, $\zeta_j^{***}(t_{r:n}), S_j(t_{i:n})$ and $O_j(t_{r:n})$ are given respectively by (25), (29) and (30).

The solution of the four nonlinear likelihood equations (24) and (31) yields the maximum likelihood estimate (MLE):

$\underline{\hat{\theta}} = (\hat{\theta}_{1,M}, \hat{\theta}_{2,M}, \hat{\alpha}_{1,M}, \hat{\alpha}_{2,M})$ of $\underline{\theta} = (\theta_1, \theta_2, \alpha_1, \alpha_2)$,

The MLE's of R(t) and H(t) are given, respectively, by (14) and (15) after replacing $(\theta_1, \theta_2, \alpha_1, \alpha_2)$ by their corresponding MLE's, $\hat{\theta}_{1,M}, \hat{\theta}_{2,M}, \hat{\alpha}_{1,M}$ and $\hat{\alpha}_{2,M}$.

Since the equations (24) and (31) are clearly transcendental equations in $\hat{\theta}_j$ and $\hat{\alpha}_j$; that is, no closed form solutions are known they must be solved by iterative numerical techniques to provide solutions (estimators), $\hat{\theta}_j$ and $\hat{\alpha}_j$, in the desired degree of accuracy. To study the variation of the MLE's $\hat{\theta}_j$ and $\hat{\alpha}_j$, the asymptotic variance of these estimators are obtained.

The asymptotic variance covariance matrix of $\hat{\theta}_j$ and $\hat{\alpha}_j$ is obtained by inverting the information matrix with elements that are negative expected values of the second order derivatives of natural logarithm of the likelihood function, for sufficiently large samples, a reasonable approximation to the asymptotic variance covariance matrix of the estimators can be obtained as;

$$\left.\begin{bmatrix} -\frac{\partial^2 l(\underline{\theta})}{\partial \theta_1^2} & -\frac{\partial^2 l(\underline{\theta})}{\partial \theta_1 \partial \theta_2} & -\frac{\partial^2 l(\underline{\theta})}{\partial \theta_1 \partial \alpha_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial \theta_1 \partial \alpha_2} \\ -\frac{\partial^2 l(\underline{\theta})}{\partial \theta_2 \partial \theta_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial \theta_2^2} & -\frac{\partial^2 l(\underline{\theta})}{\partial \theta_2 \partial \alpha_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial \theta_2 \partial \alpha_2} \\ -\frac{\partial^2 l(\underline{\theta})}{\partial \alpha_1 \partial \theta_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial \alpha_1 \partial \theta_2} & -\frac{\partial^2 l(\underline{\theta})}{\partial \alpha_1^2} & -\frac{\partial^2 l(\underline{\theta})}{\partial \alpha_1 \partial \theta_2} \\ -\frac{\partial^2 l(\underline{\theta})}{\partial \alpha_2 \partial \theta_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial \alpha_2 \partial \theta_2} & -\frac{\partial^2 l(\underline{\theta})}{\partial \alpha_2 \partial \theta_1} & -\frac{\partial^2 l(\underline{\theta})}{\partial \alpha_2^2} \end{bmatrix}^{-1} \right|_{\hat{\theta}_j, \hat{\alpha}_j}$$

$$\cong \begin{bmatrix} \mathrm{v}(\hat{\theta}_1) & cov(\hat{\theta}_1, \hat{\theta}_2) & cov(\hat{\theta}_1, \hat{\alpha}_1) & cov(\hat{\theta}_1, \hat{\alpha}_2) \\ cov(\hat{\theta}_1, \hat{\theta}_2) & \mathrm{v}(\hat{\theta}_2) & cov(\hat{\theta}_2, \hat{\alpha}_1) & cov(\hat{\theta}_2, \hat{\alpha}_2) \\ cov(\hat{\alpha}_1, \hat{\theta}_1) & cov(\hat{\alpha}_1, \hat{\theta}_2) & \mathrm{v}(\hat{\alpha}_1) & cov(\hat{\alpha}_1, \hat{\alpha}_2) \\ cov(\hat{\alpha}_2, \hat{\theta}_1) & cov(\hat{\alpha}_2, \hat{\theta}_2) & cov(\hat{\alpha}_2, \hat{\alpha}_1) & \mathrm{v}(\hat{\alpha}_2) \end{bmatrix} \tag{32}$$

