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TRUNCATED TWO PIECE BIVARIATE NORMAL DISTRIBUTION

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ABSTRACT: *In this paper we define doubly truncated two piece bivariate normal distribution. We have estimated the mean, variance and correlation coefficient of this distribution by method of moments. Results regarding singly truncated two piece bivariate normal distribution have been shown as a particular case of doubly truncated two piece bivariate normal distribution.*

KEYWORDS: Doubly truncated two piece bivariate normal distribution, method of moments, lower truncation, upper truncation, Recurrence relation.

INTRODUCTION

Dickson (1886) has demonstrated a possible genesis for the bivariate normal distribution as the vector combinations of independently normally distributed components on oblique axes. When a target is rectangular and data are collected on the location of hits within the target area only, a truncated bivariate normal model will most often fit the data very well. An analysis was carried out for the estimation of the parameters of this distribution and computational estimates by Dyer, Danny, D. (1970). The expected values of Spearman's ρ_s and Kendall's t are obtained for the singly truncated bivariate normal distribution. Tables of $E(\rho_s)$ and $E(t)$ for various values of *n*, and ρ and '*a*' – the point of truncation are given by Aitkin et al (1965). Inference is considered for the marginal distribution of X when (X, Y) has a truncated bivariate normal distribution; the X-values are observed by Arnold and etal. (1993). Rosenbaum (1961) has obtained first and second order moments of singly truncated standard bivariate normal distribution. Shah and Parikh (1964) has given recurrent relations between moments of singly and doubly truncated standard bivariate normal distributions. The two piece normal (TPN) distribution has been shown to provide an adequate approximation to a number of naturally occurring skewed data sets, verifying its robust nature in actual use.

In this chapter we introduce truncated two piece bivariate normal (TTPBN) distribution. In section 2 we have given the p.d.f. of TTPBN distribution. The general recurrence relation for the moments of this distribution is obtained in section 3. Mean, variance and correlation coefficient has been obtained in this section. Singly truncated distributions and its mean, variance and correlation coefficient are derived in section 4.

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The density function

The density function of doubly truncated standard two-piece bivariate normal distribution is

$$f(z_{11},z_{21}) = \begin{cases} C_1 K \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[z_{11}^2 - 2\rho z_{11} z_{21} + z_{21}^2 \right] \right\} & h < z_{11} < 0,\ k < z_{21} < 0 \\ C_2 K \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[z_{11}^2 - 2\rho z_{11} \frac{z_{21}}{k_2} + \left(\frac{z_{21}}{k_2}\right)^2 \right] \right\} & h < z_{11} < 0,\ 0 < z_{21} < m \\ C_3 K \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[\left(\frac{z_{11}}{k_1}\right)^2 - 2\rho \left(\frac{z_{11}}{k_1}\right) z_{21} + z_{21}^2 \right] \right\} & 0 < z_{11} < l,\ k < z_{21} < 0 \\ C_4 K \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[\left(\frac{z_{11}}{k_1}\right)^2 - 2\rho \left(\frac{z_{11}}{k_1}\right)\left(\frac{z_{21}}{k_2}\right) + \left(\frac{z_{21}}{k_2}\right)^2 \right] \right\} & 0 < z_{11} < l,\ 0 < z_{21} < m \end{cases}$$

(2.1)

where $K = \frac{2}{\pi} \left[(1+k_1)(1+k_2)\sigma_{11}\sigma_{21}\sqrt{1-\rho^2} \right]^{-1}$

$$C_1^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\frac{1}{2} - \Phi\left(\frac{h-\rho z_{21}}{\sqrt{1-\rho^2}}\right)\right]\left[\frac{1}{2} - \Phi(k)\right]$$

$$C_2^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\frac{1}{2} - \Phi\left(\frac{h-\rho \frac{z_{21}}{k_2}}{\sqrt{1-\rho^2}}\right)\right]\left[\Phi\left(\frac{m}{k_2}\right) - \frac{1}{2}\right]$$

$$C_3^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\Phi\left(\frac{\frac{l}{k_1}-\rho z_{21}}{\sqrt{1-\rho^2}}\right) - \frac{1}{2}\right]\left[\frac{1}{2} - \Phi(k)\right]$$

$$C_4^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\Phi\left(\frac{\frac{l}{k_1}-\rho z_{21}}{\sqrt{1-\rho^2}}\right) - \frac{1}{2}\right]\left[\Phi\left(\frac{m}{k_2}\right) - \frac{1}{2}\right]$$

Let $G_S(t,\mu,\sigma) = t^{s-1}\sigma Z\left(\frac{t-\mu}{\sigma}\right) + (s-1)\sigma G_{s-2}(t,\mu,\sigma) + \mu G_{s-1}(t,\mu,\sigma)$

(2.2)

and $G_o(t,\mu,\sigma) = \int_t^\infty \frac{1}{\sigma} Z\left(\frac{x-\mu}{\sigma}\right) dx = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \Phi\left(\frac{t-\mu}{\sigma}\right).$

Also,

$$G_s(t,p,\mu,\sigma)=\int_t^p \frac{1}{\sqrt{2\pi\sigma}}x^s e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

$$=\sigma\left[p^{s-1}Z\left(\frac{p-\mu}{\sigma}\right)\ t^{s-1}Z\left(\frac{t-\mu}{\sigma}\right)\right]+(s-1)\sigma G_{s-2}(t,p,\mu,\sigma)+\mu G_{s-1}(t,p,\mu,\sigma)$$

(2.3)

and

$$G_o(t,p,\mu,\sigma)=\int_t^p \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx=\Phi\left(\frac{t-\mu}{\sigma}\right)\ \Phi\left(\frac{p-\mu}{\sigma}\right)$$

3 Estimation of Parameters

Let $V_{r,s}=E\left[z_{11}^r\ z_{21}^s\right]$

$$=V_{r,s}^1+V_{r,s}^2+V_{r,s}^3+V_{r,s}^4.$$

Now,

$$V_{r,s}^1=\int_h^0\int_k^0 C_1Kz_{11}^r z_{21}^s\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[z_{11}^2-2\rho z_{11}z_{21}+z_{21}^2\right]\right\}dz_{11}dz_{21}$$

$$\therefore V_{r,s}^1=C_1Kh^{r-1}Z(h)(2\pi)(1-\rho^2)^{3/2}G_s\left(k,o,h\rho,\sqrt{1-\rho^2}\right)+(r-1)(1-\rho^2)V_{r-2,s}^1+\rho\ V_{r-1,s+1}^1$$

(3.1)

$$V_{r,s}^2=\int_h^0\int_0^m C_2K\ z_{11}^r\ z_{21}^s\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[z_{11}^2-2\rho z_{11}\left(\frac{z_{21}}{k_2}\right)+\left(\frac{z_{21}}{k_2}\right)^2\right]\right\}dz_{11}\ dz_{21}$$

$$\therefore\ V_{r,s}^{2} = C_2 K\, h^{r-1} Z(h)\, k_2^{s}\ (2\pi)\left(1-\rho^2\right)^{3/2} G_s\left(o,m,h\rho,\sqrt{1-\rho^2}\right) + (r-1)\left(1-\rho^2\right)V_{r-2,s}^{2} + \frac{\rho}{k_2}V_{r-1,s+1}^{2}$$

