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TWO STAGE IMPLICIT HYBRID R-K SCHEME FOR TREATMENT OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: *In this paper, family of two-stage hybrid Runge-Kutta schemes were developed, analyzed and computerized to solve second order ordinary differential equations. Their development and analysis make use of Taylor and Binomial series expansion, Dalhquist stability model test equation and Pade's approximation techniques respectively. The theoretical results shown that the schemes are Consistent, Convergent, A-stable and. A(α) stable with large interval of absolute stability (-∞,0).*

Keywords: *A-stable, Accurate, Semi-Implicit, Hybrid*

1.0 Introduction

There are many processes in the field of Sciences, Management and Engineering which involve rate of change of one or more quantities (dependent variable) in relation to another, which may result to general Ordinary Differential Equations (ODEs) or Partial Differential Equations (PDEs). A differential equation is called ODEs if the dependent variable is a function of a single independent variable otherwise it is a PDEs .

The most general form of ODEs to be considered is of the form

$$y'' = f(x, y, y') = 0 \tag{1}$$

with initial condition

$$y'(x_o) = y_o^1, \; y(x_o) = y_o \tag{2}$$

The second order initial value problem (1) and (2) can be reduced to system of two first order equation of the form

$$y_1 = y, \quad y_1(x_o) = \eta_o$$

$$y'' = y_2 = y_1' = f(x, y, y'), \quad y_2(x_o) = \eta_1 \tag{3}$$

which in vector form can be written as

$$y_2 = \frac{dy_1}{dx} = f(x, y, y') \quad (4)$$

In an attempt to solve (4) it will be assumed that $f(x, y, y')$ satisfied the following conditions.

(i) $f(x, y, y')$ is a real valued vector

(ii) $f(x, y)$ is defined and continuous in the region D of x – y plane as defined by

$$D = (x, y) / a \le x \le b, \quad -\infty < y < \infty) \quad (5)$$

(iii) There exist a real constant L such that for any $x \in$ (a,b) and pairs of number (x, y_1) and (x, y_2) in D $\left| f(x, y_1) - f(x, y_2) \right\| \le L \| y_1 - y_2 \|$

Where L is the Lipschitz constant of order one with L = Sup $\left\| \partial f / \partial y \right\|$ $j = \partial f / \partial y$ is the Jacobian of f . D with respect to y. That is, 2 x 2 matrix whose i -j[th] element is $\partial f_2 / \partial y_2$ (x, y_1, y_2) and $\mathrm{II} \cdot \mathrm{II}$ denote a matrix norm of vector norm employed in (6) as defined in Mitchell (1969).

Hong Yuanfu (1982) proposed a general form of Runge-Kutta called Rational Runge Kutta of the form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \quad (7)$$

where,

$$K_i = hf\left(x_n + C_1 h, y_n + \sum_{j=1}^{i} a_{ij} K_j \right) \quad (8)$$

$$H_i = hg\left(x_n + d_i h, \ z_n + \sum_{j=1}^{i} b_{ij} H_j \right)$$

with

$$g(x_n,\ z_n) = -z_n^2 f(x_n y_n)$$
$$and\ \ z_n = 1/y_n \tag{9}$$

In his development he considered explicit family of method, that is the case $a_{ij} = b_{ij} = 0$, for $j \geq 1$ and develop families of orders one, two and three of the schemes.

During analysis of these schemes he discovered that the schemes are A-stable. This properties of the scheme stimulated Okunbor (1985) to extend the scheme to family of order four. This performance on stiff ODEs is satisfactory but their performance on stiff oscillatory problem is not encouraging. This prompted Babatola (1999) to consider the Implicit family of the method of which incidentally performed well.

Although the method is suitable, accurate and stable, but it is bedeviled by some computational difficulties which makes it more cumbersome.