The appropriate (32) is used to derive the 100 (1- γ) % confidence intervals of the parameters as in following forms :

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$\hat{\theta}_j \pm Z_{\frac{\gamma}{2}}\sqrt{V(\hat{\theta}_j)}\ , \hat{\alpha}_j \pm Z_{\frac{\gamma}{2}}\sqrt{V(\hat{\alpha}_j)}\ , \qquad for\ j = 1,2$

where, $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ percentile of the standard normal distribution.

The asymptotic variance – covariance matrix will be obtained by inverting the information matrix with the elements that are negative of the observed values of the second order derivate of the logarithm of the likelihood functions .using the logarithm of the likelihood functions ,the elements of the information matrix are given by:

$$l_{12} = l_{21} = \frac{\partial l_1}{\partial \theta_2} = \frac{\partial l_2}{\partial \theta_1}$$

$$= -pp_2\left\{\sum_{i=1}^{r} \varphi^*(t_{i:n}) + (n-r)\omega^{*2}(t_{r:n})\Psi^*(t_{r:n})\right\}, \tag{33}$$

where l_1 is the first derivatives of the natural logarithm of the likelihood function with respect to $\theta_1$and l_2 is the first derivatives of the natural logarithm of the likelihood function with respect to θ_2.

and for i = 1, 2, …, r

where

$$\varphi^*(t_{i:n}) = k_1^*(t_{i:n})k_2^*(t_{i:n})\,\zeta_1^{**}(t_{i:n})\ \zeta_2^{**}(t_{i:n}), \tag{34}$$

$$\Psi^*(t_{r:n}) = \zeta_1^{***}(t_{r:n})\,\zeta_2^{***}(t_{r:n}), \tag{35}$$

$$l_{jj} = \frac{\partial l_j}{\partial \theta_j} = -p_j\left\{\sum_{i=1}^{r} A_j^*(t_{i:n}) + (n-r)\omega^*(t_{r:n})B_j^*(t_{r:n})\right\}, \tag{36}$$

$$A_j^*(t_{i:n}) = \zeta_j^{**}(t_{i:n})\theta_j^{-2} - p_s k_j^{*^2}(t_{i:n})\,\zeta_j^{**}(t_{i:n})\,\zeta_s^{**}(t_{i:n}),$$

and

$$B_j^*(t_{r:n}) = \zeta_j^{***}(t_{r:n})\omega^*(t_{r:n}) + p_j\,\zeta_j^{***^2}(t_{r:n})\omega^*(t_{r:n}),$$

where l_{jj} is the second derivatives of the natural logarithm of the likelihood function with respect to θ_j, for $j = 1,2$, $s = 1,2, j \neq s$ the functions $k_j^*(.)$ and ω^* (.) are given by (22) and (23), $\zeta_j^{**}(.)$ and $\zeta_j^{***}(.)$ by (25).

$$\frac{\partial^2 l}{\partial \alpha_1 \alpha_2} = \frac{\partial^2 l}{\partial \alpha_2 \alpha_1}$$

$$= -p_1p_2\left\{\sum_{i=1}^{r} \varphi^{**}(t_{i:n}) + (n-r)\Psi^{**}(t_{r:n})\right\}, \tag{37}$$

where

$$\varphi^{**}(t_{i:n}) = \zeta_1^{**}(t_{i:n})\,\zeta_2^{**}(t_{i:n})S_1(t_{i:n})S_2(t_{i:n}), \tag{38}$$