(3.2)

$$\therefore\ V_{r,s}^{3} = \int_{o}^{l}\int_{k}^{o} C_3 K\, z_{11}^{r} z_{21}^{s} \exp\left\{\frac{-1}{2\left(1-\rho^2\right)}\left[\left(\frac{z_{11}}{k_1}\right)^2 - 2\rho\left(\frac{z_{11}}{k_1}\right)z_{21} + z_{21}^2\right]\right\} dz_{11}\, dz_{21}$$

$$\therefore\ V_{r,s}^{3} = -C_3 K k_1^2 l^{r-1} Z\left(\frac{l}{k_1}\right)(2\pi)\ G_s\left(k,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)\left(1-\rho^2\right)^{3/2} + (r-1)k_1^2\left(1-\rho^2\right) V_{r-2,s}^{3} + k_1\rho\ V_{r-1,s+1}^{3}$$

(3.3)

and

$$V_{r,s}^{4} = \int_{0}^{l}\int_{0}^{m} C_4 K z_{11}^{r} z_{21}^{s}\ \exp\left\{\frac{-1}{2\left(1-\rho^2\right)}\left[\left(\frac{z_{11}}{k_1}\right)^2 - 2\rho\left(\frac{z_{11}}{k_1}\right)\left(\frac{z_{21}}{k_2}\right) + \left(\frac{z_{21}}{k_2}\right)^2\right]\right\} dz_{11}\, dz_{21}$$

$$\therefore\ V_{r,s}^{4} = -C_4 K\ k_2^{s}\ k_1^{2}\ l^{r-1} Z\left(\frac{l}{k_1}\right)(2\pi)\ G_s\left(o,m,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)\left(1-\rho^2\right)^{3/2}$$

$$+(r-1)\ k_1^{2}\left(1-\rho^2\right) V_{r-2,s}^{4} + \frac{k_1}{k_2}\rho\ V_{r-1,s+1}^{4}$$

(3.4)

Hence, using (3.1) to (3.4), we have

$$\therefore\ V_{r,s} = K(2\pi)\left(1-\rho^2\right)^{3/2}\left[h^{r-1}Z(h)\left\{C_1 G_s\left(k,o,h\rho,\sqrt{1-\rho^2}\right) + C_2 k_2^{s} G_s\left(o,m,h\rho,\sqrt{1-\rho^2}\right)\right\}\right.$$

$$\left.-l^{r-1}Z\left(\frac{l}{k_1}\right)k_1^2\left\{C_3 G_s\left(k,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right) + C_4 k_2^{s} G_s\left(o,m,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)\right\}\right]$$

$$+(r-1)\left(1-\rho^{2}\right)\left[V^{1}_{r-2,s}+V^{2}_{r-2,s}+k_{1}^{2}\left(V^{3}_{r-2,s}+V^{4}_{r-2,s}\right)\right]$$

$$+\rho\left[V^{1}_{r-1,s+1}+\frac{1}{k_{2}}V^{2}_{r-1,s+1}+k_{1}\left(V^{3}_{r-1,s+1}+\frac{1}{k_{2}}V^{4}_{r-1,s+1}\right)\right]$$

(3.5)

Changing the order of integration, we get

$$V^{1}_{r,s}=C_{1}K\int_{h}^{0}z_{11}^{r}\left\{\int_{k}^{0}z_{21}^{s-1}\exp\left[\frac{1}{2}\left(\frac{z_{11}-\rho z_{21}}{\sqrt{1-\rho^{2}}}\right)^{2}\right]z_{21}\exp\left[\frac{1}{2}z_{21}^{2}\right]dz_{21}\right\}dz_{11}$$

$$=C_{1}K\,k^{s-1}Z(k)(2\pi)\left(1-\rho^{2}\right)^{3/2}G_{r}\left(h,o,k\rho,\sqrt{1-\rho^{2}}\right)+\left(1-\rho^{2}\right)(s-1)V^{1}_{r,s-2}+\rho V^{1}_{r+1,s-1}$$

(3.6)

$$V^{2}_{r,s}=C_{2}Kk_{2}\int_{h}^{0}z_{11}^{r}\left\{\int_{0}^{m}z_{21}^{s-1}\exp\left[\frac{1}{2}\left(\frac{z_{11}-\frac{\rho z_{21}}{k_{2}}}{\sqrt{1-\rho^{2}}}\right)^{2}\right]\frac{z_{21}}{k_{2}}\exp\left[\frac{1}{2}\left(\frac{z_{21}}{k_{2}}\right)^{2}\right]dz_{21}\right\}dz_{11}$$

$$=-C_{2}K\,k_{2}^{2}\,m^{s-1}Z\left(\frac{m}{k_{2}}\right)(2\pi)\,G_{r}\left(h,o,\frac{\rho m}{k_{2}},\sqrt{1-\rho^{2}}\right)\left(1-\rho^{2}\right)^{3/2}$$

$$+(s-1)k_{2}^{2}\left(1-\rho^{2}\right)V^{2}_{r,s-2}+\rho k_{2}V^{2}_{r+1,s-1}$$

(3.7)

$$V^{3}_{r,s}=C_{3}K\int_{0}^{l}z_{11}^{r}\left\{\int_{k}^{0}z_{21}^{s-1}\exp\left[\frac{1}{2}\left(\frac{\frac{z_{11}}{k_{1}}-\rho z_{21}}{\sqrt{1-\rho^{2}}}\right)^{2}\right]z_{21}\exp\left[\frac{1}{2}z_{21}^{2}\right]dz_{21}\right\}dz_{11}$$

$$=C_{3}K\,k^{s-1}\,Z(k)\,k_{1}^{r}\,(2\pi)\,G_{r}\left(o,l,\rho k,\sqrt{1-\rho^{2}}\right)\left(1-\rho^{2}\right)^{3/2}+(s-1)\left(1-\rho^{2}\right)V^{3}_{r,s-2}+\frac{\rho}{k_{1}}V^{3}_{r+1,s-1}$$

(3.8)

$$V_{r,s}^{4} = C_4 K k_2 \int_0^m z_{11}^{r} \left\{ \int_0^l z_{21}^{s-1} \exp\left[\frac{1}{2}\left(\frac{\frac{z_{11}}{k_1} - \frac{\rho z_{21}}{k_2}}{\sqrt{1-\rho^2}} \right)^2 \right] \frac{z_{21}}{k_2} \exp\left[\frac{1}{2}\left(\frac{z_{21}}{k_2}\right)^2 \right] dz_{21} \right\} dz_{11}$$

$$= -C_4 K\, m^{s-1} k_1^{r} k_2^{2}\, G_r\left(o, l, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)(2\pi) Z\left(\frac{m}{k_2}\right)\left(1-\rho^2\right)^{3/2}$$

$$+ (s-1)\left(1-\rho^2\right) k_2^2\, V_{r,s-2}^{4} + \left(\frac{k_2}{k_1}\right)\rho\, V_{r+1,s-1}^{4}$$

(3.9)

