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TESTING FOR ORDER RESTRICTION ON MEAN VECTORS OF MULTIVARIATE NORMAL POPULATIONS

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ABSTRACT: *Multivariate isotonic regression theory plays a key role in the field of testing statistical hypotheses under order restriction for vector valued parameters. This kind of statistical hypothesis testing has been studied to some extent, for example, by Kulatunga and Sasabuchi (1984) when the covariance matrices are known and also Sasabuchi et al. (2003) and Sasabuchi (2007) when the covariance matrices are unknown but common. In the present paper, we are interested in a general testing for order restriction of mean vectors against all possible alternatives based on a random sample from several p−dimensional normal populations when the unknown covariance matrices are common. In fact, this problem of testing is an extension of Bazyari and Chinipardaz's (2012) problem. We propose a test statistic by likelihood ratio method based on orthogonal projections on the closed convex cones, study its upper tail probability under the null hypothesis and estimate its critical values for different significance levels by using Monte Carlo simulation. The problem of testing and obtained results is illustrated with a real example where this inference problem arises to evaluate the effect of Vinylidene fluoride on liver damage.*

KEYWORDS: Monte Carlo simulation; Multivariate isotonic regression; Multivariate normal population; Testing order restriction.

AMS (2000) subject classification: Primary 62F30; secondary 62F03, 62H15.

INTRODUCTION

Problems concerning estimation of parameters and determination the statistic, when it is known a priori that some of these parameters are subject to certain order restrictions, are of considerable interest. There are many sizeable literatures dealing with means testing problem under order restrictions. Bartholomew (1959), considered the problem of testing the homogeneity of several univariate normal means against an order restricted alternative hypothesis.

In many applications researchers are interested in testing for inequality constraints among population means vectors $\boldsymbol{\mu}_i$, $i = 1,2,\ldots,k$, after adjusting for covariates. For instance, toxicologists are often interested in studying the effect of a chemical on the mean weight of a specific organ of an animal after adjusting for its body weight (Kanno et al. 2002a, 2002b).

Instead of the usual two-sided alternative $\boldsymbol{\mu}_i \neq \boldsymbol{\mu}_j$ $i \neq j$, researchers are often interested in testing against inequalities among the parameters (known as order restrictions). Some common order restrictions of interest in the multivariate distributions (with at least one strict inequality) are; (a) Simple order $\boldsymbol{\mu}_i \leq \boldsymbol{\mu}_j$, for $i < j$, where this unequal means that all the elements of $\boldsymbol{\mu}_j - \boldsymbol{\mu}_i$ are non-negative. (b) Simple tree order $\boldsymbol{\mu}_1 \leq \boldsymbol{\mu}_j$, for $1 < j$, (c) Umbrella order (with peak at i)

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$\boldsymbol{\mu}_1$ □ $\boldsymbol{\mu}_2$ □□□ $\boldsymbol{\mu}_i$ □ $\boldsymbol{\mu}_{i□1}$ □□□ $\boldsymbol{\mu}_k$. The null hypothesis being H_0□ :$\boldsymbol{\mu}_1$ □ $\boldsymbol{\mu}_2$ □□□ $\boldsymbol{\mu}_k$

(with at least one strict inequality).

Robertson and Wegman (1978), obtained the likelihood ratio test statistic for testing the isotonicness of several univariate normal means against all alternative hypotheses. They calculated its exact critical values at different significance levels for some of the normal distributions and simulated the power by Monte Carlo experiment. Also they considered the test of trend for an exponential class of distributions.

Sasabuchi et al. (1983), extended Bartholomew's (1959) problem to multivariate normal mean vectors with known covariance matrices. They computed the likelihood ratio test statistic and proposed an iterative algorithm for computing the bivariate isotonic regression. Sasabuchi et al. (2003), generalized Bartholomew's (1959) problem to that of several multivariate normal mean vectors with unknown covariance matrices. They proposed a test statistic, studied its upper tail probability under the null hypothesis and estimated its critical values. Sasabuchi (2007), provided some tests, which are more powerful than Sasabuchi et al. (2003).

Bazyari (2012), presented some properties of testing homogeneity of multivariate normal mean vectors against an order restriction for two cases, the covariance matrices are known, and the case that they have an unknown scale factor. He computed the critical values for the proposed test statistic by Kulatunga and Sasabuchi (1984) for the first case at different significance levels for some of the two and three dimensional normal distributions. The power and p□value of test statistic are computed using Monte Carlo simulation. Also when the covariance matrices have an unknown scale factor the specific conditions are given which under those the estimator of the unknown scale factor does not exist and the unique test statistic is obtained. Bazyari and Chinipardaz (2012), generalized Robertson and Wegman's (1978) problem to that of several multivariate normal mean vectors with unknown covariance matrices. They proposed a test statistic, studied its upper tail probability under the null hypothesis and estimated its critical values using Monte Carlo simulation. Bazyari and Pesarin (2013), considered testing the homogeneity of k mean vectors against two-sided restricted alternatives separately in multivariate normal distributions and examined the problem of testing under two separate cases. One case is that covariance matrices are known, the other one is that covariance matrices are unknown but common. In two cases, the test statistics are proposed, the null distributions of test statistics are derived and its critical values are computed at different significance levels. The power of tests studied via Monte Carlo simulation. Bazyari (2016), considered testing homogeneity of multivariate normal mean vectors under an order restriction when the covariance matrices are completely unknown, arbitrary positive definite and unequal. The bootstrap test statistic proposed and because of the main advantage of the bootstrap test is that it avoids the derivation of the complex null distribution analytically and is easy to implement, the bootstrap p□value defined and an algorithm presented to estimate it. The power of the test estimated for some of the p□dimensional normal distributions by Monte Carlo simulation. Also, the null distribution of test statistic evaluated using kernel density. The problem of estimating the unknown parameter $\boldsymbol{\mu}_i$, i □1,2,□, p , under inequality constraints has received considerable attention in many books. For an excellent review on this subject one may refer to the books by Silvapulle and Sen (2005) and van Eeden (2006).