$$\Psi^{**}(t_{r:n}) = \zeta_1^{***}(t_{r:n})\,\zeta_2^{***}(t_{r:n})O_1(t_{r:n})O_2(t_{r:n}), \tag{39}$$

$$\frac{\partial^2 l}{\partial \alpha_j^2} = -p_j\left\{\sum_{i=1}^{r} A_j^{**}(t_{i:n}) + (n-r)B_j^{**}(t_{r:n})\right\}, \tag{40}$$

where

$A_j^{**}(t_{i:n}) = \zeta_j^{**}(t_{i:n})S_j^{\backslash}(t_{i:n}) - p_s S_j^2(t_{i:n})\,\zeta_j^{**}(t_{i:n})\,\zeta_s^{**}(t_{i:n})$, *j=1, 2, s=1, 2, s≠j,*

$B_j^{**}(t_{r:n}) = \zeta_j^{***}(t_{r:n})\ O_j(t_{r:n})\ Z_j(t_{r:n}) + \tau^{**}(t_{r:n})\ O_j(t_{r:n})$

$Z_j(t_{r:n}) = \ln(t_{r:n}) - t_{r:n}^{\alpha_j}\ln(t_{r:n}) - t_{r:n}^{\alpha_j} e^{-t_{r:n}^{\alpha_j}} \ln(t_{r:n})\,(1-e^{-t_{r:n}^{\alpha_j}})^{-1}$,

$S_j^{\backslash}(t_{i:n}) = -\alpha_j^{-2} + t_{i:n}^{\alpha_j}\ln^2(t_{i:n}) - \{(\theta_j - 1)\ln(t_{i:n})\,[e^{-2t^{\alpha_j}} t^{2\alpha_j}\ln(t_{i:n})(1-e^{-t_{i:n}^{\alpha_j}})^{-2}]\}$,

and

$$\frac{\partial^2 l}{\partial\theta_1\partial\alpha_1} = p_1\left\{\sum_{i=1}^{r} A_1^{\Delta}(t_{i;n}) + (n-r)B_1^{\Delta}(t_{r;n})\right\}, \qquad (41)$$

where

$A_1^{\Delta}(t_{i:n}) = p_2\,\zeta_1^{**}(t_{i:n})\,\zeta_2^{**}(t_{i:n})k_1^*(t_{i:n})S_1(t_{i:n}) + \zeta_1^{**}(t_{i:n})D_1^*(t_{i:n})$,

$B_1^{\Delta}(t_{r:n}) = \zeta_1^{***}(t_{r:n})D_1^*(t_{r:n}) + \tau_1^{\Delta}(t_{r:n})\omega^*(t_{r:n})$,

where

$D_1^*(t_{i:n}) = e^{-t_{i:n}^{\alpha_j}} t_{i:n}^{\alpha_1}(1-e^{-t_{i:n}^{\alpha_1}})^{-1}\ln(t_{i:n})$,

$$\tau_1^{\Delta}(t_{r:n}) = \zeta_1^{***}(t_{r:n})O_1(t_{r:n}) + p_1\,\zeta_1^{***^2}(t_{r:n})O_1(t_{r:n}), \qquad (42)$$

$$\frac{\partial^2 l}{\partial\theta_2\partial\alpha_1} = -p_1p_2\left\{\sum_{i=1}^{r} A_1^{\Delta\Delta}(t_{i:n}) + (n-r)\,B_1^{\Delta\Delta}(t_{r:n})\right\}, \qquad (43)$$

where

$A_1^{\Delta\Delta}(t_{i:n}) = -\,\zeta_1^{**}(t_{i:n})\,\zeta_2^{**}(t_{i:n})k_2^*(t_{i:n})S_1(t_{i:n})$,