Hence, using (3.6) to (3.9) we get

$$V_{r,s} = K(2\pi)\left(1-\rho^2\right)^{3/2}\left[k^{s-1} Z(k)\left\{C_1 G_r\left(\eta, o, k\rho, \sqrt{1-\rho^2}\right) + C_3 k_1^{r} G_r\left(o, l, \rho k, \sqrt{1-\rho^2}\right)\right.\right.$$

$$- m^{s-1} Z\left(\frac{m}{k_2}\right) k_2^2 \left\{ C_2 G_r\left(\eta, o, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right) + C_4 k_1^{r} G_r\left(o, l, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)\right.$$

$$+ (s-1)\left(1-\rho^2\right)\left[V_{r,s-2}^{1} + V_{r,s-2}^{3} + k_2^2\left(V_{r,s-2}^{2} + V_{r,s-2}^{4}\right)\right]$$

$$+ \rho\left[V_{r+1,s-1}^{1} + \frac{1}{k_1} V_{r+1,s-1}^{3} + k_2\left(V_{r+1,s-1}^{3} + \frac{1}{k_1} V_{r+1,s-1}^{4}\right)\right]$$

(3.10)

Putting (r-2) for r, in (3.6) to (3.9) we get

$$V_{r-2,s}^{1} = C_1 K k^{s-1} Z(k)(2\pi)\left(1-\rho^2\right)^{3/2} G_{r-2}\left(\eta, o, k\rho, \sqrt{1-\rho^2}\right)$$

$$+ \left(1-\rho^2\right)(s-1)\, V_{r-2,s-2}^{1} + \rho\, V_{r-1,s-1}^{1}$$

$$V_{r-2,s}^{2} = -C_2 K\, k_2^2 m^{s-1} Z\left(\frac{m}{k_2}\right)(2\pi)\left(1-\rho^2\right)^{3/2} G_{r-2}\left(\eta, o, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)$$

$$+ \left(1-\rho^2\right)(s-1) k_2^2\, V_{r-2,s-2}^{2} + \rho k_2\, V_{r-1,s-1}^{2}$$

$$V^3_{r-2,s} = C_3 K\, k^{s-1} k_1^{r-2} Z(k)(2p)\left(1-r^2\right)^{3/2} G_{r-2}\left(o,l,rk,\sqrt{1-r^2}\right)$$

$$+(s-1)\left(1-r^2\right) V^3_{r-2,s-2} + \frac{r}{k_1} V^3_{r-1,s-1}$$

$$V^4_{r-2,s} = -C_4 K\, m^{s-1} k_1^{r-2} k_2^{\,2} G_{r-2}\left(o,l,\frac{rm}{k_2},\sqrt{1-r^2}\right)(2p) Z\left(\frac{m}{k_2}\right)\left(1-r^2\right)^{3/2}$$

$$+(s-1)\left(1-r^2\right)k_2^2\, V^4_{r-2,s-2} + \frac{rk_2}{k_1} V^4_{r-1,s-1}.$$

(3.11)

Also putting (r -1) for r and (s +1) for s, in (3.6) to (3.9) we get

$$V^1_{r-1,s+1} = C_1 K\, k^s Z(k)(2p)\left(1-r^2\right)^{3/2} G_{r-1}\left(h,o,kr,\sqrt{1-r^2}\right) + \left(1-r^2\right)(s)\, V^1_{r-1,s-1} + r\, V^1_{r,s}$$

$$V^2_{r-1,s+1} = -C_2 K\, k_2^2\, m^s Z\left(\frac{m}{k_2}\right)(2p)\left(1-r^2\right)^{3/2} G_{r-1}\left(h,o,\frac{rm}{k_2},\sqrt{1-r^2}\right)$$

$$+\left(1-r^2\right)(s)k_2^2\, V^2_{r-1,s-1} + r\, k_2\, V^2_{r,s}$$

$$V^3_{r-1,s+1} = C_3 K\, k^s\, k_1^{r-1} Z(k)(2p)\left(1-r^2\right)^{3/2} G_{r-1}\left(o,l,rk,\sqrt{1-r^2}\right) + s\left(1-r^2\right) V^3_{r-1,s-1} + \frac{r}{k_1} V^3_{r,s}$$

$$V^4_{r-1,s+1} = -C_4 K\, m^s\, k_1^{r-1} k_2^2\, G_{r-1}\left(o,l,\frac{rm}{k_2},\sqrt{1-r^2}\right)(2p) Z\left(\frac{m}{k_2}\right)\left(1-r^2\right)^{3/2}$$

$$+s\left(1-r^2\right)k_2^2\, V^4_{r-1,s-1} + \frac{rk_2}{k_1} V^4_{r,s}$$

(3.12)

Substituting the values of $V^i_{r-2,s}$ and $V^i_{r-1,s+1}$ (i = 1,2,3,4) from (3.11) & (3.12) respectively in (3.5) we get the recurrence relation

$$V_{r,s} = C_1 K (2p)\sqrt{1-r^2}\Big[h^{r-1} Z(h)\, G_s\left(k,o,hr,\sqrt{1-r^2}\right) + k^{s-1} Z(k)(r-1)\left(1-r^2\right)$$

$$G_{r-2}\left(h,o,kr,\sqrt{1-r^2}\right) + k^s\, Z(k) r\, G_{r-1}\left(h,o,kr,\sqrt{1-r^2}\right)\Big]$$

$$+C_2K(2\pi)\sqrt{1-\rho^2}\Big[\eta^{r-1}Z(\eta)k_2^sG_s\left(o,\mu,\eta\rho,\sqrt{1-\rho^2}\right)-k_2^2\mu^{s-1}Z\left(\frac{\mu}{k_2}\right)(r-1)(1-\rho^2)$$

$$G_{r-2}\left(\eta,o,\frac{\rho\mu}{k_2},\sqrt{1-\rho^2}\right)-k_2\mu^sZ\left(\frac{\mu}{k_2}\right)\rho\,G_{r-1}\left(\eta,o,\frac{\rho\mu}{k_2},\sqrt{1-\rho^2}\right)\Big]\Big\}$$

$$-C_3K(2\pi)\sqrt{1-\rho^2}\Big\{\Big[\lambda^{r-1}k_1^2Z\left(\frac{\lambda}{k_1}\right)G_s\left(\kappa,o,\frac{\rho\lambda}{k_1},\sqrt{1-\rho^2}\right)-k_1^r\kappa^{s-1}Z(\kappa)(1-\rho^2)(r-1)$$

$$G_{r-2}\left(o,\lambda,\rho\kappa,\sqrt{1-\rho^2}\right)-k_1^r\kappa^sZ(\kappa)\rho\,G_{r-1}\left(o,\lambda,\rho\kappa,\sqrt{1-\rho^2}\right)\Big]\Big\}$$

$$-C_4K(2\pi)\sqrt{1-\rho^2}\Big\{\Big[k_1^2k_2^s\lambda^{r-1}Z\left(\frac{\lambda}{k_1}\right)G_s\left(o,\mu,\frac{\rho\lambda}{k_1},\sqrt{1-\rho^2}\right)$$

$$+k_1^rk_2^2\mu^{s-2}Z\left(\frac{\mu}{k_2}\right)(1-\rho^2)(r-1)G_{r-2}\left(o,\lambda,\frac{\rho\mu}{k_2},\sqrt{1-\rho^2}\right)$$

$$+k_1^r\frac{\mu^s}{k_2}Z\left(\frac{\mu}{k_2}\right)\rho\,G_{r-1}\left(o,\lambda,\frac{\rho\mu}{k_2},\sqrt{1-\rho^2}\right)\Big]\Big\}$$

$$+\rho(r+s-1)\left[V^1_{r-1,S-1}+k_2V^2_{r-1,S-1}+k_1V^3_{r-1,S-1}+k_1k_2V^4_{r-1,S-1}\right]$$

$$+(r-1)(s-1)(1-\rho^2)\left[V^1_{r-2,S-2}+k_2^2V^2_{r-2,S-2}+k_1^2V^3_{r-2,S-2}+k_1^2k_2^2V^4_{r-2,S-2}\right]\Big\}$$

