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THE SOLUTION OF THE INTEGRAL EQUATIONS OF ASTROSTATISTICS

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ABSTRACT: *In the present paper, a hybrid method of practical and analytical techniques was devolved for the solution of the integral equations of Astrostatistics. For the practical technique we used the least squares method to obtain the representations of the involving observational functions. While for the analytical technique we used Fourier integrals to obtain analytical solutions of the density and the frequency functions. Computational procedures are also included to illustrates the implementations' of the developments for analytical as well as numerical computations*

KEYWORDS: Integral equations of the Astrostatistics, Fourier integrals, Least Sqare, Computation procedure, numerical computions

INTRODUCTION

Modern astrophysics is characterized by the extreme rise in the amount of data available, such that, here was a call for the formation of a new area of astronomy and astrophysics called *astroinformatics* [1]. Astrostatistics (or Statistical

Astronomy) has emerged since 1980s. It is a blending of statistical analysis with astronomical data. There are many references dealing with Astrostatistics, of these are for examples [2,3,4]

The least-squares method is the most powerful techniques that has been devised for the problems of statistical data analysis. On the other hand, the analytical formulae are usually offering much deeper insight into the nature of the problems to which they refer.

In the present paper a hybrid method of practical and analytical techniques was devolved for the solution of the integral equations of Astrostatistics. For the practical technique we used the least squares method to obtain the representations of the involving observational functions. While for the analytical technique we used Fourier integrals to obtain analytical solutions of the density and the frequency functions. Computational procedures are also included to illustrates the implementations' of the developments for analytical as well as numerical computations

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BASIC FORMULATIONS

The integral equations

If we assume that the interstellar space is perfectly transparent, then the basic integral equations of the Astrostatistics are [5]:

$$b(x) = \int_{-\infty}^{\infty} \Delta(\rho)\,\Phi(x+\rho)\,d\rho \tag{1}$$

and

$$a(x) = \int_{-\infty}^{\infty} \exp\{k\,\rho\}\,\Delta(\rho)\,\Phi(x+\rho)\,d\rho\,, \tag{2}$$

where , $a(x)$ and $b(x)$ are functions given from observations, while

$$\Delta(\rho) \equiv c\,S\,\exp\{-3c\rho\}\,D\left(e^{-c\rho}\right), \tag{3}$$

$c = 0.4605$, $D(r)$ the density function at the distance $r\ (= e^{-c\rho})$ in a small region of the sky subtends a solid angle S, finally $\Phi(z)$ is the frequency function of the random variable z.

The Fourier integrals

The integrals

$$f^*(\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\exp\{i\beta\,x\}\,dx \tag{4}$$

and

$$f(x) = \int_{-\infty}^{\infty} f^*(\beta)\exp\{-i\,\beta\,x\}\,d\,\beta\,, \tag{5}$$

are respectively the Fourier transfer of the function f(x) and its inverse. The function f(x) satisfies Dirchlet's condition and

$$\int_{-\infty}^{\infty} f(x)\,dx\,, \tag{6}$$

is absolutely convergent.

The representations of the b(x) and a(x)

The functions b and a which are known from observations could be represented analytically by the least squares method [6] in the forms:

$$a(x) = \sum_{i=1}^{n} A_i \varphi_i(x)\,, \tag{7.1}$$

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$$b(x) = \sum_{i=1}^{n} B_i \psi_i (x), . \tag{7.2}$$

where φ's and ψ's are both linearly independent functions of x.

SOLUTION OF THE INTEGRAL EQUATIONS

Case 1: The functions b(x), or a(x) , and Φ(x) being known.

Multiply Equation (1) by $\exp\{i\,\omega\,x\}$ and integrate between $-\infty$ and $+\infty$ Then

$$\int_{-\infty}^{\infty} b(x) \exp\{i\,\omega\,x\} dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta(\rho)\Phi(x+\rho)\exp\{i\,\omega\,x\} dx d\rho$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta(\rho)\Phi(x+\rho)\exp\{i\,\omega\,x\}\exp\{-i\,\omega\,\rho\}.\exp\{i\,\omega\,\rho\} dx\, d\rho,$$

that is:

$$\int_{-\infty}^{\infty} b(x) \exp\{i\,\omega\,x\}\, dx = \int_{-\infty}^{\infty} \Delta(\rho) \exp\{-i\,\omega\,\rho\} d\rho \int_{-\infty}^{\infty} \Phi(x+\rho) \exp\{i\,\omega\,(x+\rho)\} dx \tag{8.1}$$

or

$$\int_{-\infty}^{\infty} b(x) \exp\{i\,\omega\,x\} dx = \int_{-\infty}^{\infty} \Delta(\rho) \exp\{-i\,\omega\,\rho\} H(\rho,\omega) d\rho\,, \tag{8.2}$$

where

$$H(\rho,\omega) = \int_{-\infty}^{\infty} \Phi(x+\rho) \exp\{i\,\omega\,(x+\rho)\, dx\,. \tag{8.3}$$