$B_1^{\Delta\Delta}(t_{r:n}) = \zeta_1^{***}(t_{r:n})\,\zeta_2^{***}(t_{r:n})O_1(t_{r:n})\omega^*(t_{r:n})$,

$$\frac{\partial^2 l}{\partial\theta_1\partial\alpha_2} = -p_1p_2\left\{\sum_{i=1}^{r} A_2^{\Delta\Delta}(t_{i:n}) + (n-r)\,B_2^{\Delta\Delta}(t_{r:n})\right\}, \qquad (44)$$

where

$A_2^{\Delta\Delta}(t_{i:n}) = \zeta_1^{**}(t_{i:n})\,\zeta_2^{**}(t_{i:n})K_1^*(t_{i:n})S_2(t_{i:n})$,

$B_2^{\Delta\Delta}(t_{r:n}) = \zeta_1^{***}(t_{r:n})\,\zeta_2^{***}(t_{r:n})O_2(t_{r:n})\omega^*(t_{r:n})$,

$$\frac{\partial^2 l}{\partial\theta_2\partial\alpha_2} = P_2\left\{\sum_{i=1}^{r} A_2^{\Delta}(t_{i:n}) + (n-r)\,B_2^{\Delta}(t_{r:n})\right\}, \qquad (45)$$

$A_2^{\Delta}(t_{i:n}) = P_1\,\zeta_1^{**}(t_{i:n})\,\zeta_2^{**}(t_{i:n})K_2^*(t_{i:n})S_2(t_{i:n}) + \zeta_2^{**}(t_{i:n})D_2^*(t_{i:n})$,

$B_2^{\Delta}(t_{r:n}) = \zeta_2^{***}(t_{r:n})D_2^*(t_{i:n}) + \tau_2^{\Delta}(t_{r:n})\omega^*(t_{r:n})$,

$\tau_2^{\Delta}(t_{r:n}) = \zeta_2^{***}(t_{r:n})O_2(t_{r:n}) + p_2\,\zeta_2^{***^2}(t_{r:n})O_2(t_{r:n})$,

and

$D_2^*(t_{i:n}) = e^{-t_{i:n}^{\alpha_2}} t_{i:n}^{\alpha_2}\ln(t_{i:n})(1-e^{-t_{i:n}^{\alpha_2}})^{-1}$,

For $j=1, 2$ the functions $S_j(.)$ and $O_j(.)$ are as given by (29) and (30), $\zeta_j^{**}(.)$ and $\zeta_j^{***}(.)$ by(25)

NUMERICAL RESULTS

Real Data Set

To illustrate the approaches developed in the previous chapter ,we consider the data set presented in Aarset (1987) to identify the bathtub hazard rate contains life time of 50 industrial devices put on life test at time zero.

Considering the data in Aarset (1987)) we fit (MTEW) distribution to the data set and summarized it in table (1) by using MATHCAD package (2011). We have presented the maximum likelihood of the vector parameters $\underline{\theta}$ of MTEW distribution. The estimations are conducted on the basis of type-II censored samples

Table (1): Maximum likelihood estimates for the four shape parameters $\alpha_1, \theta_1, \alpha_2$ and θ_2 of MTEW distribution for Aarset (1987) data.

Paramete	**MLE**	**MSE**	**Var**
$\alpha_1 = 1$	1.040	0.14	0.138
$\theta_1 = 0.5$	0.540	0.030	0.023
$\alpha_2 = 1$	0.930	0.09	0.0851
$\theta_2 = 0.5$	0.490	0.02	0.020

Simulation Study:

In the chapter three, the maximum likelihood estimators of the vector parameter $\underline{\theta}$ of MTEW distribution is presented . In order to assess the statistical performances of thes estimates, a simulation study is conducted. The computations are carried out for censoring percentages of 60% for each sample size (n =10, 15, 20, 25, 30, 40 and 50). The mean square errors (MSE's) using generated random samples of different sizes are computed for each estimator.

- Simulation study for classical method (MLE's)

Simulation studies have been performed using MATHCAD for illustrating the new results for estimation problem. We obtained the performance of the proposed estimators using Maximum likelihood estimation method through a simulation study. 1000 random sample of size n=10, 15, 20,30, 40 and 50, were generated from MTEW distribution and used to study the properties of maximum likelihood estimators, with diferent values of the parameter to study the properties of MLE's estimators.