These difficulties include large amount of function evaluation, requirement of large amount of computer facilities, Difficult computer manipulation and large amount of computational efforts. This method which we are considering in this work hopes to reduce the problem to the bearest minimum by reducing the number of function evaluation. The method is of the form

$$y_{n+1} = \frac{y_n}{1 + y_n \sum_{i=1}^{R} V_i H_i} \tag{10}$$

where,

$$H_i = hg\left(x_n + d_i h,\ z_n + \sum_{j=1}^{R} b_{ij} H_i \right) \tag{11}$$

and

$$g(x_n\ z_n) = -Z_n^2 f(x_n, y_n)$$

While V_i, b_{ij} and d_i are parameters to be determined. This modification (10) is called R- Stage Hybrid Runge-Kutta method.

Adopting Fatunla (1987) and Ademiluyi and Babatola (2001) approach, the above formular can be classified into Explicit, Semi-Implicit and Implicit. In this work, we shall consider Two-stage, Implicit Hybrid Runge Kutta for treatment of Second Order ODEs.

The numerical values of these coefficients can be obtained from the set of non-linear equations generated by adopting the following steps.

Step 1: Obtain the Taylor series of H_i's about point (x_n, y_n) for i = 1(1)r

Step 2: Insert the series expansion to (10)

Step 3: Compare the final expansion with the Taylor series expansion of y_{n+1}

about $(x_{n,}y_n)$ in the powers of $0h^{n+1}$.

The number of parameters normally exceeds the number of equations but in the spirit of King (1966), Gill (1951) and Blum (1952), these parameters are chosen to ensure that one or more of the following condition are satisfied.

i. Adequate order of the accuracy of the scheme is achieved.

ii. Minimum bound of local truncation exist

iii. The method has maximize interval of absolute stability.

iv. Minimum computer storage facilities are utilized.

Development of the Scheme

Now setting R = 2 in equation (5) then Two-stage Implicit Hybrid R – K scheme is of the form

$$y_{n+1} = \frac{y_n}{1 + y_n\left(V_1 H_1 + V_2 H_2\right)} \qquad (12)$$

where,

$$H_i = hg\left(x_n + d_1 h,\ z_n + \sum_{j=1}^{2} b_{ij} H_j\right) \quad ,i=1,2 \qquad (13)$$

with

$$g\left(x_n\ z_n\right) = -z_n^2 f(x_n, y_n) \text{ and } z_n = 1/y_n$$

Adopting Binomial expansion theorem on the right side of equation (12) and ignoring higher terms we obtained

$$y_{n+1} = y_n - y_n^2 \sum_{i=1}^{2} V_i H_i \qquad (14)$$

with

$$\mathrm{H_i} = hg\left(x_n + d_i h,\ \mathrm{z_n} + \sum_{i=1}^{2} b_{ij} H_j \right), i = (1)2 \qquad (15)$$

with constraints

$$d_i = \sum_{j=1}^{2} b_{ij} \qquad (16)$$

Expanding H_i's i = 1 (1)2 in equation (13) about (x_n z_n), we obtain

$$H_i = hN_i + h^2M_i + h^3R_i + h^4L_i + 0h^6, i=1,2 \qquad (17)$$

Where

$N_i = g\ (x_n, z_n) = g_n$

$M_I = d_i\, gx + (b_{i1}\, N_i + b_{i2}\, N_2) = d_i\, Dg_n$

$$\begin{aligned}
R_i &= (b_{i1}M_1 + b_{i2}M_2)g_z + \tfrac{1}{2} d_i^2 g_{xx} + d_i(b_{i1}N_1 + b_{i2}N_2)g_{xz} \\
&+ \tfrac{1}{2}(b_{i1}N_1 + b_{i2}N_2)^2 g_{xz} = (b_{i1}N_1 + b_{i2}N_2)g_z Dg_n + \tfrac{1}{2} d_i D^2 g_n \\
L_i &= (b_{i1}R_1 + b_{i2}R_2)g_Z + (b_{i1}M_1 + b_{i2}M_2)g_{xz} \\
&+ (b_{i1}N_1 + b_{i2}N_2)(b_{i1}M_1 + b_{i2}M_2))g_{zz} + \tfrac{1}{6} d_1^2 g_{xxx} \\
&+ \tfrac{1}{2} d_1^2 (b_{i1}N_1 + b_{i2}N_2)g_{xxz} + \tfrac{1}{2} d_i (b_{i1}N_1 + b_{i2}N_2)^2 g_{xzz} + \\
&\tfrac{1}{2} d_i (b_{i1}N_1 + b_{i2}N_2)g_{xzz} + \tfrac{1}{6}(b_{i1}N_1 + b_{i2}N_2)g_{xzz} + \tfrac{1}{2} d_1^2 (b_{i1}N_1 + b_{i2}N_2)g_{xxz} \\
&+ \tfrac{1}{2} d_i (b_{i1}N_1 + b_{i2}N_2)g_{xzz} + \tfrac{1}{6}(b_{i1}N_1 + b_{i2}N_2)^3 g_{zzz} \\
&= b_{i1}(b_{i1}d_1 + b_{i2}d_2) + b_{i2}(b_{i1}d_1 + b_{22}d_2))g_2^2 Dg_n + (d_i(b_{ij}d_i + b_{i2}d_2)Dg_n Dg_z \\
&+ \tfrac{1}{2}(b_{i1}d_1^2 + b_{i2}d_2^2)g_z D^2 g_n + \tfrac{1}{6} d_1^3 D^2 g_n, \quad \mathrm{i} = (1)2
\end{aligned} \qquad (18)$$