(3.13)

Considering $s=0$ in (3.5) we get

$$V_{r,0}=E\left[Z_{11}{}^r\right]$$

$$=K(2\pi)(1-\rho^2)^{3/2}\Big[\eta^{r-1}Z(\eta)\left\{C_1G_o\left(\kappa,o,\eta\rho,\sqrt{1-\rho^2}\right)+C_2G_o\left(o\mu,\eta\rho,\sqrt{1-\rho^2}\right)\right\}$$

$$-\lambda^{r-1}Z\left(\frac{\lambda}{k_1}\right)k_1^2\left\{C_3G_o\left(\kappa,o,\frac{\rho\lambda}{k_1},\sqrt{1-\rho^2}\right)+C_4G_o\left(o,\mu,\frac{\rho\lambda}{k_1},\sqrt{1-\rho^2}\right)\right\}\Big]$$

$$+(r-1)\left(1-\rho^2\right)\left[V^1_{r-2,o}+V^2_{r-2,o}+k_1^2\left(V^3_{r-2,o}+V^4_{r-2,o}\right)\right]$$
$$+\rho\left[V^1_{r-1,1}+\frac{1}{k_2}V^2_{r-1,1}+k_1\left(V^3_{r-1,1}+\frac{1}{k_2}V^4_{r-1,1}\right)\right]\Bigg]$$

(3.14)

Now, considering $s = 0$ in (3.12) we get

$$V^1_{r-1,1}=C_1K\,Z(k)(2\pi)(1-\rho^2)^{3/2}G_{r-1}\left(o,h,kr,\sqrt{1-\rho^2}\right)+\rho V^1_{r,o}$$

$$V^2_{r-1,1}=-C_2K\,k_2^2Z\left(\frac{m}{k_2}\right)(2\pi)(1-\rho^2)^{3/2}G_{r-1}\left(o,h,\frac{r\,m}{k_2},\sqrt{1-\rho^2}\right)+\rho\,k_2\,V^2_{r,o}$$

$$V^3_{r-1,1}=C_3K\,k_1^rZ(k)(2\pi)(1-\rho^2)^{3/2}G_{r-1}\left(l,o,r\,k,\sqrt{1-\rho^2}\right)+\frac{\rho}{k_1}\,V^3_{r,o}$$

$$V^4_{r-1,1}=-C_4K\,k_2^2k_1^rZ\left(\frac{m}{k_2}\right)(2\pi)(1-\rho^2)^{3/2}G_{r-1}\left(l,o,\frac{r\,m}{k_2},\sqrt{1-\rho^2}\right)+\frac{\rho\,k_2}{k_1}\,V^4_{r,o} \qquad \textbf{(3.15)}$$

Using (3.15) in (3.14), we get

$$\therefore\; V_{r,0}=K(2\pi)\sqrt{1-\rho^2}\Bigg[h^{r-1}Z(h)\left\{C_1G_o\left(k,o,hr,\sqrt{1-\rho^2}\right)+C_2G_o\left(o,m,hr,\sqrt{1-\rho^2}\right)\right\}$$

$$-\,l^{r-1}Z\left(\frac{l}{k_1}\right)k_1^2\left\{C_3G_o\left(k,o,\frac{r\,l}{k_1},\sqrt{1-\rho^2}\right)+C_4G_o\left(o,m,\frac{r\,l}{k_1},\sqrt{1-\rho^2}\right)\right\}\Bigg]$$

$$+(r-1)\left[V^1_{r-2,o}+V^2_{r-2,o}+k_1^2\left(V^3_{r-2,o}+V^4_{r-2,o}\right)\right]$$

$$+C_1K\rho\,Z(k)(2\pi)\sqrt{1-\rho^2}\,G_{r-1}\left(o,h,kr,\sqrt{1-\rho^2}\right)$$

$$-\,C_2K\rho\,k_2Z\left(\frac{m}{k_2}\right)(2\pi)\sqrt{1-\rho^2}\,G_{r-1}\left(o,h,\frac{r\,m}{k_2},\sqrt{1-\rho^2}\right)$$

$$+C_3K\rho\,k_1^{r+1}Z(k)(2\pi)\sqrt{1-\rho^2}\,G_{r-1}\left(l,o,r\,k\sqrt{1-\rho^2}\right)$$

$$- C_4 K r k_2 k_1^{r+1} Z\left(\frac{m}{k_2}\right)(2p)\sqrt{1-r^2}\; G_{r-1}\left(l, o, \frac{r m}{k_2}, \sqrt{1-r^2}\right)$$

(3.16)

By symmetry, we obtain

$$V_{0s} = K(2p)\sqrt{1-r^2}\Big[k^{s-1} Z(k)\left\{C_1 G_o\left(h,o,kr,\sqrt{1-r^2}\right) + C_2 G_o\left(h,o,kr,\sqrt{1-r^2}\right)\right\}$$

$$- m^{s-1} Z\left(\frac{m}{k_2}\right)k_2^2 \left\{ C_3 G_o\left(o,l,\frac{r m}{k_2},\sqrt{1-r^2}\right) + C_4 G_o\left(o,l,\frac{r m}{k_2},\sqrt{1-r^2}\right)\right\}\Big]$$

$$+ (s-1)\left[V^1_{o,s-2} + V^2_{o,s-2} + k_2^2\left(V^3_{o,s-2} + V^4_{o,s-2}\right)\right]$$

$$+ C_1 K r Z(h)(2p)\sqrt{1-r^2}\, G_{s-1}\left(o,k,hr,\sqrt{1-r^2}\right)$$

$$- C_2 K r k_1 Z\left(\frac{l}{k_1}\right)(2p)\sqrt{1-r^2}\, G_{s-1}\left(m,o,\frac{r l}{k_1},\sqrt{1-r^2}\right)$$

$$+ C_3 K r k_2^{s+1} Z(h)(2p)\sqrt{1-r^2}\, G_{s-1}\left(o,k,r h,\sqrt{1-r^2}\right)$$

$$- C_4 K r k_1 k_2^{s+1} Z\left(\frac{l}{k_1}\right)(2p)\sqrt{1-r^2}\; G_{s-1}\left(m,o,\frac{r l}{k_1},\sqrt{1-r^2}\right) \qquad \textbf{(3.17)}$$

Using (3.16) and (3.17) we can obtain the first and second order moments which are given below:

$$V_{10} = K(2p)\sqrt{1-r^2}\Big[Z(h)\left\{C_1 G_o(k,o,hr,\sqrt{1-r^2} + C_2 G_o\left(o,m,hr,\sqrt{1-r^2}\right)\right\}$$

$$- Z\left(\frac{l}{k_1}\right)k_1^2 \left\{ C_3 G_o\left(k,o,\frac{r l}{k_1},\sqrt{1-r^2}\right) + C_4 G_o\left(o,m,\frac{r l}{k_1},\sqrt{1-r^2}\right)\right\}\Big]$$

$$+ C_1 K r Z(k)(2p)\sqrt{1-r^2}\, G_o\left(o,h,kr,\sqrt{1-r^2}\right)$$

$$- C_2 K r k_2 Z\left(\frac{m}{k_2}\right)(2p)\sqrt{1-r^2}\, G_o\left(o,h,\frac{r m}{k_2},\sqrt{1-r^2}\right)$$

$$+ C_3 K r k_1^2 Z(k)(2p)\sqrt{1-r^2}\, G_o\left(l,o,r k,\sqrt{1-r^2}\right)$$

$$- C_4 K r k_2 k_1^2 Z\left(\frac{m}{k_2}\right)(2p)\sqrt{1-r^2}\; G_o\left(l,o,\frac{r m}{k_2},\sqrt{1-r^2}\right)$$

