International Research Journal of Pure and Applied Physics

Vol.5, No.3, pp.27-32, August 2017

Published by European Centre for Research Training and Development UK (www.eajournals.org)

WKB SOLUTIONS FOR QUANTUM MECHANICAL GRAVITATIONAL POTENTIAL PLUS HARMONIC OSCILLATOR POTENTIAL

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ABSTRACT: *We have obtained the exact energy spectrum for a quantum mechanical gravitational potential, plus a harmonic oscillator potential, via the WKB approach. Also a special case of the potential has been considered and their energy eigen value obtained.*

KEYWORDS: Schrodinger Equation, Harmonic Oscillator Potential, Quantum Mechanical Gravitational Potential, Bohr Sommerfeld Wkb Approximation.

INTRODUCTION

The bound state solutions of the Schrodinger equation (SE) are only possible for some potentials of physical interest [1-3]. These solutions could be exact or approximate and they normally contain all the necessary information for the quantum system. Quite recently, several authors have tried to solve problems that involve obtaining the exact or approximate solutions of the Schrodinger equation for a number of special potentials using different methods [4-7].

One of the earliest and simplest methods of obtaining approximate eigenvalues to the one-dimensional Schrodinger equation in the limiting case of large quantum numbers was originally proposed by Wentzel, Kramers, and Brillouin known as the WKB approximation method [8]. Considering the one dimensional radial Schrodinger equation of the form [9] for s-wave case where $l = 0$, it means that the problem has no minimum value and also doesn't have the left turning point from the physical point of view [10] and the energy obtained would not produce a stable bound state. In order for the physical system to have a stable bound state, we use the Langer correction $l(l+1) \to \left(l+\frac{1}{2}\right)^2$ in the centrifugal term of the radial Schrodinger equation [8]. The replacement of $l(l+1) \to \left(l+\frac{1}{2}\right)^2$ regularizes the radial WKB wave function at the origin and ensure correct asymptotic behaviour at large quantum numbers.

It was observed by Langer [11] that the reason for this modification arose from the fact that the quantization condition for the one-dimensional problem was derived under the assumption that the wave function approached zero for $\to \pm\infty$, whereas the radial part of the solution approached zero for $r \to 0$ and $r \to \infty$.

In this work, our aim is to solve the Schrodinger equation for the quantum mechanical gravitational potential (QMGP) plus the harmonic oscillator potential (HOP) via the WKB approximation method. The QMGHOP potential takes the form:

$$V(r) = mgz + \delta e^{-kz} + \frac{1}{2}\mu\omega^2 z^2 \qquad (1)$$

Where k is the momentum, z is the displacement, m is the mass, g is the gravitational acceleration, δ is an adjustable parameter, μ is the reduced mass, and ω is the angular frequency. The QMGP could be used to calculate the energy of a body falling under gravity from quantum mechanical point of view. Recently, Ita *et al* [12], have applied the NU method to the QMGHOP, where they obtained bound state s-wave solution of the SE equation. Berberan-Santos *et al*. [13] have studied the motion of a particle in a gravitational field using the QMGP without the exponential term. They obtained the classical and quantum mechanical position probability distribution function for the particle. The HOP has been widely studied in the literature. For example, Amore and Fernandez [14] studied the two-particle harmonic oscillator in a one-dimensional box and obtained energy eigen values which were comparable with energies from variational and perturbation method. Jasmin et al [15] also investigated the single particle level density in a harmonic oscillator potential well and obtained very interesting result.

This paper is organized as follows: Section 1 has the introduction, the semiclassical Bohr-Sommerfeld quantization condition is reviewed in section 2, The WKB radial solution to the QMGHOP is solved in section 3. Finally, we give a brief discussion in section 4 before the conclusion in section 5

Semiclassical Quantization and the WKB Approximation

In this section we consider quasiclassical solution of the Schrodinger's equation for the spherically symmetric potentials. Given the Schrodinger equation for a spherically symmetric potentials $V(r)$ of eq. (2) as:

$$(-i\hbar)^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi(r,\theta,\phi) = \left[2m\big(E - V(r)\big)\right]\psi(r,\theta,\phi) \qquad (2)$$

The total wave function in eq. (2) can be defined as

$$\psi(r,\theta,\phi) = [rR(r)]\left[\sqrt{sin\theta}\,\Theta(\theta)\Phi(\phi)\right] \qquad (3)$$

And by decomposing the spherical wave function in eq. (2) using eq. (3) we obtain the following equations:

$$\left(-i\hbar\frac{\mathrm{d}}{\mathrm{dr}}\right)^2 R(r) = \left[2m\big(E - V(r)\big) - \frac{\vec{M}^2}{r^2}\right]R(r), \qquad (4)$$

$$\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^2 \Theta(\theta) = \left[\vec{M}^2 - \frac{M_z^2}{\sin^2\theta}\right]\Theta(\theta), \qquad (5)$$

$$\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}\phi}\right)^2 \Phi(\phi) = M_z^2\Phi(\phi) \qquad (6)$$

