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THE GRAY OPTIMIZATION MODEL APPLICABLE TO NON-HOMOGENOUS EXPONENTIAL SERIES

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ABSTRACT: *This paper proposes a new background which applicable to the non-homogenous exponential series, and give the best reducing value which conditioned by the error quadratic sum of fitting original series least. This method made the model more matching, and through verification, the modeling effect is good, which has certain practical value*
KEYWORDS: non-homogenous exponential; background; reducing value ; Optimization

INTRODUCTION

After nearly thirty years of development,the gray system theory has been widely used in every field of national product[1-2]. Many scholars make a lot of research around to improve the accuracy of predictions model and which have made the modeling precision of GM(1,1) model improved greatly[3-10]. But those optimization methods are based on that the original data is based on the homogenous exponential sequence, however in real applications, the original data is not homogeneous exponential sequence, otherwise the data will lose the modeling significance. The reference [11] presented the new gray forecasting model and white equations for the non-homogenous exponential sequence, which have higher modeling accuracy. But the new method of reference [11] just optimized the model expression, not change the background of the model, which is builded based on the homogenous exponential sequence, this lead to modeling accuracy of the new model is not high. This paper proposes a new background which applicable to the non- homogenous exponential series, and give the best reducing value which conditioned by the error quadratic sum of fitting original series least. This method made the model more matching, and through verification, the modeling effect is good, which has certain practical value.

2、the re-optimization of the GM(1,1) which applicable to non-homogenous exponential sequence

Definition[11]: Let $x^{(0)}(k)$ is a non-negative and non-homogenous exponential sequence， $x^{(1)}(k)$ is the 1－AGO sequence of $x^{(0)}(k)$, and $x^{(1)}(k)=Be^{Ak}+Ck+D$（Especially，if C=0， $x^{(0)}(k)$ is a homogenous exponential sequence）， z^{I} is the background value， then $x^{(0)}(k)+az^{(1)}(k)=\frac{1}{2}(2k-1)b+c$ is the NHGM（1，1，k）,and $\frac{dx^{(1)}}{dt}+ax^{(1)}=tb+c$ is the white equation of NHGM(1,1,k).

Theorem 1：Let $x^{(1)}(k)=Be^{Ak}+Ck+D$ is the1－AGO series of non-homogenous exponential series $x^{(0)}(k)$, (Especially，if C=0，$x^{(0)}(k)$ is a homogenous exponential sequence），then the background $z^{(1)}(k)=\frac{B(e^{A}-1)}{A}e^{A(k-1)}+C(2k-1)+D$ ， among them $A=\ln\frac{\alpha^{(1)}x^{(0)}(k)}{\alpha^{(1)}x^{(0)}(k-1)}$ 、

$B=\frac{\alpha^{(1)}x^{(0)}(k)}{(e^{A}-1)^{2}e^{A(t-2)}}$ 、 $C=x^{(0)}(k)-\frac{e^{A}\alpha^{(1)}x^{(0)}(k)}{e^{A}-1}$ 、

$D=x^{(1)}(k)-kx^{(0)}(k)-\frac{ke^{A}\alpha^{(1)}x^{(0)}(k)}{e^{A}-1}-\frac{e^{2A}\alpha^{(1)}x^{(0)}(k)}{(e^{A}-1)^{2}}$ k=3、4…………

Proof： $x^{(1)}(k)=Be^{Ak}+Ck+D$ is expression（1）。

Then $x^{(0)}(k)=B(e^{A}-1)e^{A(k-1)}+C$ ……………….. （2）

$\alpha^{(1)}x^{(0)}(k)=x^{(0)}(k)-x^{(0)}(k-1)=B(e^{A}-1)^{2}e^{A(k-2)}$ ……………….. （3）

$\frac{\alpha^{(1)}x^{(0)}(k)}{\alpha^{(1)}x^{(0)}(k-1)}=e^{A}$， $A=\ln\frac{\alpha^{(1)}x^{(0)}(k)}{\alpha^{(1)}x^{(0)}(k-1)}$ ……………….. （4）

By （3）, $B=\frac{\alpha^{(1)}x^{(0)}(k)}{(e^{A}-1)^{2}e^{A(t-2)}}$ ……………….. （5）

（5）into （2）,get $C=x^{(0)}(k)-\frac{e^{A}\alpha^{(1)}x^{(0)}(k)}{e^{A}-1}$ ……………….. （6）