Let $x+\rho=\alpha$ then Equation (8.3) reduces to:

$$H(\rho,\omega) \equiv H(\omega) = \int_{-\infty}^{\infty} \Phi(\alpha) \exp\{i\,\omega\,\alpha\} d\alpha\,, \tag{8.4}$$

by Equation (4), Equation (8.4) becomes:

$$H(\omega) = \int_{-\infty}^{\infty} \Phi(\alpha) \exp\{i\,\omega\,\alpha\} d\alpha = 2\pi\Phi^{*}(\omega), \tag{8.5}$$

also

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$$\int_{-\infty}^{\infty} \Delta(\rho) \exp\{-i\,\omega\,\rho\} d\rho = 2\pi\,\Delta^*(-\omega), \tag{8.6}$$

$$\int_{-\infty}^{\infty} b(x) \exp\{i\,\omega\,x\} dx = 2\pi\, b^*(\omega). \tag{8.7}$$

Using Equations (8.5), (8.6) and (8.7) into Equation (8.1) we get:

$$2\pi\, b^*(\omega) = \{2\pi\,\Delta^*(-\omega)\}\{2\pi\Phi^*(\omega)\},$$

that is

$$b*(\omega) = 2\pi\,\Delta^*(-\omega)\Phi^*(\omega), \tag{8.8}$$

also

$$b^*(-\omega) = 2\pi\,\Delta^*(\omega)\Phi^*(-\omega). \tag{8.9}$$

Since

$$\Delta(x) = \int_{-\infty}^{\infty} \Delta^*(\omega) \exp\{-i\,\omega\,x\} d\omega, \tag{8.10}$$

then from Equation (8.9) into Equation (8.10) we get:

$$\Delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b^*(-\omega)}{\Phi^*(-\omega)} \exp\{-i\,\omega\,x\} d\omega \Rightarrow$$

$$\Delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b^*(\omega)}{\Phi^*(\omega)} \exp\{i\,\omega\,x\} d\omega. \tag{8.11}$$

Since $b(\omega)$ and $\Phi(\omega)$ are presumed known, the functions $b^*(\omega)$ and $\Phi^*(\omega)$ can be obtained. Formulae (8.9) then gives Δ(x), from which the density function D(x) is easily deduce from Equation (3).

Solution procedure of Case 1

● **Given**: $\mathbf{b(\,x\,)}$ and $\mathbf{\Phi(x)}$

● **Required** : : $\Delta(x)$

● **Computational Steps :**

1- From the known $b(\,x\,)$ and $\Phi(x)$ find the Fourier integrals :

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$$\Phi^*(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\alpha)\, e^{i\omega\omega}d\alpha \qquad ; \qquad b^*(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} b(\alpha)\, e^{i\omega\omega}d\alpha$$

2- In the second integral the function b is known analytically from Equation(7.2)

Find $\Delta(x)$ from:

$$\Delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b^*(\omega)}{\Phi^*(\omega)}\, e^{\,i\omega x}\; d\omega$$

3- End

Case 2:General solution of the two integral Equations

We assume that the functions b(x) and a(x) are obtained from observations and it is required to find the functions $\Delta(x)$ and $\Phi(x)$.

Consider Equation (2) which is:

$$a(x)=\int_{-\infty}^{\infty}\exp\{k\,\rho\}\Delta(\rho)\Phi(x+\rho)\,d\,\rho. \tag{8.12}$$

Multiply by $\exp\{i\,\omega\,x\}\,dx$ and integrate between $-\infty$ to $+\infty$. Then

$$\int_{-\infty}^{\infty}a(x)\exp\{i\,\omega\,x\}dx=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(\rho)\Phi(x+\rho)\exp\{k\,\rho\}\exp\{i\,\omega\,x\}dx\,d\rho=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta(\rho)\Phi(x+\rho)\exp\{i\,\omega\,x\}\exp\{k\,\rho\}\cdot\exp\{-i\omega\,\rho\}.\exp\{i\omega\,\rho\}dx\,d\rho,$$

that is:

$$\int_{-\infty}^{\infty}a(x)\exp\{i\omega\,x\}\,dx=\int_{-\infty}^{\infty}\exp\{-i\omega\,\rho+k\,\rho\}\Delta(\rho)d\rho\int_{-\infty}^{\infty}\exp\{i\omega\,(x+\rho\}\,\Phi(x+\rho)dx. \tag{8.13}$$

According to Equation (4), Equation (8.13) reduces to

$$2\,\pi\,a^*(\omega)=2\pi\,\Phi^*(\omega)\int_{-\infty}^{\infty}\exp\{-i\,\rho(\omega+i\,k\,)\}\,\Delta(\rho)d\rho\,,$$

hence

$$2\pi\,a^*(\omega)=\left(2\pi\,\Phi^*(\omega)\right)\left(2\pi\,\Delta^*(-\omega-ik)\right)$$

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or

$$a^*(\omega) = 2\pi\Phi^*(\omega)\Delta^*(-\omega - ik). \qquad 8.14)$$

Similarly,

$$a^*(-\omega) = 2\pi\Phi^*(-\omega)\Delta^*(\omega - ik). \qquad (8.15)$$

In deriving Equations (8.14) and (8.15) we are assuming that the Fourier integrals hold for complex values of the argument.