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

Where $\overrightarrow{M}^2$, M_z^2 are the constants of separation and, at the same time, integrals of motion. The squared angular momentum is defined as: $\overrightarrow{M}^2 = \gamma^2\hbar^2$ and $\gamma = l + \frac{1}{2}$. [8]

Considering eq. (4), the leading order WKB quantization condition appropriate to eq. (1) is

$$\int_{r_1}^{r_2} \sqrt{P^2(r)}\,dr = \pi\hbar\left(n + \frac{1}{2}\right), \text{ n=0, 1, 2.. .} \tag{7}$$

where r_2& r_1 are the classical turning point known as the roots of the equation

$$P^2(r) = 2m\big(E - V(r)\big) - \frac{\gamma^2\hbar^2}{r^2} = 0 \tag{8}$$

eq. (7) is the WKB quantization condition which is subject for application in the preceding section. Consider Eq. (4)-(6) in the framework of the quasiclassical method, the solution of each of these equations in the leading ħ approximation can be written in the form

$$\Psi^{WKB}(r) = \frac{A}{\sqrt{P(r,\lambda)}} exp\left[\pm\frac{i}{\hbar}\int\sqrt{P^2(r)}\,dr\right] \tag{9}$$

Solutions to the Radial Schrödinger Equation

The radial Schrodinger equation for the QMGHOP can be solved approximately using the WKB quantization condition eq. (7) for $\gamma = 0$. To obtain the exact solution, we consider two turning points.

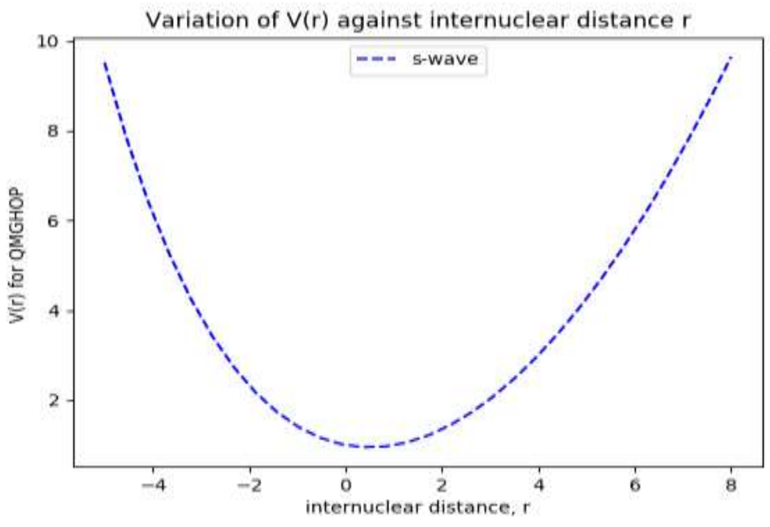

Fig 1: The plot of the quantum mechanical gravitational potential plus harmonic oscillator potential as a function of internuclear distance 'r'.

The potential in eq. (1) can be written as

$$V(r) = \beta r + V_0 e^{-\alpha r} + \frac{1}{2}\mu\omega^2 r^2, \tag{10}$$

where $\beta = mg$, $\alpha = k$, $z = r$, $\delta = V_0$

We can also write eq. (10) as

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

$$V(r) = \beta r + V_0(1 - \alpha r + \alpha^2 r^2) + \tfrac{1}{2}\mu\omega^2 r^2, \tag{11}$$

On arranging eq. (11), we get our working potential amendable to the WKB method as

$$V(r) = V_0 + (\beta - \alpha V_0)r + \left(\alpha^2 V_0 + \tfrac{1}{2}\mu\omega^2\right)r^2, \tag{12}$$

We show in fig. 1, the plot the QMGHOP as a function of r with $V_0 = 1, \beta = 0.2\, fm^{-1}, \omega = 0.5$ for $\alpha = 0.4$

Subs. Eq. (12) into eq. (7), we have

$$\int_{r_1}^{r_2} \sqrt{P^2(r)}\, dr = \int_{r_1}^{r_2} \sqrt{2\mu\left(E - V_0 - (\beta - \alpha V_0)r - \left(\alpha^2 V_0 + \tfrac{1}{2}\mu\omega^2\right)r^2\right)}\, dr \tag{13}$$

Factoring out $\sqrt{2\mu}$, we have

$$\sqrt{2\mu}\int_{r_1}^{r_2} \sqrt{\left((E - V_0) - (\beta - \alpha V_0)r - \left(\alpha^2 V_0 + \tfrac{1}{2}\mu\omega^2\right)r^2\right)}\, dr = \pi\hbar\left(n + \tfrac{1}{2}\right) \tag{14}$$

$$\sqrt{2\mu}\int_{r_1}^{r_2} \sqrt{-\left(\alpha^2 V_0 + \tfrac{1}{2}\mu\omega^2\right)r^2 - (\beta - \alpha V_0)r + (E - V_0)dr} = \pi\hbar\left(n + \tfrac{1}{2}\right) \tag{15}$$