And （5）、（6）into （1），

$D=x^{(1)}(k)-kx^{(0)}(k)-\frac{ke^{A}\alpha^{(1)}x^{(0)}(k)}{e^{A}-1}-\frac{e^{2A}\alpha^{(1)}x^{(0)}(k)}{(e^{A}-1)^{2}}$ ……………….. （7）

Get the new background ：

$$Z(k)=\int_{k-1}^{k}x^{(1)}(t)dt=\int_{k-1}^{k}(Be^{At}+Ct+D)dt=\frac{B}{A}e^{At}+Ct^{2}+Dt\Big|_{k-1}^{k}=\frac{B(e^{A}-1)}{A}e^{A(k-1)}+C(2k-1)+D$$

among them， $A=\ln\frac{\alpha^{(1)}x^{(0)}(k)}{\alpha^{(1)}x^{(0)}(k-1)}$ 、 $B=\frac{\alpha^{(1)}x^{(0)}(k)}{(e^{A}-1)^{2}e^{A(t-2)}}$ 、 $C=x^{(0)}(k)-\frac{e^{A}\alpha^{(1)}x^{(0)}(k)}{e^{A}-1}$ 、

$D=x^{(1)}(k)-kx^{(0)}(k)-\frac{ke^{A}\alpha^{(1)}x^{(0)}(k)}{e^{A}-1}-\frac{e^{2A}\alpha^{(1)}x^{(0)}(k)}{(e^{A}-1)^{2}}$ k=3、4…………

Theorem 2[11] Let $\hat{a}=[a,b,c]^T$ is a parameter array,

$$Y=\begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix} \quad B=\begin{bmatrix} -z^{(1)}(2) & \frac{3}{2} & 1 \\ -z^{(1)}(3) & \frac{5}{2} & 1 \\ \vdots & \vdots & \vdots \\ -z^{(1)}(n) & \frac{1}{2}(2n-1) & 1 \end{bmatrix}$$

, then Least square parameter estimation of NHGM（1,1,k） which is $x^{(0)}(k)+az^{(1)}(k)=\frac{1}{2}(2k-1)b+c$ meet the condition $\hat{a}=(B^TB)^{-1}B^TY$.

Theorem 3 Let the white equation of NHGM(1,1,k) is $\frac{df(t)}{dt}+af(t)=tb+c$,whose solution is $f(t)=Ce^{-at}+\frac{b}{a}t+\frac{c}{a}+\frac{b}{a^2}$ (among them, C is a undetermined coefficient), then conditioned by the error quadratic sum of fitting original series least (the expression of error quadratic sum is $S=\sum_{k=1}^{n}[\hat{x}^{(0)}(k)-x^{(0)}(k)]^2$), the best reducing value is

$$\hat{x}^{(0)}(k)=-\frac{\sum_{i=1}^{n}\left[be^{-ai}-ax^{(0)}(k)e^{-ai}\right]}{\sum_{i=1}^{n}ae^{-2ai}}e^{-ak}+\frac{b}{a} \quad k=1,2,3,\ldots$$

Proof： By the white equation $\frac{df(t)}{dt}+af(t)=tb+c$, getting its solution is $f(t)=Ce^{-at}+\frac{b}{a}t+\frac{c}{a}+\frac{b}{a^2}$ (among them, C is a undetermined coefficient), its derivative is $f'(t)=-aCe^{-at}+\frac{b}{a}$. And $\hat{x}^{(0)}(t)=f'(t)=-aCe^{-at}+\frac{b}{a}$

Let $S=\sum_{k=1}^{n}[\hat{x}^{(0)}(k)-x^{(0)}(k)]^2$ is the error quadratic sum of modeling reducing value,

and $S=\sum_{k=1}^{n}[\hat{x}^{(0)}(k)-x^{(0)}(k)]^2=\sum_{k=1}^{n}[-aCe^{-ak}+\frac{b}{a}-x^{(0)}(k)]^2$,

If $S=\sum_{k=1}^{n}[\hat{x}^{(0)}(k)-x^{(0)}(k)]^2$ is the least , $\frac{dS}{dC}=0$ 。

That is $2\sum_{t=1}^{n}\left[-ae^{-at}C+\frac{b}{a}-x^{(0)}(k)\right]-ae^{-at}=0$