Suppose that $\mathbf{X}_{i1}$,$\mathbf{X}_{i2}$,□,$\mathbf{X}_{in_i}$ are random vectors from a p□dimensional normal distribution N_p□$\boldsymbol{\mu}_i$,□□ with unknown mean vector $\boldsymbol{\mu}_i$, i □1,2,□,k , and nonsingular covariance matrix □. We assume that □ is unknown. Consider the problem of testing

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$H_0 : \boldsymbol{\mu}_1 \square\ \boldsymbol{\mu}_2 \square\square\square\ \boldsymbol{\mu}_k$,

against the hypothesis H_1 , where H_1 is all possible alternatives on the mean vectors. Still consider p □dimensional normal distributions $\mathbf{X}_i \sim N_p \square \boldsymbol{\mu}_i, \square\square$, $i \square 1,2,\square,k$, where $\boldsymbol{\mu}_i \square (\boldsymbol{\mu}\square_{i(1)}, \boldsymbol{\mu}\square_{i(2)})\square$, $i \square 1,2,\square,k$. In general, we say that $\boldsymbol{\mu} \square (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \square, \boldsymbol{\mu}_k) \square R^{p\square k}$, for any $\boldsymbol{\mu}_i \square (\square_{1i}, \square_{2i}, \square, \square_{pi}) \square\square R^p$, $i \square 1,2,\square,k$, is ordered on columns, or simply $\boldsymbol{\mu}$ is on ordered matrix, if $\boldsymbol{\mu}_1 \square \boldsymbol{\mu}_2 \square\square\square \boldsymbol{\mu}_k$. Suppose that the dimension of $\boldsymbol{\mu}\square_{i(1)}$'s is r and the dimension of $\boldsymbol{\mu}\square_{i(2)}$ is $p \square r$. In the present paper, we are interested in the problem of testing

$$H0 : \boldsymbol{\mu}1(1) \square \boldsymbol{\mu}2(1) \square\square\square \boldsymbol{\mu}k(1), \boldsymbol{\mu}1(2) \square \boldsymbol{\mu}2(2) \square\square\square \boldsymbol{\mu}k(2),$$

against all alternative hypotheses on the mean vectors.

Also in the present paper, we suppose that the common covariance matrices are unknown. It is clear that if $r \square 0$, this testing problem is the testing problem given in Bazyari and Chinipardaz (2012). Therefore this testing problem is an extension of Bazyari and Chinipardaz (2012). Such tests may be used in some fields. This kind of testing representation is common, for instance, in selection and ranking problem for finding the largest element of several normal means (see Shimodaira, 2000). Sarka et al. (1995) and Silvapulle and Sen (2005) discuss other examples from different areas, especially in medicine. Also their applications can be found in clinical trails design to test superiority of a combination therapy (Laska and Meisner, 1989 and Sarka et al., 1995). Consider the following example.

Example 1. A survey is conducted among the students in 4th grad, 5th grad and mixed grads in distinct I, and among the students in 4th grad and 5th grad in distinct II. Observations on four variables: the age, the household income, the height and the number of hours for non-academic activities per week in schools are collected. The means are represented as elements in matrix $\boldsymbol{\mu} \square R^{4\square 5}$ and given in Table 1.

Table 1. Structure of the mean vector elements in experiment on the students

	4th grad Dstinct I	5th grad Dstinct I	Mixed grads Dstinct I	4th grad Dstinct II	5th grad Dstinct II
Age	$\square_{11}$	$\square_{12}$	$\square_{13}$	$\square_{14}$	$\square_{15}$
Income	$\square_{21}$	$\square_{22}$	$\square_{23}$	$\square_{24}$	$\square_{25}$
Height	$\square_{31}$	$\square_{32}$	$\square_{33}$	$\square_{34}$	$\square_{35}$
Play hours	$\square_{41}$	$\square_{42}$	$\square_{43}$	$\square_{44}$	$\square_{45}$

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One may assume that the inequalities

□11 □□12 □□13 □□14 □□15, □21 □□22 □□23 □□24 □□25

□31 □□32 □□33 □□34 □□35, □41 □□42 □□43 □□44 □□45,

with at least one strict inequality in one of them is established. So we have the ordered hypothesis H_0□□ : $\boldsymbol{\mu}_1$ □ $\boldsymbol{\mu}_2$ □ $\boldsymbol{\mu}_3$ □ $\boldsymbol{\mu}_4$, with at least one strict inequality.

The rest of this paper is organized as follows. In Section 2, the problem of testing is described, two definitions are given and a test statistic is proposed. In Section 3, the null distribution of the test, two lemmas and main theorem are given. In Section 4, the critical values of the test statistic when the sample sizes are identical and also when they are different are estimated using Monte Carlo simulation. The problem of testing is applied to an application example in Section 5. Concluding remarks are given in Section 6. The complete source programs are written in software S □ $PLUS$.

The problem of testing

Consider p□dimensional normal distributions $\mathbf{X}_i \sim N_p$□$\boldsymbol{\mu}_i$,□□, with observations $\mathbf{X}_{ij}$, j □1,2,□n_i , i □1,2,□,k . In the present paper, we are interested in testing

$$H0 :\boldsymbol{\mu}1(1) □\boldsymbol{\mu}2(1) □□□\boldsymbol{\mu}k(1),\boldsymbol{\mu}1(2) □\boldsymbol{\mu}2(2) □□□\boldsymbol{\mu}k(2),$$

against all alternative hypotheses on the mean vectors when the unknown covariance matrices are common.

1 n_i k n_i

Let $\bar{\mathbf{X}}_i$ □ n_i□j□1 $\mathbf{X}_{ij}$ and S □ □i□1 □j□1 ($\mathbf{X}_{ij}$ □ $\bar{\mathbf{X}}_i$)($\mathbf{X}_{ij}$ □ $\mathbf{X}_i$)□ be the sample mean vector of

i th population and sample mean variance covariance matrix respectively.