MATHCAD package is used to evaluate the ML estimators under censored type-II using equations (24) and (31) for diffrent values of the parameters: $(\theta_1 = 2.5, \theta_1 = 1.1\ , \alpha_2 = 2.5, \theta_2 = 1.3)$ and $(\alpha_1 = 2.5,\ \theta_1 = 1.4\ ,\ \alpha_2 = 2.5,\ \theta_2 = 1.7)$ and mixing proportion (p = 0.3, 0.2). The performance of the resulting estimates of the parameters has been considered in terms of the mean square error (MSE). Furthermore, for each estimators the skewness, kurtosis and Pearson type of distributions will be obtained. The simulation procedures will be described below:

Step 1:1000 random samples of size 10, 15, 20,30, 40 and 50, were generated from MTEW distribution. If U has a uniform (0, 1) random number, then $x_{i,j} = px1_{i,j} + (1-p)x2_{i,j}$ where $x1_{i,j} = [-\ln(1-(u_{i,j})^{\frac{1}{\theta_1}}]^{\frac{1}{\alpha_1}},\ x2_{i,j} = [-\ln[1-(u_{i,j})^{\frac{1}{\theta_2}}]]^{\frac{1}{\alpha_2}}$ follows MTEW distribution.

Step 2: Choose the number of failure *r*, we choose r to be less than the sample size n.

Step 3: Newton-Raphson method was used for solving the equations (24) and (31), respectively, to obtain the ML estimators of the unknown parameters $\alpha_1,\ \theta_1\ ,\ \alpha_2\ and\ \theta_2$.

Step 4: The MSE, and the moment about the mean are obtained to compute the skewness, kurtosis and Pearson criterion K_P to determine Pearson type of the estimators. We report average estimates obtained by solving the method of maximum likelihood with mean squared error in parentheses MSE for $\underline{\hat{\theta}} = \frac{\sum(\underline{\hat{\theta}}-\underline{\theta})^2}{1000} + [(\underline{\hat{\theta}} - \underline{\theta})^2]$.

Table 2: MLE for the parameters of mixture exponentiated Weibull distribution using type II censoring when $p = 0.3$, $r = 6$, $\alpha_1 = 2.5$, $\theta_1 = 1.1$, $\alpha_2 = 2.5$ and θ_2 =1.3 and different sized samples.

n		MLE	Bias	MSE	Var	SE	Skewness	Kurtosis	Pearson type
10	α_1	3.523	1.023	4.322	3.275	0.181	1.995	6.835	I
	θ_1	1.211	0.111	0.180	0.167	0.041	0.714	4.717	IV
	α_2	3.366	0.867	2.788	2.036	0.143	1.524	4.939	I
	θ_2	1.236	0.064	0.155	0.151	0.039	0.839	5.098	IV
15	α_1	2.836	0.336	1.752	1.639	0.085	2.076	7.085	I
	θ_1	1.235	0.135	0.115	0.097	0.021	0.321	2.927	I
	α_2	2.705	0.205	1.134	1.092	0.070	1.998	6.677	I
	θ_2	1.351	0.051	0.121	0.118	0.023	0.571	3.530	VI
20	α_1	2.648	0.148	0.322	0.300	0.027	0.752	5.577	IV
	θ_1	1.207	0.107	0.067	0.056	0.012	0.869	5.090	IV
	α_2	2.551	0.051	0.268	0.265	0.026	0.427	3.682	IV
	θ_2	1.293	0.007	0.048	0.048	0.011	0.576	5.432	IV
30	α_1	2.465	0.035	0.184	0.183	0.014	-1.027	3.303	I
	θ_1	1.242	0.142	0.070	0.051	0.007	1.634	5.493	I
	α_2	2.365	0.135	0.176	0.158	0.013	-0.845	3.204	I
	θ_2	1.356	0.056	0.057	0.054	0.008	1.437	4.671	I
40	α_1	2.468	0.032	0.247	0.246	0.012	-0.218	3.801	IV
	θ_1	1.244	0.144	0.062	0.041	0.005	1.569	5.362	I
	α_2	2.349	0.151	0.234	0.211	0.011	-0.416	3.342	IV
	θ_2	1.457	0.157	0.188	0.634	0.010	2.096	7.604	I
50	α_1	2.528	0.028	0.190	0.189	0.009	-1.375	5.221	I
	θ_1	1.189	0.089	0.026	0.018	0.003	0.832	3.869	I
	α_2	2.394	0.106	0.199	0.188	0.009	-1.361	4.302	I
	θ_2	1.381	0.081	0.128	0.121	0.007	2.546	9.719	I