From Equations (8.15) and (8.9) we get:

$$\frac{b^*(-\omega)}{a^*(-\omega)} = \frac{\Delta^*(\omega)}{\Delta^*(\omega - ik)}. \qquad (8.16)$$

Let

$$\omega = i\,k\,(\xi + 1), \qquad (8.17)$$

then, from Equations (8.16) and (8.17) we get:

$$\frac{b^*(-ik\,(\xi+1))}{a^*(-ik\,(\xi+1))} = \frac{\Delta^*(\,ik\,(\xi+1))}{\Delta^*(\,ik\xi)}. \qquad (8.18)$$

Let

$$\frac{b^*(-ik\,(\xi+1))}{a^*(-ik\,(\xi+1))} = \exp\{\,F(\,\xi\,)\,\}. \qquad (8.19)$$

Since b(x) and a(x) are supposed to be known functions, the corresponding Fourier transfers can be determined. Thus, the right-hand side of Equation (8.19) is a known function of ξ. Similarly, let

$$\Delta^*(ik\xi) = \exp\{\,G\,(\,\xi\,)\}. \qquad (8.20)$$

From Equations (8.19) and (8.20), then Equation (8.18) becomes

$$\frac{\exp\{\,G\,(\xi+1)\}}{\exp\{\,G\,(\,\xi\,)\}} = \exp\{F(\,\xi\,)\}, \qquad (8.21)$$

that is

$$G(\xi+1) - G(\xi) = F(\xi)\,. \qquad (8.22)$$

The solution of this difference equation can be obtained as follows. From Equation (8.22) we have:

$$G(\xi+1) - G(\,\xi\,) = F(\,\xi\,) \Rightarrow \Delta G(\,\xi\,) = (E-1)\,G(\,\xi\,),$$

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where Δ and E are respectively the difference and the shift operators defined as:

$$E\,f(x) = f(x+1) \quad \text{and} \quad \Delta\,f(x) = f(x+1) - f(x)\,.$$

Consequently, $E^n\, f(x) = f(x+n)$, where n is a positive integer.

From the above equation we get:

$$G(\xi) = -(1-E)^{-1} F(\xi) \xrightarrow{\text{By binomial theorem}} G(\xi) = -(1 - E + E^2 - E^3 + \cdots)F(\xi)$$

that is:

$$G(\xi) = -(F(\xi) - F(\xi+1) - F(\xi+2) + \cdots) \Rightarrow$$

$$G(\xi) = -\sum_{s=0}^{\infty} (-1)^s F(\xi+s) \tag{8.23}$$

This is a functional difference Equation, the solution of which gives us $\Delta^*(\xi)$. The function $\Delta(\xi)$ is then given by:

$$\Delta(\xi) = \int_{-\infty}^{\infty} \Delta^*(\xi) \exp\{-i\omega\,\xi\}\, d\omega\,. \tag{8.24}$$

Also, from the known solution of $\Delta^*(\xi)$, the function $\Phi^*(x)$ may be obtained from Equation (8.15) as:

$$\Phi^*(x) = \frac{b^*(x)}{2\,\pi\,\Delta^*(-x)}\,. \tag{8.25}$$

Finally, $\Phi(x)$ may then be obtained from:

$$\Phi(x) = \int_{-\infty}^{\infty} \Phi^*(x) \exp\{-i\,\omega\,x\} d\omega\,. \tag{8.26}$$

The method described above will be summarized in the following procedure

Solution procedure of Case 2

- **Given**: $\mathbf{a(x)}$ and $\mathbf{b(x)}$
- **Required** : $\mathbf{\Delta(x)}$ and $\mathbf{\Phi(x)}$
- **Steps :**

1-From the known $a(x)$ and $b(x)$ find the Fourier integrals

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$$a^*(\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(x)\, e^{i\beta x} dx \qquad ; \qquad b^*(\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(x)\, e^{i\beta x} dx$$

2- $F(\xi) = \ln b^*\left(-i\,k\,(\xi+1)\right) - \ln a^*\left(-i\,k\,(\xi+1)\right)$

3- $G(\xi) = -\sum_{s=0}^{\infty}(-1)^s F(\xi+s)$

4- $\Delta^*(\omega) = e^{G(-i\omega/k)}$

5- $\Phi^*(x) = \dfrac{b^*(x)}{2\,\pi\,\Delta^*(-x)}$

6- $\Delta(x) = \int_{-\infty}^{\infty}\Delta^*(\omega)e^{-i\omega x}d\omega$

7- $\Phi(x) = \int_{-\infty}^{\infty}\Phi^*(\omega)e^{-i\omega x}d\omega$

8- End

Finally, the density function D(x) could be determined for both cases from

Equation (3)

In concluding the present paper, the solutions of the integral equations of Astrostatistics are developed using a hybrid method of practical and analytical techniques. For the practical technique the least squares method was used to obtain the representations of the involving observational functions. While for the analytical technique we used Fourier integrals to obtain analytical solutions for the density and the frequency functions. Computational procedures are also included to illustrates the implementations' of the developments for analytical as well as numerical computations.

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