Definition 1 (Sasabuchi et al., 1983). Given p□variate real vectors $\mathbf{X}_1$, $\mathbf{X}_2$,□, $\mathbf{X}_k$ and p□ p positive definite matrices □$_1$,□$_2$,□,□$_k$, a p□k real matrix

($\hat{\boldsymbol{\mu}}_1$,$\hat{\boldsymbol{\mu}}_2$,□,$\hat{\boldsymbol{\mu}}_k$) is said to be the multivariate isotonic regression (MIR) of

$\mathbf{X}_1$, $\mathbf{X}_2$,□, $\mathbf{X}_k$ with weights □$_1^{□1}$,□$_2^{□1}$,□,□$_k^{□1}$, if ($\hat{\boldsymbol{\mu}}_1$ □$\hat{\boldsymbol{\mu}}_2$ □□□$\hat{\boldsymbol{\mu}}_k$) and

($\hat{\boldsymbol{\mu}}_1$,$\hat{\boldsymbol{\mu}}_2$,□,$\hat{\boldsymbol{\mu}}_k$) satisfies

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$$\min_{\boldsymbol{\mu}_1 \le \boldsymbol{\mu}_2 \le \cdots \le \boldsymbol{\mu}_k} \sum_{i=1}^{k} (\mathbf{X}_i - \boldsymbol{\mu}_i)'\Sigma_i^{-1}(\mathbf{X}_i - \boldsymbol{\mu}_i) = \sum_{i=1}^{k} (\mathbf{X}_i - \hat{\boldsymbol{\mu}}_i)'\Sigma_i^{-1}(\mathbf{X}_i - \hat{\boldsymbol{\mu}}_i),$$

where $\hat{\boldsymbol{\mu}}_i$'s can be computed by the iterative algorithm proposed by Sasabuchi et al. (1983).

In fact, this definition includes the definition given in Barlow et al. (1972) for univariate variables.

Definition 2. c is called a convex cone if $x, y \in c$, $\alpha \ge 0, \beta \ge 0$, then $\alpha x + \beta y \in c$. Also c is called a closed convex cone if it is convex cone and close set. Define two closed convex cones c_0 and c_1 in R^{pk} by

$$c_0 = \left\{ \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_k \end{pmatrix} \middle| \; \boldsymbol{\mu}_1(1) \le \boldsymbol{\mu}_2(1) \le \cdots \le \boldsymbol{\mu}_k(1), \; \boldsymbol{\mu}_1(2) \le \boldsymbol{\mu}_2(2) \le \cdots \le \boldsymbol{\mu}_k(2), \; \boldsymbol{\mu}_i \in R^p, i = 1,2,\ldots,k \right\},$$

$$c_1 = \left\{ \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_k \end{pmatrix} \middle| \; \boldsymbol{\mu}_i \in R^p, i = 1,2,\ldots,k \right\},$$

where under the closed convex cone c_1 there is no any restriction on the mean vectors $\boldsymbol{\mu}_i$.

Suppose that $\hat{\boldsymbol{\mu}}_i$, $i = 1,2,\ldots,k$, is the MIR of unknown parameter $\boldsymbol{\mu}_i$ under the closed convex cone c_0. Then we have

$$\sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i)' S^{-1} (\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i) = \min_{\boldsymbol{\mu} \in c_0} \left\{ \sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \boldsymbol{\mu}_i)' S^{-1} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}_i) \right\}.$$

For pk-dimensional real vectors $\mathbf{x} = (\mathbf{x}_1', \mathbf{x}_2', \ldots, \mathbf{x}_k')'$ and $\mathbf{y} = (\mathbf{y}_1', \mathbf{y}_2', \ldots, \mathbf{y}_k')'$ their inner product in R^{pk} is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\boldsymbol{\Lambda}} = \sum_{i=1}^{k} n_i \mathbf{x}_i' \boldsymbol{\Lambda}^{-1} \mathbf{y}_i$$

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□ □n1□ □1 0 □ □ □ □**y**1 □ □ □ (**x**$_1$□,**x**□$_2$,□,**x**□$_k$)□ □ □□ □ □.

□0 n □ □1 □ □ □**y**k □ □

□ k □

Also define a norm ‖ .‖□ in R^{pk} by ‖ **x** ‖^2□ □**x**,**x**□$_\Lambda$. Suppose that for **x**□R^{pk},

□$_\Lambda$(**x**,c) be the point which minimizes ‖ **x**□**w** ‖$_\Lambda$, where **w**□c. We note that, since c is a closed convex cone, so the uniqueness of □$_\Lambda$(**x**,c) is clear.

Let A□B be the Kronecker product of matrices $A_{r□m}$ □ (a_{ij}) and $B_{h□s}$ □ (b_{kl}) and defined as

□a11B □ a1mB□

□ □

A□B□ □ □ □ □.

□ □ar1B □ $armB$□ □

Therefore

□**x**,**y**□$_\Lambda$ □ **x**□(D□$\Lambda^{□1}$)**y**,

where

□n_1 0 □ 0 □

□ □ 0

□

D□ □□□0

.

□

□ □ □

□0

□

□

0 □ nk

□ □

The test statistic

The likelihood function for testing H_0 versus H_1 is

L(**μ**1,**μ**2 ,□,**μ**k ,□) □ □i□k1 □ □ □ □ □ □ □ □ □ ______(2□1) p/2 □ □ni □n2i exp□ □ □ □
_1 □n□i (**X**ij □**μ**i)□ □ □1(**X**ij □**μ**i)□ □ □ □ □

□ □ | □ | 2 j 1 □ □ □

n

□ 1 ______ □ _ _n2 exp□ 1 k n □

$$= (2\pi)^{-np/2}|\Sigma|^{-n/2}\exp\left\{-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(\mathbf{X}_{ij}-\boldsymbol{\mu}_i)'\Sigma^{-1}(\mathbf{X}_{ij}-\boldsymbol{\mu}_i)\right\}$$

$$= (2\pi)^{-np/2}|\Sigma|^{-n/2}\exp\left\{-\frac{1}{2}tr\,\Sigma^{-1}\left[\sum_{i=1}^{k}n_i(\bar{\mathbf{X}}_i-\boldsymbol{\mu}_i)(\bar{\mathbf{X}}_i-\boldsymbol{\mu}_i)'+S\right]\right\},$$

where S is distributed with Wishart distribution $W_p(n-k,\Sigma)$ and $n=\sum_{i=1}^{k}n_i$.