Table (3): MLE for the parameters of mixture exponentiated Weibull distribution using type- II censoring when $p = 0.2$, $r = 6, \alpha_1 = 2.5,\ \theta_1 = 1.4,\ \alpha_2 = 2.5,$ θ_2 =1.7 and different sized samples.

n		MLE	Bias	MSE	Var	SE	Skewness	Kurtosis	Pearson type
10	α_1	3.859	1.359	7.098	5.251	0.229	1.781	6.112	I
	θ_1	1.529	0.129	0.347	0.330	0.057	0.074	2.741	I
	α_2	3.678	1.178	4.618	3.230	0.180	1.387	4.476	I
	θ_2	1.600	0.010	0.333	0.323	0.057	-0.020	2.448	I
15	α_1	3.158	0.658	3.085	2.652	0.109	2.282	8.075	I
	θ_1	1.620	0.220	0.222	0.173	0.028	0.331	3.342	IV
	α_2	3.0467	0.547	2.186	1.887	0.092	1.947	6.926	I
	θ_2	1.688	0.012	0.159	0159	0.027	0.224	3.133	IV
20	α_1	2.863	0.363	1.621	1.489	0.061	2.953	14.609	I
	θ_1	1.516	0.116	0.102	0.089	0.015	0.170	4.513	IV
	α_2	2.777	0.277	1.390	1.313	0.057	3.021	15.677	I
	θ_2	1.666	0.034	0.124	0.123	0.017	0.111	3.949	IV
30	α_1	2.715	0.215	0.796	0.751	0.029	1.873	8.896	VI
	θ_1	1.503	0.103	0.069	0.058	0.008	-0.243	6.164	IV
	α_2	2.574	0.074	0.398	0.393	0.021	1.000	4.874	VI
	θ_2	1.711	0.011	0.115	0.115	0.011	1.225	5.027	I
40	α_1	2.709	0.209	1.210	1.166	0.027	2.928	13.715	I
	θ_1	1.508	0.108	0.094	0.082	0.007	0.008	5.412	IV
	α_2	2.470	0.030	0.480	0.480	0.017	0.988	0.988	IV
	θ_2	1.765	0.065	0.177	0.173	0.010	1.039	4.261	I
50	α_1	2.511	0.012	0.222	0.222	0.009	-0.552	3.706	IV
	θ_1	1.558	0.158	0.079	0.054	0.005	1.179	3.825	I
	α_2	2.399	0.101	0.268	0.258	0.010	-0.705	3.912	VI
	θ_2	1.814	0.114	0.194	0.181	0.009	1.713	5.722	I

Table (4): Asymptotic variances and covariances of estimators of the mixture exponentiated Weibull distribution under type II Censoring.