If $\left(\alpha^2 V_0 + \tfrac{1}{2}\mu\omega^2\right) = \tilde{V}$ and factoring out $\tilde{V}$, we obtain

$$\sqrt{2\mu\tilde{V}}\int_{r_1}^{r_2} \sqrt{-r^2 - \left(\tfrac{\beta - \alpha V_0}{\tilde{V}}\right)r + \left(\tfrac{E - V_0}{\tilde{V}}\right)}\, dr = \pi\hbar\left(n + \tfrac{1}{2}\right) \tag{16}$$

let $-\left(\frac{\beta - \alpha V_0}{\tilde{V}}\right) = B$, and $\left(\frac{E - V_0}{\tilde{V}}\right) = -C$, we have

$$\sqrt{2\mu\tilde{V}}\int_{r_1}^{r_2} \sqrt{(-r^2 + Br - C)}dr = \pi\hbar\left(n + \tfrac{1}{2}\right) \tag{17}$$

$$\sqrt{2\mu\tilde{V}}\int_{r_1}^{r_2} \sqrt{(r - r_1)(r_2 - r)}dr = \pi\hbar\left(n + \tfrac{1}{2}\right) \tag{18}$$

Where we obtain the turning points $r_2 \& r_1$ from the terms inside the square roots as

$$r_1 = \frac{-B - \sqrt{B^2 - 4C}}{2}$$

$$r_2 = \frac{-B + \sqrt{B^2 - 4C}}{2}$$

Solving the integral of eq. (18) explicitly, we obtain

$$\sqrt{2\mu\tilde{V}}\left(-\left(\tfrac{\beta - \alpha V_0}{\tilde{V}}\right) - 2\sqrt{\left(\tfrac{E - V_0}{\tilde{V}}\right)}\right) = 2\hbar\left(n + \tfrac{1}{2}\right) \tag{19}$$

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

$$\sqrt{E-V_0} = -\frac{1}{2\sqrt{2\mu}}\left[2\hbar\left(n+\frac{1}{2}\right)+\frac{\sqrt{2\mu}(\beta-\alpha V_0)}{\sqrt{\left(\alpha^2 V_0+\frac{1}{2}\mu\omega^2\right)}}\right] \quad (20)$$

$$E-V_0 = \frac{1}{4}\left[\frac{2\hbar\left(n+\frac{1}{2}\right)}{\sqrt{2\mu}}+\frac{(\beta-\alpha V_0)}{\sqrt{\left(\alpha^2 V_0+\frac{1}{2}\mu\omega^2\right)}}\right]^2 \quad (21)$$

$$E_{n,l} = V_0+\frac{\hbar^2}{2\mu}\left\{\left(n+\frac{1}{2}\right)\left[\left(n+\frac{1}{2}\right)+\frac{2\mu/\hbar^2(\beta-\alpha V_0)}{\sqrt{2\mu/\hbar^2\left(\alpha^2 V_0+\frac{1}{2}\mu\omega^2\right)}}\right]+\frac{\left((2\mu/\hbar^2)(\beta-\alpha V_0)\right)^2}{4\left[(2\mu/\hbar^2)\left(\alpha^2 V_0+\frac{1}{2}\mu\omega^2\right)\right]}\right\} \quad (22)$$

Eq. (22) turns out to be similar to eq. (22) of Ref. [12].

And from eq. (9), the exact ground state wave function expression of eq. (10) for the classically allowed region ($r_1 < r < r_2$) is given as

$$R(r) \propto \frac{A}{\left((E-V_0)-(\beta-\alpha V_0)r-\left(\alpha^2 V_0+\frac{1}{2}\mu\omega^2\right)r^2\right)^{1/4}} \cos\left(\int_{r_2}^{r}\sqrt{(E-V_0)-(\beta-\alpha V_0)r-\left(\alpha^2 V_0+\frac{1}{2}\mu\omega^2\right)r^2}\,dr-\frac{\pi}{4}\right) \quad (23)$$

where A is the normalization constant.

DISCUSSION

Equation (22) is the energy eigen value for the QGMP+ HOP. If $\beta = V_0 = 0$ in eq. (1), the potential model turns back into the harmonic oscillator potential and the energy equation (22) yields the energy eigen values for the harmonic oscillator potential as

$$E_{n,l} = \frac{\hbar^2}{2\mu}\left(n+\frac{1}{2}\right)^2 \quad (24)$$

Eq. (24) is similar to eq. (35) of Ref. [16] obtained for the harmonic oscillator potential in electric field when the field strengthε was set to zero.

CONCLUSIONS

The WKB approximation method is general for all types of problems in quantum mechanics, simple from the physical point of view, and its correct application results in the exact energy eigenvalues for all solvable potentials.In summary, we have obtained the exact energy eigen value and its corresponding un-normalized wave function using the WKB approximation method for the radial Schrodinger equation with the quantum mechanical gravitational potential plus the harmonic oscillator potential for s-state, excluding the centrifugal term.

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

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ISSN 2055-009X(Print), ISSN 2055-0103(Online)