Suppose that A is a $p\times p$ non-negative definite real (symmetric) matrix, $\lambda_1,\lambda_2,\ldots,\lambda_p$ are the characteristic roots of A and ϵ is a positive number, then

$$|I_p+\epsilon A| = \prod_{i=1}^{p}(1+\epsilon\lambda_i) = 1+\epsilon\sum_{i=1}^{p}\lambda_i + O(\epsilon^2) = 1+tr(\epsilon A)+O(\epsilon^2). \quad (1)$$

By Anderson (1984), it is well known that the supremum of the function $L(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\ldots,\boldsymbol{\mu}_k,\Sigma)$ on $\Sigma>0$ which is the supremum for all the $p\times p$ positive definite matrices given by

$$\max_{\Sigma>0} L(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\ldots,\boldsymbol{\mu}_k,\Sigma) = \left(\frac{2\pi e}{n}\right)^{-\frac{np}{2}}\left|S+\sum_{i=1}^{k}n_i(\bar{\mathbf{X}}_i-\boldsymbol{\mu}_i)(\bar{\mathbf{X}}_i-\boldsymbol{\mu}_i)'\right|^{-\frac{n}{2}}, \quad (2)$$

Therefore we have

$$\max_{H_0,\Sigma>0} L(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\ldots,\boldsymbol{\mu}_k,\Sigma) = \left(\frac{2\pi e}{n}\right)^{-\frac{np}{2}}\left|S+\sum_{i=1}^{k}n_i(\bar{\mathbf{X}}_i-\hat{\boldsymbol{\mu}}_i)(\bar{\mathbf{X}}_i-\hat{\boldsymbol{\mu}}_i)'\right|^{-\frac{n}{2}}, \quad (3)$$

and also it is completely clear that

$$\max L(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\ldots,\boldsymbol{\mu}_k,\Sigma) = \left(\frac{2\pi e}{n}\right)^{-\frac{np}{2}}|S|^{-\frac{n}{2}}. \quad (4)$$

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□ H_1,□□0□ □ n □

From equations (1) and (2) we get that

np

max□□0 $Ln\, L(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, □, \boldsymbol{\mu}_k, □)$ □ Ln□□□ 2□ne□□□□ 2 □ $\bar{n}\, Ln$ □□ $\bar{S}$ □ □□k n_i $(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)$□□□

2 □ i 1□

np

Ln□ □ □

□ □ 2□ne□□□ 2 n2 Ln □□□□S I_p □ S□12 □i□k1 n_i $(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)$□S □□□ $\frac{1}{2}$

□

np

□ 2□ne□□ 2 n2 □$Ln\, S$ □ Ln □□I_p □ S□12 □□k n_i $(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)$□S□□□□□□□□□ $\frac{1}{2}$

□ Ln□ □ □ □ □i 1□ □ □ □□ □

np

2

□ Ln□□ 2□e□□□ 2 □ n □□$Ln\, S$ □

$$□ Ln\left(\frac{2□e}{n}\right)^{□\frac{np}{2}} □ \frac{n}{2} Ln|S| □ \frac{n}{2} Ln\left[1 □ tr □ \sum n\ S\right.$$

$$□ Ln\left(\frac{2□e}{n}\right)^{□\frac{np}{2}} □ \frac{n}{2} Ln|S| □ \frac{n}{2} Ln\left[1 □ □ \sum n_i (\bar{\mathbf{X}}_i □ \boldsymbol{\mu}_i\right.$$

On the other hand

$$Ln\left|S □ \sum_{i□1}^{k} n_i (\bar{\mathbf{X}}_i □ \hat{\boldsymbol{\mu}}_i)(\bar{\mathbf{X}}_i □ \hat{\boldsymbol{\mu}}_i)'\right|$$

np

2 □2□e □2

Ln□□□□□□□1□tr □□□□□k n_i S□1 $(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)$□□□□□□□□□□□□ □□□□

□ n □ □□ □□□ □ i 1 □□

□ □ k □1 $(\bar{\mathbf{X}}_i$ □ $\boldsymbol{\mu}_i)(\mathbf{X}_i$ □ $\boldsymbol{\mu}_i)$□□□□□□□ i

$$\sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \boldsymbol{\mu}_i)' \Sigma^{-1} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}_i) \Big\}, \quad \Sigma = S$$

$$\max_{H_0} Ln\, L(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_k, \Sigma) \tag{5}$$

$$= -\frac{n}{2}\Big\{ Ln|S| + \min_{\boldsymbol{\mu} \in H_0} \sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \boldsymbol{\mu}_i)' S^{-1} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}_i) \Big\}$$

$$= -\frac{n}{2}\Big\{ Ln|S| + \sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i)' S^{-1} (\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i) \Big\}.$$

By equations given in (3) and (4), the likelihood ratio test statistic is

$$\Lambda = \frac{\left| S + \sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i)(\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i)' \right|^{-\frac{n}{2}}}{|S|^{-\frac{n}{2}}}.$$

Therefore

$$-2Ln\Lambda = n\left\{ Ln\left| S + \sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i)(\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i)' \right| - Ln|S| \right\}$$

Then from equation (5) we have

$$-2Ln\Lambda \approx n \sum_{i=1}^{k} n_i (\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i)' S^{-1} (\bar{\mathbf{X}}_i - \hat{\boldsymbol{\mu}}_i)$$

$$= n\|\bar{\mathbf{X}} - \hat{\boldsymbol{\mu}}\|^2_S,$$

where $\bar{\mathbf{X}} = (\bar{\mathbf{X}}_1', \bar{\mathbf{X}}_2', \ldots, \bar{\mathbf{X}}_k')'$ and $\hat{\boldsymbol{\mu}} = (\hat{\boldsymbol{\mu}}_1', \hat{\boldsymbol{\mu}}_2', \ldots, \hat{\boldsymbol{\mu}}_k')'$. Thus the test statistic by LRT method given by

$$T = n\|\bar{\mathbf{X}} - \hat{\boldsymbol{\mu}}\|^2_S.$$

For given significance level α, we reject the null hypothesis H_0, when $T \geq t_\alpha$, where t_α is a positive constant depending on the significance level.