n	**Case 1**				**Case 2**			
	$(\alpha_1=2.5,\theta_1=1.1,\alpha_2=2.5,\theta_2=1.3)$				$(\alpha_1=2.5,\theta_1=1.4,\alpha_2=2.5,\theta_2=1.7)$			
	$\hat{\alpha}_1$	$\hat{\theta}_1$	$\hat{\alpha}_2$	$\hat{\theta}_2$	$\hat{\alpha}_1$	$\hat{\theta}_1$	$\hat{\alpha}_2$	$\hat{\theta}_2$
10	3.275	-0.4810	-0.6740	-0.8730	5.251	-1.7738	-0.8956	-0.9657
	-	0.167	0.4213	0.4720	-	0.330	0.9679	0.8739
	-	-	2.036	-0.3210	-	-	3.230	-1.3418
	-	-	-	0.151	-	-	-	0.323
15	1.639	-0.3320	-0.4320	-0.6520	2.652	-0.9184	-0.7834	-0.8534
	-	0.097	0.3512	0.4105	-	0.173	0.7856	0.6526
	-	-	1.092	-0.1210	-	-	1.887	-1.1173
	-	-	-	0.118	-	-	-	0.159
20	0.300	-0.1095	-0.4095	-0.3211	1.489	-0.7871	-0.6311	-0.7128
	-	0.056	0.1705	0.2205	-	0.089	0.5782	0.4815
	-	-	0.265	-0.0982	-	-	1.313	-0.9452
	-	-	-	0.048	-	-	-	0.123
30	0.286	-0.0754	-0.2162	-0.1834	0.534	-0.5637	-0.4732	-0.5407
	-	0.051	0.0534	0.0863	-	0.058	0.1052	0.1243
	-	-	0.158	-0.0536	-	-	0.393	-0.6578
	-	-	-	0.054	-	-	-	0.115
40	0.246	-0.0315	-0.0729	-0.0694	1.166	-0.1967	-0.0869	-0.1824
	-	0.041	0.0378	0.0713	-	0.082	0.0935	0.1031
	-	-	0.211	-0.0213	-	-	0.480	-0.4325
	-	-	-	0.634	-	-	-	0.173
50	0.189	-0.0180	-0.0514	-0.0403	0.222	-0.0831	-0.0721	-0.0857
	-	0.018	0.0125	0.0471	-	0.054	0.0714	0.0927
	-	-	0.188	-0.0094	-	-	0.258	-0.2189
	-	-	-	0.121	-	-	-	0.181

CONCLUDING REMARKS

Simulation results are displayed in tables 2, 3and 4, which give the posterior mean and MSE. Simulation studies are adopted for different sized samples. We have presented the maximum likelihood estimators of the vector parameters $\underline{\theta}$, of the life times follow MTEW distribution. Our observations about the results are stated in the following Points:1-Tables 2, 3 shows the maximum likelihood estimators, of the unknown parameters, MSE, skewness, kurtosis and Pearson type distribution. From these tables, we conclude that the MLE's estimators have the minimum MSE for most sample sizes. As the sample size increases MSE's and bias of the estimated parameters $(\alpha_1, \theta_1, \alpha_2, \theta_2)$ decreases. This indicates that the MLE's estimators provide asymptotically normally distributed and consistent estimators for the parameters. Generally, we observed that the estimators for the unknown parameters $\alpha_1$and α_2 have Pearson type I distribution. Also, we observed that the estimators for the unknown parameters $\theta_1$and θ_2 have Pearson type IV distribution .

2-Table 4 shows that the variances for all parameters in case-1 less than the variances for all parameters in case-2. It is immediate to note that the average estimate of $\mathrm{Cov}(\hat{\alpha}_1, \hat{\alpha}_2)$ less than the average estimate of $\mathrm{Cov}(\hat{\alpha}_1, \hat{\theta}_1)$, the average estimate of $\mathrm{Cov}(\hat{\theta}_1, \hat{\alpha}_2)$ is less than the average estimate of $\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)$, when n increases the covariance decrease. The values of $\mathrm{Cov}(\hat{\alpha}_1, \hat{\theta}_2)$, $\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)$ and $\mathrm{Cov}(\hat{\alpha}_1, \hat{\alpha}_2)$ are very small and converge to zero.

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