where

$$Dg_n = g_x + g_n g_z$$

$$D^2 g_n = g_{xX} + 2g_n g_{Xz} + g_n^2 g_{zz}$$

$$D^3 g_n = g_{xXx} + 3g_n g_{xXz} + 3g_n^2 g_{xzz} + g^3 g_{zzz}$$

$$Dg_z = g_{xz} + g_z^2 + g_n g_{xz} \qquad (19)$$

Using equation

Thus expressing g and its partial derivatives in terms of f and its partial derivatives. To facilitates the comparison of coefficients

That is,

$$g_n = \frac{-f_n}{y_n^2},\ \mathrm{g}_{\mathrm{x}} = \frac{-f_x}{y_n^2},\ \mathrm{g}_{\mathrm{xx}} = \frac{-f_{xx}}{y_n^2}$$

$$g_{xxx} = \frac{-f_{xxx}}{y_n^2}$$

$$\mathrm{g}_{\mathrm{z}} = \frac{-2f_n}{y_n} + f_y,\quad \mathrm{g}_{\mathrm{xz}} = \frac{-2f_x}{y_n} + f_{xy}$$

$$\mathrm{g}_{\mathrm{xxz}} = \frac{-2f_{xx}}{y_n} + f_{xxy,}\quad \mathrm{g}_{\mathrm{zz}} = -2\mathrm{f}_{\mathrm{n}} - \mathrm{y}_{\mathrm{n}}^2\mathrm{f}_{\mathrm{yy}}$$

$$\mathrm{g}_{\mathrm{xzz}} = -2yf_x - y_n^2 f_{xyy}$$

$$g_{zzz} = 4y_n^2 f_y + 6y_n^2 f_{yy} + y_n^4 f_{yyy} \qquad (20)$$

Using equation (20) in (18), we get

$$\mathrm{N}_1 = \frac{-f_n}{y_n^2},\ \mathrm{M}_1 = \frac{-\mathrm{d}_1}{\mathrm{y}_{\mathrm{n}}^2}\left(Df_n + \frac{2\mathrm{f}_{\mathrm{n}}^2}{\mathrm{y}_{\mathrm{n}}^2}\right)$$

$$R_i = \frac{1}{y_n^2}(((b_{i1}d_1 + b_{i2}d_2)\left(\frac{-2f_n}{y_n} + f_y\right)\Big[Df_n + 2\frac{f_n^2}{y_n}\Big) + \tfrac{1}{2}\left(d_1^2\left(D^2 f_n + \frac{2f_n}{y_n}\left(f_x + \frac{2f_n}{y_n}\right)\right)\right)$$

$$L_i = \frac{-1}{y_n^2}\big[b_{11}(b_{i1}d_1 + b_{i2}d_2) + b_{i2}(b_{21} + d_1 + b_{22}d_2)\left(\frac{-2f_n}{y_n} + f_y\right)\left(Df_n + \frac{2f_n^2}{y_n}\right) + d_i(b_{i1}d_1 + b_{i2}d_2)$$