The null distribution of the test statistic

To obtain the null distribution of the statistic T, first we denote $\frac{T}{n}$ by $\bar{T}$. Then

$$\bar{T} = \|\bar{\mathbf{X}} - \hat{\boldsymbol{\mu}}\|^2_S = \|\bar{\mathbf{X}} - \Pi_S(\bar{\mathbf{X}}, C_0)\|^2_S \tag{6}$$

If $H_2: \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \cdots = \boldsymbol{\mu}_k$, then H_2 is the least favorable among hypotheses satisfying H_0 with the largest type I error probability (Silvapulle and Sen, 2005). Therefore for given the significance level α, we have $\alpha = \sup P_{\boldsymbol{\mu}_0, \Sigma}(\bar{T} \geq t_\alpha)$, where

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□□0

$\boldsymbol{\mu}_0$ is the common value of $\boldsymbol{\mu}_1,□,\boldsymbol{\mu}_k$ under H_2.

Now, easily we have the following theorem.

Theorem 1. Under the hypothesis H_2, the distribution of T given in (6) is independent of $\boldsymbol{\mu}_0$.

Proof. Define the random vector $\mathbf{Y}$ by

□$\mathbf{Y}_1$ □ □$\mathbf{X}_1$ □$\boldsymbol{\mu}_0$ □ □$\boldsymbol{\mu}_0$ □

□ □ □ □ □ □

$\mathbf{Y}$ □ □ □ □ □ □ □ □ □ $\mathbf{X}$ □ □ □ □.

□□$\mathbf{Y}k$ □□ □□$\mathbf{X}_k$ □$\boldsymbol{\mu}_0$ □□ □□$\boldsymbol{\mu}0$ □□

Then it is clear that the distribution of $\mathbf{Y}$ in independent of $\boldsymbol{\mu}_0$ and is distributed with

$N_p(0, \_\_^□)$. On the other hand

n_i

$$T □\| \mathbf{X}□□_S(\mathbf{X},c_0)\|^2_S □\| \mathbf{Y}□□_S(\mathbf{Y},c_0)\|^2_S.$$

Since the distribution of $\|□_S(\mathbf{Y},c_0) □\mathbf{Y}\|^2_S$ is independent of $\boldsymbol{\mu}_0$, so the distribution of T statistic is independent of $\boldsymbol{\mu}_0$ and this completes the proof. Define the closed convex cone c_2 as

□ □$\boldsymbol{\mu}_1$ □ □

□ □ | □□

c2 □ □$\boldsymbol{\mu}$ □ □ □ □$\boldsymbol{\mu}1(1)$ □ $\boldsymbol{\mu}2(1)$ □□□ $\boldsymbol{\mu}k(1)$,□1(2)1 □ □2(2)1 □□□ □k(2)1□,

□□ □□$\boldsymbol{\mu}_k$ □□ □□

where $□_{i(2)1} □e_r□_{□1}\boldsymbol{\mu}_i$, $i □1,2,□,k$ and $e_{r□1}$ is a p□dimensional vector with the $(r □1)□th$ element being one, others are zero. Also we define another statistic

$$T^* □\| \mathbf{X}□□_S(\mathbf{X},c_0)\|^2_S □\| \mathbf{X}□□_S(\mathbf{X},c_2)\|^2_S.$$

Since the computation of the critical values from the formula □□supsup$P_{\boldsymbol{\mu},□}(T □ t_□)$

□ H_0

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is difficult, we will show that the distribution of T^* is independent of $\boldsymbol{\mu}_0$ and Σ, where $\boldsymbol{\mu}_0$ is the common value of $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_k$.

If $\hat{\hat{\boldsymbol{\mu}}} = (\hat{\hat{\boldsymbol{\mu}}}_1', \hat{\hat{\boldsymbol{\mu}}}_2', \ldots, \hat{\hat{\boldsymbol{\mu}}}_k')'$ is the multivariate isotonic regression of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ under the closed convex cone c_2, then

$$T^* = \sum_{i=1}^{k} n_i (\mathbf{X}_i - \hat{\boldsymbol{\mu}}_i)' S^{-1} (\mathbf{X}_i - \hat{\boldsymbol{\mu}}_i) - \sum_{i=1}^{k} n_i (\mathbf{X}_i - \hat{\hat{\boldsymbol{\mu}}}_i)' S^{-1} (\mathbf{X}_i - \hat{\hat{\boldsymbol{\mu}}}_i).$$

Suppose that M is a $p \times p$ nonsingular positive definite matrix given by

$$M = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}, \quad (7)$$

where M_{11} is a $r \times r$ dimension matrix and M_{22} is a $(p-r)\times(p-r)$ dimension matrix. Also put

$$D = \begin{pmatrix} I & 0 \\ 0 & M_{22}^{-1} \end{pmatrix} \quad \text{and} \quad \Gamma = DM.$$

Then we get that

$$\Gamma = \begin{pmatrix} M_{11} & 0 \\ M_{22}^{-1} M_{21} & I \end{pmatrix}.$$

Put

$$\begin{pmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{pmatrix} = \Gamma^{-1} = \begin{pmatrix} M_{11}^{-1} & 0 \\ -M_{22}^{-1} M_{21} M_{11}^{-1} & I \end{pmatrix}. \quad (8)$$

Lemma 1. For matrix M given in (7), we have

a) For any $p \times p$ orthogonal matrix H,

$$(I \otimes (HM))c_0 = c_0.$$

b) There exists a $p \times p$ orthogonal matrix H which satisfies:

$$(I \otimes (HM))c_2 = c_2.$$

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Proof. The proof of part (a) is easy to derive. We only prove the part (b). Put

I_r^* □[I_r 0], then by (8) it is clear that I_r^* □$^{□1}$ □[$M_{11}^{□1}$ 0] . Let □$_r$□$_{□1}$ □e_r□$_{□1}$ □ . Then

□□*T* **μ**$_1$ □ □

(*I* □□)c2 □ □□□□ □ □□ **μ**1(1) □ **μ**2(1) □□□ **μ***k* (1) ,**μ**1(2)1 □ **μ**2(2)1 □□□ **μ***k* (2)1□□

□□□□*T* **μ**$_k$ □□ □□

□ □*k*□1 □□□□□□*T*□**μ**1 □□□ I_r* **μ**$_i$ □ *Ir** **μ**(*i*□1) ,*er*□□1 **μ***i* □*er*□□1 **μ**(*i*□1) □□□ *i*□1 □□□□*T* **μ**$_k$ □□ □□