Solve it, $C=\dfrac{\sum_{t=1}^{n}\left[be^{-at}-ae^{-at}x^{(0)}(t)\right]}{\sum_{t=1}^{n}a^2e^{-at}}$

C into $\hat{x}^{(0)}(t)=-aCe^{-at}+\frac{b}{a}$

$$\hat{x}^{(0)}(t)=-a\frac{\sum_{t=1}^{n}\left[be^{-at}-ae^{-at}x^{(0)}(t)\right]}{\sum_{t=1}^{n}a^2e^{-at}}e^{-at}+\frac{b}{a}=-\frac{\sum_{t=1}^{n}\left[be^{-at}-ae^{-at}x^{(0)}(t)\right]}{\sum_{t=1}^{n}ae^{-at}}e^{-at}+\frac{b}{a}$$

3、The modeling process

Let $x^{(1)}(k)=Be^{Ak}+Ck+D$ is the1—AGO series of non-homogenous exponential series $x^{(0)}(k)$.

a、Solving the background value $z^{(1)}(k)=\frac{B(e^A-1)}{A}e^{A(k-1)}+C(2k-1)+D$, among them,

$$A=\ln\frac{\alpha^{(1)}x^{(0)}(k)}{\alpha^{(1)}x^{(0)}(k-1)}\text{、}\quad B=\frac{\alpha^{(1)}x^{(0)}(k)}{(e^A-1)^2e^{A(t-2)}}\text{、}\quad C=x^{(0)}(k)-\frac{e^A\square\alpha^{(1)}x^{(0)}(k)}{e^A-1}\text{、}$$

$$D=x^{(1)}(k)-k\square x^{(0)}(k)-\frac{k\square e^A\square\alpha^{(1)}x^{(0)}(k)}{e^A-1}-\frac{e^{2A}\square\alpha^{(1)}x^{(0)}(k)}{(e^A-1)^2}\qquad k=3、4…………$$

b、Solving parameters of the white equation a,b,c, $\hat{a}=[a,b,c]^T=(B^TB)^{-1}B^TY$, among them,

$$Y=\begin{bmatrix}x^{(0)}(2)\\x^{(0)}(3)\\\vdots\\x^{(0)}(n)\end{bmatrix}\quad B=\begin{bmatrix}-z^{(1)}(2)&\frac{3}{2}&1\\-z^{(1)}(3)&\frac{5}{2}&1\\\vdots&\vdots&\vdots\\-z^{(1)}(n)&\frac{1}{2}(2n-1)&1\end{bmatrix}$$

c、solving the best initial condition $C=\dfrac{\sum_{t=1}^{n}\left[be^{-at}-ae^{-at}x^{(0)}(t)\right]}{\sum_{t=1}^{n}a^2e^{-at}}$.

d、Though the forecasting formula

$$\hat{x}^{(0)}(t)=-aCe^{-at}+\frac{b}{a}=-\frac{\sum_{t=1}^{n}\left[be^{-at}-ae^{-at}x^{(0)}(t)\right]}{\sum_{t=1}^{n}ae^{-at}}e^{-at}+\frac{b}{a}$$, getting the fitted values

4、The example analysing

Example 1 $X^{(0)}=\{1+e^{0.6\times 1}, 1+e^{0.6\times 2}, 1+e^{0.6\times 3}, 1+e^{0.6\times 4}, 1+e^{0.6\times 5}, 1+e$ is a non-homogenous exponential series, let the original GM(1,1) is Model 1， the model from reference [12] is Model 2， the model from reference [11] is Model 3， the model from reference [13] is Model 4， the method of this paper is Model 5, and fitting the first six value of the series $X^{(0)}$,forecasting the last value.

Table 1 The fitting accuracy comparison

Original value		4.3201	7.0496	12.0323	21.0855	37.5982	Average relative error(%)
Model 1	Fitting value	3.70882	6.40842	11.073	19.133	33.0597	10.496
	Relative error(%)	14.1501	9.0958	7.9025	9.2601	12.0712	
Model 2	Fitting value	4.26	6.84	11.46	19.73	34.55	3.9
	Relative error(%)	1.32	2.93	4.65	6.39	8.09	
Model 3	Fitting value	4.19	6.71	11.24	19.23	33.83	5.4
	Relative error(%)	2.93	4.69	6.51	8.28	10	
Model 4	Fitting value	3.871	6.831	12.057	21.282	37.562	2.9631
	Relative error(%)	10.4086	3.096	0.2853	0.9297	0.0959	
Model 5	Fitting value	4.3907	7.1782	12.2574	21.5124	38.376	1.8997
	Relative error(%)	1.633	1.8238	1.9485	2.0244	2.0687	