(3.18)

$$
\begin{aligned}
V_{01} = {} & K(2\pi)\sqrt{1-r^2}\left[Z(k)\left\{C_1 G_o\left(h,o,kr,\sqrt{1-r^2}\right) + C_2 G_o\left(h,o,kr,\sqrt{1-r^2}\right)\right\}\right. \\
& \left. - Z\left(\frac{m}{k_2}\right)k_2^2\left\{C_3 G_o\left(o,l,\frac{rm}{k_2},\sqrt{1-r^2}\right) + C_4 G_o\left(o,l,\frac{rm}{k_2},\sqrt{1-r^2}\right)\right\}\right] \\
& + C_1 Kr\, Z(h)(2\pi)\sqrt{1-r^2}\, G_o\left(o,k,hr,\sqrt{1-r^2}\right) \\
& - C_2 Kr\, k_1 Z\left(\frac{l}{k_1}\right)(2\pi)\sqrt{1-r^2}\, G_o\left(m,o,\frac{rl}{k_1},\sqrt{1-r^2}\right) \\
& + C_3 Kr\, k_2^2 Z(h)(2\pi)\sqrt{1-r^2}\, G_o\left(o,k,rh,\sqrt{1-r^2}\right) \\
& - C_4 Kr\, k_1 k_2^2 Z\left(\frac{l}{k_1}\right)(2\pi)\sqrt{1-r^2}\, G_o\left(m,o,\frac{rl}{k_1},\sqrt{1-r^2}\right)
\end{aligned}
$$

(3.19)

and

$$
\begin{aligned}
V_{2o} = {} & K(2\pi)\sqrt{1-r^2}\left[hZ(h)\left\{C_1 G_o\left(k,o,hr,\sqrt{1-r^2}\right) + C_2 G_o\left(o,m,hr,\sqrt{1-r^2}\right)\right\}\right. \\
& \left. - l\, Z\left(\frac{l}{k_1}\right)k_1^2\left\{C_3 G_o\left(k,o,\frac{rl}{k_1},\sqrt{1-r^2}\right) + C_4 G_o\left(o,m,\frac{rl}{k_1},\sqrt{1-r^2}\right)\right\}\right] \\
& + V_{o,o}^{1} + V_{o,o}^{2} + k_1^2\left(V_{o,o}^{3} + V_{o,o}^{4}\right) \\
& + C_1 Kr\, Z(k)(2\pi)\sqrt{1-r^2}\, G_1\left(o,h,kr,\sqrt{1-r^2}\right) \\
& - C_2 Kr\, k_2 Z\left(\frac{m}{k_2}\right)(2\pi)\sqrt{1-r^2}\, G_1\left(o,h,\frac{rm}{k_2},\sqrt{1-r^2}\right) \\
& + C_3 Kr\, k_1^3 Z(k)(2\pi)\sqrt{1-r^2}\, G_1\left(l,o,rk,\sqrt{1-r^2}\right) \\
& - C_4 Kr\, k_2 k_1^3 Z\left(\frac{m}{k_2}\right)(2\pi)\sqrt{1-r^2}\, G_1\left(l,o,\frac{rm}{k_2},\sqrt{1-r^2}\right)
\end{aligned}
$$

(3.20)

$$V_{02} = K(2\pi)\sqrt{1-\rho^2}\Bigg[kZ(k)\left\{C_1G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)+C_2G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)\right\}$$
$$-\mu Z\left(\frac{\mu}{k_2}\right)k_2^2\left\{C_3G_o\left(o,l,\frac{\rho\mu}{k_2},\sqrt{1-\rho^2}\right)+C_4G_o\left(o,l,\frac{\rho\mu}{k_2},\sqrt{1-\rho^2}\right)\right\}\Bigg]$$
$$+\ V_{o,o}^1+V_{o,o}^2+k_2^2\left(V_{o,o}^3+V_{o,o}^4\right)$$
$$+C_1K\rho Z(h)(2\pi)\sqrt{1-\rho^2}G_1\left(o,k,h\rho,\sqrt{1-\rho^2}\right)$$
$$-C_2K\rho k_1Z\left(\frac{\lambda}{k_1}\right)(2\pi)\sqrt{1-\rho^2}G_1\left(m,o,\frac{\rho\lambda}{k_1},\sqrt{1-\rho^2}\right)$$
$$+C_3K\rho k_2^3Z(h)(2\pi)\sqrt{1-\rho^2}G_1\left(o,k,\rho h,\sqrt{1-\rho^2}\right)$$
$$-C_4K\rho k_1k_2^3Z\left(\frac{\lambda}{k_1}\right)(2\pi)\sqrt{1-\rho^2}G_1\left(m,o,\frac{\rho\lambda}{k_1},\sqrt{1-\rho^2}\right)$$

(3.21)

Putting $r = 1$, $s = 1$, in (3.13), we get

$$V_{1,1} = C_1K(2\pi)\sqrt{1-\rho^2}\left[Z(h)G_1\left(k,o,h\rho,\sqrt{1-\rho^2}\right)+k\,Z(k)\rho\;G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)\right]$$
$$+C_2K(2\pi)\sqrt{1-\rho^2}\left[k_2Z(h)G_1\left(o,m,h\rho,\sqrt{1-\rho^2}\right)-k_2\mu\,Z\left(\frac{\mu}{k_2}\right)\rho\,G_o\left(h,o,\frac{\rho\mu}{k_2},\sqrt{1-\rho^2}\right)\right]$$
$$-C_3K(2\pi)\sqrt{1-\rho^2}\left[k_1^2Z\left(\frac{\lambda}{k_1}\right)G_1\left(k,o,\frac{\rho\lambda}{k_1},\sqrt{1-\rho^2}\right)-k_1k\,Z(k)\rho\,G_o\left(o,l,\rho k,\sqrt{1-\rho^2}\right)\right]$$
$$-C_4K(2\pi)\sqrt{1-\rho^2}\left[k_1^2k_2Z\left(\frac{\lambda}{k_1}\right)G_1\left(o,m,\frac{\rho\lambda}{k_1},\sqrt{1-\rho^2}\right)+\frac{k_1\mu}{k_2}Z\left(\frac{\mu}{k_2}\right)\rho\;G_o\left(o,l,\frac{\rho\mu}{k_2},\sqrt{1-\rho^2}\right)\right]$$

(3.22)

Now, $E(z_{11}) = V_{10}$, $E(z_{21}) = V_{01}$, $V(z_{11}) = V_{20} - (V_{10})^2$, $V(z_{21}) = V_{02} - (V_{01})^2$.