□*k*□1 □□□□1 □□* □1 *Ir** □□1□(*i*□1) ,*er*□□1 □□1□*i* □*er*□□1 □□1□(*i*□1) □□□ □

□ □□ □ □*Ir* □ □*i* □

i□1 □□□□*k* □□ □□

□

□□□

□ □*k*□1 □□□□ □1 □□□ *M*11□1□*i*(1) □ *M*11□1□(*i*□1)(1) ,*er*□□1 □□1□*i* □*er*□□1 □□1□(*i*□1) □□□

i□1 □□□□$_k$ □□ □□

□

k□1 □□□□□1 □□ □□

□ □□□ □ □□*i*(1) □□(*i*□1)(1) ,□*r*□□1 □*i* □□*r*□□1 □(*i*□1) □ *i*□1 □□□□$_k$ □□ □□

□

k□1 □□□□1 □□□□

□

□ □□□ □ □□*i*(1) □□(*i*□1)(1) , □*i*(2)1 □ □(*i*□1)(2)1□ □ c2. *i*□1 □□□□□$_k$ □□ □□

On the other hand

(*I* □ *M*)c$_2$ □ (*I* □ (*D*$^{□1}$*T*)) c$_2$ □ (*I* □ *D*$^{□1}$)c$_2$

k□1 □□□*D*□1 **μ**1 □□ □□

□

□ □□□ □ □ $\boldsymbol{\mu}i(1)$ □ $\boldsymbol{\mu}(i$□$1)(1)$,$\boldsymbol{\mu}i(2)$ □ $\boldsymbol{\mu}$ $(i$□$1)(2)$ □ i□1 □□□□D□1
$\boldsymbol{\mu}_k$ □□ □□

□k□1 □□□D□1 $\boldsymbol{\mu}1$ □□ * Ir*$\boldsymbol{\mu}(i$□$1)$,er□□1 $\boldsymbol{\mu}i$ □ er□□1 $\boldsymbol{\mu}$
$(i$□$1)$ □□□ □

□ □□ □ □Ir $\boldsymbol{\mu}i$ □

i□1 □□□□D□1 $\boldsymbol{\mu}_k$ □□ □□

□k□1 □□□□□1 □□* D□i □ Ir* D□$(i$□$1)$,er□□1 D□i □
er□□1 D□$(i$□$1)$ □□□

□ □□ □ □Ir

i□1 □□□□k □□ □□

□

□

□k□1 □□□□1 □□* Ir*□$(i$□$1)$,er□□1 D□i □er□□1 D□$(i$□$1)$ □□□.

□ □□ □ □Ir □i □

i□1 □□□□k □□ □□

□

Put a_1 □□$_{11}$$^{□1}$ $M_{22}$$^{□1}$ $e_{1,(p□r)□1}$, where $e_{1,(p□r)□1}$ is a $(p$ □$r)$ -dimensional vector of the form $(1,0,□,0)$, and □$_{11}$ is the positive number such that a_1□a_1 □1. There exists a $(p$ □ $r)$□$(p$ □ $r)$ orthogonal matrix H_{22} such that $H_{22}\, a_1$ □$e_{1,(p□r)□1}$. Put

□I 0 □

H □□$_0$ H_{22} □□ . Then for matrix H , we have

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(I □(HM))c2 □ (I □H)(I □M)c2

□□H□

□k□1 | □□ 1 □□* Ir* □(i□1) ,er□□1 D□i □er□□1
D□(i□1) □□□

□ □□ □ □Ir □i □

i□1 □□H□k □□ □□

□□

□□□

□ |

□k□1 □ 1 □□* H□□i □ Ir* H□ □(i□1) ,er□□1 D H□□i □er□□1
D H□□(i□1) □□□

□ □□ □ □Ir

i□1 □□□□k □□ □□

□

□□□

k□1 □□ 1 □□ * I * ,e M □1 H e M
□1 H □

I □

□ □□□ □ □ | r □i □ r □(i□1) 1□,(p□r)□1 22 22□ □i(2) □ 1□,(p□r)□1 22 22□
□(i□1)(2) □ i□1 □□□k □□ □□

□□

□□□

□k□1 | □□ 1 □□* Ir* □(i□1) ,□11 e1□,(p□r)□1 □i(2) □□11
e1□,(p□r)□1□(i□1)(2) □□□

□ □□ □ □Ir □i □

i□1 □□□□k □□ □□

□

□□□

□ |

□*k*□1 □ 1 □□* *Ir** □(*i*□1) , *er*□□1 □*i* □ *er*□□1□(*i*□1) □□□ □
c2,

□ □□ □ □I_r □$_i$ □

i□1 □□□□$_k$ □□ □□

□

and this completes the proof.

Lemma 2. Under the hypothesis H_2, the distribution of T^* is independent of $\boldsymbol{\mu}_0$ and □, where $\boldsymbol{\mu}_0$ is the common value of $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, □, \boldsymbol{\mu}_k$.

Proof. It is clear that the distribution of T^* is independent of $\boldsymbol{\mu}_0$.

Put $W □ (\mathbf{X} □ \boldsymbol{\mu}_0) □ J$, where J is the vector of k 1's. To arbitrary positive definite real matrix □, there exists lower triangular non-singular matrix M with positive diagonal elements satisfying $M□M□ □ I_p$. Let H be the orthogonal matrix which satisfies the part (b) of lemma 1. Then

$$T^* □\|W □□_S(W,c_0)\|^2_S □\|W □□_S(W,c_2)\|^2_S$$
$$□\|(I □ (HM))W □□_{HMSM□H□}((I □ (HM))W,(I □ (HM))c_0)\|^2_{HMSM□H□}$$
$$□ \|(I □ (HM))W □□_{HMSM□H□}((I □ (HM))W,(I □ (HM))c_2)\|^2_{HMSM□H□} .$$

Put

□Z_1 □ □HMW_1 □ □ □ □ □

□Z^2 □ □HMW^2 □ * $HMSM□H□$.