Table 2 The prediction accuracy comparison

Original value		Model 1	Model 2	Model 3	Model 4	Model 5
67.6863	Prediction value	57.1234	61.08	89.8	65.2977	69.1036
	Relative error (%)	15.6057	9.76	11.65	2.598	2.0938

Form the table 1 and table 2, for the non-homogenous exponential series $X^{(0)}=\{1+e^{0.6\times1},1+e^{0.6\times2},1+e^{0.6\times3},1+e^{0.6\times4},1+e^{0.6\times5},1+e^{0.6\times6},1+e^{0.6\times7}\}$, the average relative error of model 1 is 10.496%, and the prediction error is reached 15.6057%, that means the model 1 is losing modeling value. And the model 2、model 3、model 4 who have been optimized, their fitting accuracy are higher than the model 1's, which are 3.9%、5.4%、2.9631%, but the prediction accuracy are not ideal. The fitting error and prediction error of model 5 are 1.8997%、2.0938% respectively, the accuracy is higher , there is a strong practical value.

But the series of example 1 is a strict same parameters non-homogenous exponential series, which is extremely rare in reality. To further verify the effectiveness of the model 5, here using the same model to fitting the different parameters non-homogenous exponential series.

Example 2 $X^{(0)}=\{1+e^{0.6\times1},1.5+e^{0.6\times2},1+e^{0.6\times3},1.5+e^{0.6\times4},1+e^{0.6\times5},1.5+e^{0.6\times6},1+e^{0.6\times7}\}$

is a the different parameters non-homogenous exponential series, let the original GM(1,1) is Model 1， the model from reference [12] is Model 2， the model from reference [11] is Model 3， the model from reference [13] is Model 4， the method of this paper is Model 5, and fitting the first six value of the series $X^{(0)}$,forecasting the last value.

Table 4 The prediction accuracy comparison

Original value		Model 1	Model 2	Model 3	Model 4	Model 5
67.6863	Prediction value	56.2622	55.76	60.57	65.0982	70.1983
	Relative error (%)	16.878	17.62	10.51	3.824	3.7112

From table 3 and table 4, for the different parameters non-homogenous exponential series $X^{(0)}=\{1+e^{0.6\times1},1.5+e^{0.6\times2},1+e^{0.6\times3},1.5+e^{0.6\times4},1+e^{0.6\times5},1.5+e^{0.6\times6},1+e^{0.6\times7}\}$, the average relative error and prediction error of model 1 are more than 10%, still no modeling value. The average relative error and prediction error of Model 2、model 3 are nearly 10%, which losing modeling value. But the fitting error of Model 4、Model 5 are 5%、4.3636% respectively, the prediction error are 3.8%、3.7% respectively, and the Model 5 is better.

Table 3 **The fitting accuracy comparison**

	Original value	4.8201	7.0496	12.5253	21.0855	38.0982	Average relative error(%)
Model 1	Fitting value	3.7775	6.4834	11.1277	19.099	32.7804	12.8371
	Relative error(%)	21.6312	8.032	11.1428	9.4214	13.9583	
Model 2	Fitting value	4.09	6.37	10.51	17.96	31.44	12.18
	Relative error(%)	15.17	9.55	16.09	14.78	17.48	
Model 3	Fitting value	4.59	6.99	11.37	19.37	33.96	5.59
	Relative error(%)	4.75	0.77	9.12	8.09	10.84	
Model 4	Fitting value	3.9284	6.9297	12.224	21.5632	38.0375	5.0028
	Relative error(%)	18.4995	1.701	2.3886	2.2654	0.1594	
Model 5	Fitting value	4.1632	6.952	12.0665	21.4465	38.6491	4.3636
	Relative error(%)	13.6285	1.3853	3.6463	1.712	1.446	

SUMMARY

This paper proposes a new background which applicable to the non- homogenous exponential series, and give the best reducing value which conditioned by the error quadratic sum of fitting original series least. This method made the model more matching, and through verification, the modeling effect is good, which has certain practical value.

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