Also, $Cov(z_{11}, z_{21}) = V_{11} - V_{10}V_{01}$ and Correlation coefficient (ρ) is $\dfrac{V_{11}-V_{10}V_{01}}{\sqrt{V(z_{11})}\sqrt{V(z_{11})}}$.

V_{10}, V_{01}, V_{20}, V_{02}, V_{11} are obtained from equations (3.18) to (3.22).

Singly truncated two piece bivariate normal distributions

Singly TTPBN distributions can be shown as a particular case of doubly truncated distribution.

If we consider $h=-\infty$ and $k=-\infty$ then the upper truncated TPBN distribution will be given as

$$f(z_{11},z_{21})=\begin{cases} C_1K\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[z_{11}^2-2\rho z_{11}z_{21}+z_{21}^2\right]\right\} & -\infty<z_{11}<0,\ -\infty<z_{21}<0 \\ C_2K\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[z_{11}^2-2\rho z_{11}\frac{z_{21}}{k_2}+\left(\frac{z_{21}}{k_2}\right)^2\right]\right\} & -\infty<z_{11}<0,\ 0<z_{21}<m \\ C_3K\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{z_{11}}{k_1}\right)^2-2\rho\left(\frac{z_{11}}{k_1}\right)z_{21}+z_{21}^2\right]\right\} & 0<z_{11}<l,\ -\infty<z_{21}<0 \\ C_4K\ \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{z_{11}}{k_1}\right)^2-2\rho\left(\frac{z_{11}}{k_1}\right)\left(\frac{z_{21}}{k_2}\right)+\left(\frac{z_{21}}{k_2}\right)^2\right]\right\} & 0<z_{11}<l,\ 0<z_{21}<m \end{cases}$$

$$C_1^{-1}=2K\sigma_{11}\sigma_{21}(2\pi)$$

$$C_2^{-1}=4K\sigma_{11}\sigma_{21}(2\pi)\left[\Phi\left(\frac{m}{k_2}\right)-\frac{1}{2}\right]$$

$$C_3^{-1}=4K\sigma_{11}\sigma_{21}(2\pi)\left[\Phi\left(\frac{\frac{l}{k_1}-\rho z_{21}}{\sqrt{1-\rho^2}}\right)-\frac{1}{2}\right]$$

$$C_4^{-1}=8K\sigma_{11}\sigma_{21}(2\pi)\left[\frac{1}{2}-\Phi\left(\frac{h-\rho\frac{z_{21}}{k_2}}{\sqrt{1-\rho^2}}\right)\right]\left[\Phi\left(\frac{m}{k_2}\right)-\frac{1}{2}\right]$$

Estimation of Parameters of upper TTPBN distribution

Let $\quad V_{r,s}=E\left[z_{11}^r\ z_{21}^s\right]$

$$\therefore V_{r,s} = -C_2 K(2\pi)\sqrt{1-\rho^2}\Bigg[k_2^2 m^{s-1} Z\left(\frac{m}{k_1}\right)(r-1)(1-\rho^2)G_{r-2}\left(\infty, o, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)$$

$$- k_2\, m^s\, Z\left(\frac{m}{k_2}\right)\rho\, G_{r-1}\left(\infty, o, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)\Bigg]$$

$$- C_3 K\, (2\pi)\sqrt{1-\rho^2}\, l^{r-1} k_1^2\, Z\left(\frac{l}{k_1}\right) G_s\left(\infty, o, \frac{\rho l}{k_1}, \sqrt{1-\rho^2}\right)$$

$$- C_4 K\, (2\pi)\sqrt{1-\rho^2}\,\Bigg[k_1^2 k_2^s\, l^{r-1}\, Z\left(\frac{l}{k_1}\right) G_s\left(o, m, \frac{\rho l}{k_1}, \sqrt{1-\rho^2}\right)$$

$$+ k_1^r\, k_2^2\, m^{s-2}\, Z\left(\frac{m}{k_2}\right)(1-\rho^2)(r-1)\, G_{r-2}\left(o, l, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)$$

$$+ k_1^r\, \frac{m^s}{k_2}\, Z\left(\frac{m}{k_2}\right)\rho\, G_{r-1}\left(o, l, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)\Bigg]$$

(4.1)

By considering $s = 0$ in (4.1), we get

$$V_{r,0} = -K(2\pi)\sqrt{1-\rho^2}\, l^{r-1} Z\left(\frac{l}{k_1}\right) k_1^2\left[C_3 G_o\left(-\infty, o, \frac{\rho l}{k_1}, \sqrt{1-\rho^2}\right) + C_4 G_o\left(o, m, \frac{\rho l}{k_1}, \sqrt{1-\rho^2}\right)\right]$$

$$+(r-1)\left[V_{r-2,o}^1 + V_{r-2,o}^2 + k_1^2\left(V_{r-2,o}^3 + V_{r-2,o}^4\right)\right]$$

$$-C_2 K\rho k_2 Z\left(\frac{m}{k_2}\right)(2\pi)\sqrt{1-\rho^2}\, G_{r-1}\left(o, -\infty, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)$$

$$-C_4 K\rho k_2 k_1^{r+1} Z\left(\frac{m}{k_2}\right)(2\pi)\sqrt{1-\rho^2}\ G_{r-1}\left(l, o, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right).$$

(4.2)

By symmetry, we obtain

$$V_{os} = -K(2\pi)\sqrt{1-\rho^2}\, m^{s-1} Z\left(\frac{m}{k_2}\right) k_2^2\left[C_3 G_o\left(o, l, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right) + C_4 G_o\left(o, l, \frac{\rho m}{k_2}, \sqrt{1-\rho^2}\right)\right]$$

$$+(s-1)\left[V_{o,s-2}^1 + V_{o,s-2}^2 + k_1^2\left(V_{o,s-2}^3 + V_{o,s-2}^4\right)\right]$$

$$- C_2 K\rho k_1 Z\left(\frac{l}{k_1}\right)(2\pi)\sqrt{1-\rho^2}\,G_{s-1}\left(m,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)$$

$$- C_4 K\rho k_1 k_2^{s+1} Z\left(\frac{l}{k_1}\right)(2\pi)\sqrt{1-\rho^2}\,G_{s-1}\left(m,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)$$

(4.3)

Using (4.2) and (4.3) we can obtain the first and second order moments which are given below:

$$V_{10} = -K(2\pi)\sqrt{1-\rho^2}\,Z\left(\frac{l}{k_1}\right)k_1^2\left[C_3 G_o\left(-\infty,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)+C_4 G_o\left(o,m,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)\right]$$

$$- C_2 K\rho k_2 Z\left(\frac{m}{k_2}\right)(2\pi)\sqrt{1-\rho^2}\,G_o\left(-\infty,o,\frac{\rho m}{k_2},\sqrt{1-\rho^2}\right)$$

$$- C_4 K\rho k_2 k_1^2 Z\left(\frac{m}{k_2}\right)(2\pi)\sqrt{1-\rho^2}\,G_o\left(l,o,\frac{\rho m}{k_2},\sqrt{1-\rho^2}\right)$$