Z □ □□□□ □ □ □ (*I* □(*HM*))*W* and *S* □

□ □ □ □ □□*Zk* □□ □□*HMWk* □□

Then by lemma 1, we have

$$T^* □\| Z □□_S(Z,c_0)\|^2_{S^*} □\| Z □□_S(Z,c_2)\|^2_{S^*} .$$

By their definitions, S^* and $Z_1, □, Z_k$ are mutually independent, S^* and Z_i distributed as $W_p(n□k, I_p)$ and $N_p(0, n_i^{□1} I_p)$, $i □1, □, k$, respectively. This completes the proof.

Suppose that

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□I^r 0 □

F_n □□□ 0 A_n □□□,

□

where I_r is the r□r identity matrix and A_n is a (p □r)□(p □r) nonsingular matrix defined by

□ 1 □ 0 □

□ □

□□n 1 □ □ A_n □□ □ □ □.

□ □

□□□n 0 □ 1□□

It is clear that if r □ 0, then F_n is given in lemma 6 of Bazyari and Chinipardaz (2012).

Now, we have the following main theorem.

Theorem 2. For the real number t□ depending on the significance level □,

□□ supsup $P_{\mu,□}(T$ □ $t_□)$ □ $P_{0,Ip}(T^*$ □ $t_□)$.

□ H_0

Proof. It is completely clear that $\bar{T}$ □T^*. Then by lemma 2, we get that

□□ supsup $P_{\mu,□}(T$ □ $t_□)$□ sup $P_{0,□}(\bar{T}$ □ $t_□)$

□ H^0 □

(9)

□ sup $P0,□(T^*$ □ $t□)$ □ $P0,Ip\ (T^*$ □ $t□)$.

□

On the other hand, we show that

□□ supsup $P_{\mu,□}(T$ □ $t_□)$□ $P_{0,I\,p}(T^*$ □ $t_□)$.

□ H_0

Using the lemmas 7 and 8 given in Bazyari and Chinipardaz (2012), it is easy to show that

$P0,□n\ (\bar{T}$ □ $t□)$ □ $P0,Ip$ □|| **X** □□S (**X**,c0) ||2S □,

where □$_n$ □ (F_n□F_n)$^{□1}$. Also

lim$P0,□n\ (\bar{T}$ □ $t□)$ □ lim□□ $P0,Ip$ □|| **X** □□S (**X**,c0) ||2S □ t□□ n□□ n

$$\square\ P_{0,I_p}\,\square\|\square_S(\mathbf{X},c_2)\ \square\ \mathbf{X}\|^2_S\ \square\ t_\square\square$$

$$\square\ P_{0,I_p}\,\square T^*\,\square\|\square_S(\mathbf{X},c_0)\ \square\ \mathbf{X}\|^2_S\,\square t_\square\ \square\ t_\square\square$$

$$\square\ P_{0,I_p}(T^*\ \square\ t_\square),$$

since $\|\square_S(\mathbf{X},c_0)\ \square\ \mathbf{X}\|^2_S\ \square\ 0$. So that

$$\square\square\ \sup_{\square}\sup_{H_0} P_{\boldsymbol{\mu},\square}(T\ \square\ t_\square)\ \square\ \sup_{\square} \bar{P}_{0,\square}(T\ \square\ t_\square) \qquad (10)$$

$$\square\ \lim_{n\square\square} \bar{P}_{0,\square_n}(T\ \square\ t_\square)\ \square\ \bar{P}_{0,I_p}(T\ \square\ t_\square).$$

From (9) and (10) the proof of theorem is complete.

Therefore to compute the critical values of the test statistic it is enough to obtain that of T^* when $\boldsymbol{\mu}=0$ and $\Sigma = I_p$.

The critical values

In this section, the critical values of the test statistic T are estimated by Monte Carlo simulation method. To obtain these values, by theorem 2, we only need to obtain that of T^* when $\boldsymbol{\mu}=0$ and $\Sigma = I_p$. In this simulation, we generate $n=\sum_{i=1}^{k} n_i$ sets of p-variate normal vectors from $N_p(0,I)$ and compute the statistic T^*. This computation is repeated 10000 times to get an estimated upper α point of T^*. We further repeat this process 10 times and compute the average of the 10 estimated upper α point for $\alpha = 0.01, 0.025, 0.05$, $(p=3, k=4, r=1)$, $(p=4, k=5, r=2)$, $(p=5, k=4, r=3)$, and $n_i = 5, 10, 15, 20, 25$, $i=1,2,\ldots,k$, respectively. The estimated critical values are given in Table 2. Also the critical values of test statistic are estimated when the sample sizes are different. The estimated critical values are given in Table 3.

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Table 2. Estimated critical values of test statistic by simulation when the sample sizes are identical

				$n_1 = n_2 = \cdots = n_k$				
	p	*k*	*r*	5	10	15	20	25
0.01	3	4	1	2.734	2.381	1.160	0.742	0.273
	4	5	2	2.916	1.049	0.825	0.535	0.414
	5	4	3	1.250	0.635	0.341	0.251	0.123
0.025	3	4	1	1.687	1.216	0.732	0.418	0.084
	4	5	2	1.662	0.841	0.615	0.416	0.240
	5	4	3	0.631	0.452	0.243	0.142	0.046
0.05	3	4	1	1.120	0.667	0.395	0.223	0.055
	4	5	2	0.547	0.623	0.352	0.335	0.071
	5	4	3	0.346	0.381	0.187	0.065	0.026

Table 3. Estimated critical values of test statistic by simulation when the sample sizes are different

	p	*k*	*r*	$n1$	$n2$	$n3$	$n4$	n_5	Critical value
0.01	3	4	1	8	12	11	18		4.012
				10	14	20	15		3.209
				16	20	12	18		2.544
	4	5	2	17	18	15	14	10	2.112
				22	21	13	20		1.730
				23	21	14	20		1.275
	5	4	3	15	18	16	31		1.015
				23	28	17	21		0.883
				26	19	29	25		0.441
0.025	3	4	1	8	12	11	18		3.725

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				10	14	20	15		2.850
				16	20	12	18		2.152
	4	5	2	17	18	15	14	10	2.006
				22	21	13	20		1.429
				23	21	14	20		0.803

	5	4	3	15	18	16	31		0.425
				23	28	17	21		0.081
				26	19	29	25		0.036
0.05	3	4	1	8	12	11	18		3.452
				10	14	20	15		2.840
				16	20	12	18		2.573
	4	5	2	17	18	15	14	10	1.861
				22	21	13	20		1.200
				23	21	14	20		0.723
	5	4	3	15	18	16	31		0.395
				23	28	17	21		0.074
				26	19	29	25		0.024