$$\left(\frac{-2f_n}{y_n} + f_y\right)\left(Df_n + \frac{2f_n^2}{y_n}\right) + d_i(b_{i1}d_1 + b_{i2}d_2)Df_n + \frac{2f_n^2}{y_n}\left(\frac{-2f_n}{y_n} + f_y\right)\Big) \qquad (\,$$

$$+\tfrac{1}{2}(b_{i1}d_1^2 + b_{i2}d_2^2)\left(D^2 f_n - \frac{2f_n}{y_n}\left(f_x - \frac{f_n^2}{y_n}\right) + \tfrac{1}{6}d_1^3(2f_n^3 + 3f_{yy})\right)$$

Recall from equation (14) (21)

$$y_{n+1} = y_n - y_n^2 (V_1 H_1 + V_2 H_2)$$

$$y_n - y_n^2 \left(V_1 \left(hN_1 + h^2 M_1 + h^3 R_1 + h^4 L_1 + 0h^5\right) + V_2 \left(hN_2 + h^2 M_2 + h^3 R_2 + h^4 L_2 + 0h^5\right)\right)$$

$$y_{n+1} = y_n - (V_1 N_1 + V_2 N_2)h - (V_1 M_1 + V_2 M_2)h^2$$

$$(V_1 R_1 + V_2 R_2)h^3 - (V_1 L_1 + V_2 L_2)h^4 + 0h^5 \qquad (22)$$

Expanding the LHS of equation (12) adopting Taylor series

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{iv} + 0h^5 \qquad (23)$$

$$\begin{aligned} y_n^1 &= f_n, \; y_n'' = Df_n, \; y_n''' = D^2 f_n + f_y Df_n \\ y_n^{iv} &= D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + D^2 f_y Df_n \end{aligned} \qquad (24)$$

With equation (24), equation (23) becomes

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} Df_n + \frac{h^3}{6}\left(D^2 f_n + f_y Df_n\right) + \frac{h^4}{24}(D^3 f_n + f_y D^2 f_n + 3Df_n$$
$$Df_y + D^2 f_y Df_n) + 0h^5 \qquad (25)$$

Comparing the coefficient of h, h^2 and h^3 in equations (22) and (25), where condition

$$T_{n+1} = 0h^4 \qquad (26)$$

is imposed. We obtained the following systems of equations for family of two stage schemes of order three.

$$V_1 + V_2 = 1$$
$$V_1 d_1 + V_2 d_2 = 1/2$$
$$V_1 (b_{11} d_1 + b_{12} d_2) + V_2 (b_{21} d_1 + b_{22} d_2) = 1/6$$
$$V_1 d_1^2 + V_2 d_2^2 = 1/3 \qquad (27)$$

With the constraints

$$b_{11} + b_{12} = d_1$$
$$b_{21} + b_{22} = d_2 \qquad (28)$$

While the associated local Truncation Error term for the family of Two-stage Implicit RR-K scheme of order three can be shown to be

$$T_{n+1} = \frac{h^4}{24}\left[Df_n + f_y D^2 f_n + 3Df_n Df_y + f^2 yDf_n\right) + \tfrac{1}{6}\left[\left(V_1(b_{11}d_1 + b_{12}d_2) + V_2(b_{21}d_1 + b_{22}d_2\right) + \right.$$

$$\left(-\frac{2f_n}{y_n}Df_n - \frac{2f_n^3}{y_n} + \frac{2f_n^2}{y_n}\right) - \tfrac{1}{2}\left(V_1 d_1^2 + V_2 d_2^2\right)\left(\frac{2f_n}{y_n}f_x - f_n\right]\right) \qquad (29)$$

From equation (27) and (28). Family of two stage schemes of order three are obtained as

Case1: With $V_1 = \frac{1}{4}, V_2 = \frac{3}{4}$

$d_1 = b_{12} = 1, b_{11} = b_{21} = 0, d_2 = b_{22} = \frac{1}{3}$,equation (12) yields

$$y_{n+1} = \frac{y_n}{1 + \frac{y_n}{4}(H_1 + 3H_2)} \qquad (30)$$

Where

$$H_1 = hg(x_n + h, \ z_n + H_2)$$
$$H_2 = hg(x_n + \tfrac{1}{3}h, \ z_n + \tfrac{1}{3}H_2) \qquad (31)$$