(4.4)

and

$$V_{01} = -K(2\pi)\sqrt{1-\rho^2}\,Z\left(\frac{m}{k_2}\right)k_2^2\left\{C_3 G_o\left(o,l,\frac{\rho m}{k_2},\sqrt{1-\rho^2}\right)+C_4 G_o\left(o,l,\frac{\rho m}{k_2},\sqrt{1-\rho^2}\right)\right\}$$

$$- C_2 K\rho k_1 Z\left(\frac{l}{k_1}\right)(2\pi)\sqrt{1-\rho^2}\,G_o\left(m,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)$$

$$- C_4 K\rho k_1 k_2^2 Z\left(\frac{l}{k_1}\right)(2\pi)\sqrt{1-\rho^2}\,G_o\left(m,o,\frac{\rho l}{k_1},\sqrt{1-\rho^2}\right)$$

(4.5)

Further,

$$V_{20} = -K(2\pi)\sqrt{1-\rho^2}\,\lambda\, Z\left(\frac{\lambda}{k_1}\right)k_1^2\left\{C_3 G_o\left(-\infty, o, \frac{\rho\lambda}{k_1}, \sqrt{1-\rho^2}\right) + C_4 G_o\left(o, \mu, \frac{\rho\lambda}{k_1}, \sqrt{1-\rho^2}\right)\right\}$$
$$+ V_{o,o}^1 + V_{o,o}^2 + k_1^2\left(V_{o,o}^3 + V_{o,o}^4\right)$$
$$- C_2 K\rho k_2 Z\left(\frac{\mu}{k_2}\right)(2\pi)\sqrt{1-\rho^2}\,G_1\left(-\infty, o, \frac{\rho\mu}{k_2}, \sqrt{1-\rho^2}\right)$$
$$- C_4 K\rho k_2 k_1^3 Z\left(\frac{\mu}{k_2}\right)(2\pi)\sqrt{1-\rho^2}\,G_1\left(\lambda, o, \frac{\rho\mu}{k_2}, \sqrt{1-\rho^2}\right)$$

(4.6)

and

$$V_{02} = -K(2\pi)\sqrt{1-\rho^2}\,\mu\, Z\left(\frac{\mu}{k_2}\right)k_2^2\left\{C_3 G_o\left(o, \lambda, \frac{\rho\mu}{k_2}, \sqrt{1-\rho^2}\right) + C_4 G_o\left(o, \lambda, \frac{\rho\mu}{k_2}, \sqrt{1-\rho^2}\right)\right\}$$
$$+ V_{o,o}^1 + V_{o,o}^2 + k_2^2\left(V_{o,o}^3 + V_{o,o}^4\right)$$
$$- C_2 K\rho k_1 Z\left(\frac{\lambda}{k_1}\right)(2\pi)\sqrt{1-\rho^2}\,G_1\left(\mu, o, \frac{\rho\lambda}{k_1}, \sqrt{1-\rho^2}\right)$$
$$- C_4 K\rho k_1 k_2^3 Z\left(\frac{\lambda}{k_1}\right)(2\pi)\sqrt{1-\rho^2}\,G_1\left(\mu, o, \frac{\rho\lambda}{k_1}, \sqrt{1-\rho^2}\right)$$

(4.7)

Putting $r = 1$, $s = 1$ in (4.1), we get

$$V_{1,1} = -C_2 K(2\pi)\sqrt{1-\rho^2}\,k_2\mu\, Z\left(\frac{\mu}{k_2}\right)\rho\, G_o\left(-\infty, o, \frac{\rho\mu}{k_2}, \sqrt{1-\rho^2}\right)$$
$$- C_3 K(2\pi)\sqrt{1-\rho^2}\,k_1^2 Z\left(\frac{\lambda}{k_1}\right)G_1\left(-\infty, o, \frac{\rho\lambda}{k_1}, \sqrt{1-\rho^2}\right)$$
$$- C_4 K(2\pi)\sqrt{1-\rho^2}\left[k_1^2 k_2 Z\left(\frac{\lambda}{k_1}\right)G_1\left(o, \mu, \frac{\rho\lambda}{k_1}, \sqrt{1-\rho^2}\right)\right.$$
$$\left. + \frac{k_1\mu}{k_2} Z\left(\frac{\mu}{k_2}\right)\rho\, G_o\left(o, \lambda, \frac{\rho\mu}{k_2}, \sqrt{1-\rho^2}\right)\right]$$

(4.8)

Now, $E(z_{11}) = V_{10}$, $E(z_{21}) = V_{01}$, $V(z_{11}) = V_{20} - (V_{10})^2$, $V(z_{21}) = V_{02} - (V_{01})^2$.

Also, $Cov(z_{11}, z_{21}) = V_{11} - V_{10}V_{01}$ and Correlation coefficient (ρ) is $\dfrac{V_{11} - V_{10}V_{01}}{\sqrt{V(z_{11})}\sqrt{V(z_{11})}}$.

V_{10}, V_{01}, V_{20}, V_{02}, V_{11} are obtained from equations (4.4) to (4.8).

If we consider $l = \infty$ and $m = \infty$ then the lower truncated TPBN distribution will be given as

$$
f(z_{11}, z_{21}) = \begin{cases}
C_1 K \exp\left\{ \dfrac{-1}{2(1-\rho^2)} \left[z_{11}^2 - 2\rho z_{11} z_{21} + z_{21}^2 \right] \right\} & h < z_{11} < 0,\ k < z_{21} < 0 \\
C_2 K \exp\left\{ \dfrac{-1}{2(1-\rho^2)} \left[z_{11}^2 - 2\rho z_{11} \dfrac{z_{21}}{k_2} + \left(\dfrac{z_{21}}{k_2} \right)^2 \right] \right\} & h < z_{11} < 0,\ 0 < z_{21} < \infty \\
C_3 K \exp\left\{ \dfrac{-1}{2(1-\rho^2)} \left[\left(\dfrac{z_{11}}{k_1} \right)^2 - 2\rho \left(\dfrac{z_{11}}{k_1} \right) z_{21} + z_{21}^2 \right] \right\} & 0 < z_{11} < \infty,\ k < z_{21} < 0 \\
C_4 K \exp\left\{ \dfrac{-1}{2(1-\rho^2)} \left[\left(\dfrac{z_{11}}{k_1} \right)^2 - 2\rho \left(\dfrac{z_{11}}{k_1} \right)\left(\dfrac{z_{21}}{k_2} \right) + \left(\dfrac{z_{21}}{k_2} \right)^2 \right] \right\} & 0 < z_{11} < \infty,\ 0 < z_{21} < \infty
\end{cases}
$$

$$C_1^{-1} = 8K\sigma_{11}\sigma_{21}(2\pi)\left[\frac{1}{2} - \Phi\left(\frac{h - \rho z_{21}}{\sqrt{1-\rho^2}} \right) \right]\left[\frac{1}{2} - \Phi(k) \right]$$

$$C_2^{-1} = 4K\sigma_{11}\sigma_{21}(2\pi)\left[\frac{1}{2} - \Phi\left(\frac{h - \rho \dfrac{z_{21}}{k_2}}{\sqrt{1-\rho^2}} \right) \right]$$