An example

The problem we are considering comes from Dietz (1989). Vinylidene fluoride is suspected of causing liver damage. An experiment was carried out to evaluate its effects. Four groups of 10 male Fischer-344 rats received, by inhalation exposure, one of several dosages of vinylidene fluoride. Among the response variables measured on the rats were three serum enzymes: SDH, SGPOT, and SGPT. It is known in the scientific considerations that the response level of the enzyme SDH would not be affected by the dosage levels of vinylidene fluoride and the responses of the other two enzymes would be affected monotonically. The data are given in Table 4. Let $\mathbf{X}_{ij} = (X_{ij1}, X_{ij2}, X_{ij3})'$ denote the observations on the three enzymes for j th subject

$(j = 1,\ldots,10)$ in treatment $i (i = 1,\ldots,4)$. Let μ_{ik} denote the mean response for i^{th} treatment (i.e. dose) and k^{th} variable and let $\boldsymbol{\mu}_i = (\mu_{1i}, \mu_{2i}, \mu_{3i})'$ for $i = 1,\ldots,4$.

Suppose that we define $\boldsymbol{\mu}_{i(1)} = \mu_{1i}$ and $\boldsymbol{\mu}_{i(2)} = (\mu_{2i}, \mu_{3i})'$. Now, one formulation of the null and alternative hypothesis is

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$$H_0 : \boldsymbol{\mu}_1(1) \leq \boldsymbol{\mu}_2(1) \leq \cdots \leq \boldsymbol{\mu}_4(1), \boldsymbol{\mu}_1(2) \leq \boldsymbol{\mu}_2(2) \leq \cdots \leq \boldsymbol{\mu}_4(2),$$

against all alternative hypotheses on the four mean vectors for significance level

$\alpha = 0.05$.

Table 4. Serum enzyme levels in rats

Dosage			1	2	3	4	5	6	7	8	9	10
			18	27	16	21	26	22	17	27	26	27
			Rat within dosage Enzyme									
0	SDH	SGPOT		101	103	90	98	101	92	123	105	92
	88	SGPT	65	67	52	58	64	60	66	63	68	56
1500	SDH	25	21	24	19	21	22	20	25	24	27	
	SGPOT		113	99	102	144	109	135	100	95	89	98
	SGPT	65	63	70	73	67	66	58	53	58	65	
5000	SDH	22	21	22	30	25	21	29	22	24	21	
	SGPOT		88	95	104	92	103	96	100	122	102	107
	SGPT	54	56	71	59	61	57	61	59	63	61	
15000	SDH	31	26	28	24	33	23	27	24	28	29	
	SGPOT		104	123	105	98	167	111	130	93	99	99
	SGPT	57	61	54	56	45	49	57	51	51	48	

From the data, we have

$$\bar{\mathbf{X}} = \begin{pmatrix} 22.7 & 22.8 & 27.3 & 27.3 \\ 99.3 & 108.4 & 100.9 & 112.9 \\ 61.9 & 63.8 & 60.2 & 52.9 \end{pmatrix}.$$

Then by iterative algorithm to compute multivariate isotonic regression given by Sasabuchi et al. (1992), under the closed convex cone c_0 the estimate of $\boldsymbol{\mu}$ is

$$\hat{\boldsymbol{\mu}} = \begin{pmatrix} 26.75 & 26.75 & 26.75 & 26.75 \\ 99.3 & 102.4 & 108.8 & 114.2 \end{pmatrix},$$

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61.90 65.3 66.03 68.1

and under the closed convex cone c_2 the estimate of $\boldsymbol{\mu}$ is

26.75 26.75 26.75 26.75 $\hat{\hat{\boldsymbol{\mu}}}$ 122 129.7 129.7 140.2.

65.33 65.33 65.33 65.33

So that

4.05 3.95 0.55 0.55

$\mathbf{X} - \hat{\boldsymbol{\mu}}$ 0 6 7.9 1.3,

1.5 5.83

15.2

0

and

4.05 3.95 0.55 0.55

$\mathbf{X} - \hat{\hat{\boldsymbol{\mu}}}$ 22.7 21.3 28.8 27.3.

1.53 5.13

12.43

3.43

The sample mean variance covariance matrix and its inverse are

3.80 1.021

47.98 10007.8 93.347, 1.888551e-004

S 3.80 93.347 109.66 Also the value of test statistic T^* is

1.021 4

and 6.153974e-006 0.00018885506 1.007236e-004 -0.00008568268.

2.084652e-002 S^{-1} -8.568268e-005 0.00919379028

6.153974e-006

4

$$T^* = \sum_{i=1}^{10} (\mathbf{X}_i - \hat{\boldsymbol{\mu}}_i)' S^{-1} (\mathbf{X}_i - \hat{\boldsymbol{\mu}}_i) - \sum_{i=1}^{10} (\mathbf{X}_i - \hat{\hat{\boldsymbol{\mu}}}_i)' S^{-1} (\mathbf{X}_i - \hat{\hat{\boldsymbol{\mu}}}_i) = 4.887.$$

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Since at significance level □□ 0.05, T^* □ 0.667, therefore we reject the null hypothesis.

CONCLUDING REMARKS

Bazyari and Chinipardaz (2012) considered the problem of testing order restriction of mean vectors against all possible alternatives based on a sample from several p□ dimensional normal distributions. They obtained a test statistic and also presented Monte Carlo simulation to estimate its critical values. In this article, the general form for this problem of testing is considered. In fact, this paper did numerical study based on the claim that the tail probability of a proposed test statistic T for testing order restricted null hypothesis can be simplified by another simpler statistic T^*. We proposed a test statistic by likelihood ratio method based on orthogonal projections on the closed convex cones. Monte Carlo simulation is used to obtain the critical values of test statistic. We also applied this test to a real example where this hypothesis problem arises to evaluate the effect of Vinylidene fluoride on liver damage. For computing the test statistic in numerical example the estimation of unknown parameter vector is done by the iterative algorithm proposed by Sasabuchi et al. (1983).

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