$$C_3^{-1} = 4K\sigma_{11}\sigma_{21}(2\pi)\left[\frac{1}{2} - \Phi(k) \right]$$

$$C_4^{-1} = 2K\sigma_{11}\sigma_{21}(2\pi).$$

Estimation of Parameters of lower TTPBN distribution

Let $V_{r,s} = E\left[z_{11}^r \, z_{21}^s \right]$

$$\therefore\ V_{r,s} = C_1K(2\pi)\sqrt{1-\rho^2}\Big[h^{r-1}Z(h)\,G_s\left(k,o,h\rho,\sqrt{1-\rho^2}\right)$$
$$+\,k^{s-1}\,Z(k)(r-1)(1-\rho^2)G_{r-2}\left(h,o,k\rho,\sqrt{1-\rho^2}\right)$$
$$+\,k^s\,Z(k)r\;G_{r-1}\left(h,o,k\rho,\sqrt{1-\rho^2}\right)\Big]$$
$$+\,C_2K\;(2\pi)\sqrt{1-\rho^2}h^{r-1}Z(h)k_2^sG_s\left(o,\infty,h\rho,\sqrt{1-\rho^2}\right)$$
$$+\,C_3K\;(2\pi)\sqrt{1-\rho^2}\Big[k_1^r k^{s-1}\,Z(k)(1-\rho^2)(r-1)G_{r-2}\left(o,\infty,\rho k,\sqrt{1-\rho^2}\right)$$
$$+\,k_1^r\;k^s\;Z(k)r\;G_{r-1}\left(o,\infty,\rho k,\sqrt{1-\rho^2}\right)\Big]$$

(4.9)

By considering s = 0 in (4.9), we get

$$\therefore\ V_{r,0} = K(2\pi)\sqrt{1-\rho^2}h^{r-1}Z(h)\left\{C_1G_o\left(k,o,h\rho,\sqrt{1-\rho^2}\right)+C_2G_{r-1}\left(o,\infty,h\rho,\sqrt{1-\rho^2}\right)\right\}$$
$$+\,(r-1)\left[V^1_{r-2,o}+V^2_{r-2,o}+k_1^2\left(V^3_{r-2,o}+V^4_{r-2,o}\right)\right]$$
$$+\,C_1K r\,Z(k)(2\pi)\sqrt{1-\rho^2}G_{r-1}\left(o,h,k\rho,\sqrt{1-\rho^2}\right)$$
$$+\,C_3K r\,k_1^{r+1}Z(k)(2\pi)\sqrt{1-\rho^2}G_{r-1}\left(o,\infty,\rho k\sqrt{1-\rho^2}\right)$$

(4.10)

By symmetry, we obtain

$$V_{0s} = K(2\pi)\sqrt{1-\rho^2}\Big[k^{s-1}Z(k)\left\{C_1G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)+C_2G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)\right\}$$
$$+\,(s-1)\left[V^1_{o,s-2}+V^2_{o,s-2}+k_2^2\left(V^3_{o,s-2}+V^4_{o,s-2}\right)\right]$$
$$+\,C_1K r\,Z(h)(2\pi)\sqrt{1-\rho^2}G_{s-1}\left(o,k,h\rho,\sqrt{1-\rho^2}\right)$$
$$+\,C_3K r\,k_2^{s+1}Z(h)(2\pi)\sqrt{1-\rho^2}G_{s-1}\left(o,k,h\rho,\sqrt{1-\rho^2}\right)$$

(4.11)

Using (4.10) and (4.11) we can obtain the first and second order moments which are given below:

$$V_{10} = K(2\pi)\sqrt{1-\rho^2}Z(h)\left\{C_1G_o\left(k,o,h\rho,\sqrt{1-\rho^2}\right)+C_2G_o\left(o,\infty,h\rho,\sqrt{1-\rho^2}\right)\right\}$$

$$+C_1K\rho\, Z(k)(2\pi)\sqrt{1-\rho^2}G_o\left(o,h,k\rho,\sqrt{1-\rho^2}\right)$$

$$+C_3K\rho\, k_1^2Z(k)(2\pi)\sqrt{1-\rho^2}G_0\left(o,\infty,k\rho,\sqrt{1-\rho^2}\right)$$

(4.12)

and

$$V_{01} = K(2\pi)\sqrt{1-\rho^2}\bigg[Z(k)\left\{C_1G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)+C_2G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)\right\}$$

$$+C_1K\rho\, Z(h)(2\pi)\sqrt{1-\rho^2}G_o\left(o,k,h\rho,\sqrt{1-\rho^2}\right)$$

$$+C_3K\rho\, k_2^2Z(h)(2\pi)\sqrt{1-\rho^2}G_o\left(o,k,h\rho,\sqrt{1-\rho^2}\right)$$

(4.13)

Similarly

$$V_{2o} = K(2\pi)\sqrt{1-\rho^2}hZ(h)\left\{C_1G_o\left(k,o,h\rho,\sqrt{1-\rho^2}\right)+C_2G_1\left(o,\infty,h\rho,\sqrt{1-\rho^2}\right)\right\}$$

$$+\; V_{o,o}^1+V_{o,o}^2+k_1^2\left(V_{o,o}^3+V_{o,o}^4\right)$$

$$+C_1K\rho\, Z(h)(2\pi)\sqrt{1-\rho^2}G_1\left(o,h,k\rho,\sqrt{1-\rho^2}\right)$$

$$+C_3K\rho\, k_1^3Z(k)(2\pi)\sqrt{1-\rho^2}G_1\left(o,\infty,k\rho,\sqrt{1-\rho^2}\right)$$

(4.14)

and

$$V_{02} = K(2\pi)\sqrt{1-\rho^2}\bigg[kZ(k)\left\{C_1G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)+C_2G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)\right\}$$

$$
\begin{aligned}
&+ V_{o,o}^{1} + V_{o,o}^{2} + k_2^2\left(V_{o,o}^{3} + V_{o,o}^{4}\right)\\
&+ C_1 K\rho\, Z(h)(2\pi)\sqrt{1-\rho^2}\,G_1\left(o,k,h\rho,\sqrt{1-\rho^2}\right)\\
&+ C_3 K\rho\, k_2^3 Z(h)(2\pi)\sqrt{1-\rho^2}\,G_1\left(o,k,h\rho,\sqrt{1-\rho^2}\right)
\end{aligned}
\tag{4.15}
$$

Putting $r=1$, $s=1$ in (4.9), we get

$$
\begin{aligned}
V_{1,1} = {} & C_1 K(2\pi)\sqrt{1-\rho^2}\left[Z(h)G_1\left(k,o,h\rho,\sqrt{1-\rho^2}\right) + k\,Z(k)\rho\; G_o\left(h,o,k\rho,\sqrt{1-\rho^2}\right)\right]\\
&+ C_2 K(2\pi)\sqrt{1-\rho^2}\,Z(h)k_2 G_1\left(o,\infty,h\rho,\sqrt{1-\rho^2}\right)\\
&+ C_3 K k_1 k\, Z(k)\rho\,(2\pi)\sqrt{1-\rho^2}\,G_o\left(o,\infty,k\rho,\sqrt{1-\rho^2}\right)
\end{aligned}
\tag{4.16}
$$

Now, $E(z_{11}) = V_{10}$, $E(z_{21}) = V_{01}$, $V(z_{11}) = V_{20} - (V_{10})^2$, $V(z_{21}) = V_{02} - (V_{01})^2$.

Also, $Cov(z_{11}, z_{21}) = V_{11} - V_{10}V_{01}$ and Correlation coefficient (ρ) is $\dfrac{V_{11} - V_{10}V_{01}}{\sqrt{V(z_{11})}\sqrt{V(z_{11})}}$.

V_{10}, V_{01}, V_{20}, V_{02}, V_{11} are obtained from equations (4.12) to (4.16).

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