Case2:With $V_1 = \frac{3}{4}, \ V_2 = \frac{1}{4}$

$b_{11} = b_{22} = \frac{1}{2}, \ b_{12} = \frac{1}{6}, \ b_{21} = -\frac{1}{2}$
$d_2 = 0, \ d_1 = \frac{2}{3}$, equation (12) yields

$$y_{n+1} = \frac{y_n}{1 + \frac{y_n}{4}(3H_2 + H_1)} \qquad (32)$$

Where

$$H_1 = hg(x_n + \tfrac{2}{3}h, \ z_n + \tfrac{1}{2}H_1 + \tfrac{1}{6}H_2)$$
$$H_2 = hg(x_n, z_n - \tfrac{1}{2}H_1 + \tfrac{1}{2}H_2) \qquad (33)$$

Imposing condition

$$T_{n+1} = O(h^5) \qquad (34)$$

Coefficient of h, h^2, h^3 and h^4 into consideration in equations (22) and (25).

We obtain the following equations for two stage family of order four.

$$
\begin{aligned}
&V_1 + V_2 = 1 \\
&V_1 d_1 + V_2 d_2 = 1/2 \\
&V_1 d_1^2 + V_2 d_2^2 = 1/3 \\
&V_1 d_1^3 + V_2 d_2^3 = 1/4
\end{aligned}
$$

$$V_1(b_{11}d_1 + b_{12}d_2) + V_2(b_{21}d_1 + b_{22}d_2) = 1/6$$
$$V_1 d_1(b_{11}d_1 + b_{12}d_2) + V_2 d_2(b_{21}d_1 + b_{22}d_2) = 1/4$$

$$V_1(b_{11}d_{1}^2 + b_{12}d_2^2) + V_2(b_{21}d_1^2 + b_{22}d_2^2) = 1/2$$

$$\begin{aligned}&V_1(b_{11}(b_{11}d_1 + b_{12}d_2) + b_{12}(b_{21}d_1 + b_{22}d_2) + \\ &V_2(b_{21}(b_{11}d_1 + b_{12}d_2) + b_{22}(b_{21}d_1 + b_{22}d_2)\} = 1/24\end{aligned} \quad (35)$$

with the constraints

$$\begin{aligned}d_1 &= b_{11} + b_{12} \\ d_2 &= b_{21} + b_{22}\end{aligned} \quad (36)$$

Examples of family of two-stage scheme or order four are obtained by setting

$$
\begin{aligned}
&V_1 = V_2 = 1/2,\ d_2 = \left(1/2 - \sqrt{3}/6\right), d_1 = (1/2 + \sqrt{\tfrac{3}{6}}) \\
&b_{11} = b_{22} = 1/4,\ b_{12} = 1/4 + \sqrt{3}/6 \\
&a_{21} = b_{21} = 1/4 - \sqrt{3}/6,
\end{aligned}
$$

Equation (12) becomes

$$y_{n+1} = \frac{y_n}{1 + \frac{y_n}{2}(H_1 + H_2)} \quad (37)$$

where

$$H_1 = hg\left(x_n + \left(1/ + \sqrt{3}/6\right)h,\ Z_n + 1/4 H_1 + \left(1/4 + \sqrt{3}/6\right)H_2\right)$$
$$H_2 = hg\left(x_n + \left(1/ + \sqrt{3}/6\right)h,\ Z_n + \left(1/4 - \sqrt{3}/6\right)H_1 + 1/4 H_2\right)$$

4.0 **Analysis of Basic Properties**

The characteristics of the basic properties of numerical schemes include errors Consistency, Convergence and Stability.

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4.1 **Error Analysis**

Errors of numerical approximation techniques of ODEs arise from different causes that can be majorly classified into discretization, truncation and Round Off errors. Other sources of errors include inherent errors often called data or model simplification error and personal error.

Round-off error is is an error introduced as a result of the computing devices.

Mathematically it can be expressed as

$$r_{n+1} = y_{n+1} - \mathrm{P}_{n+1} \tag{38}$$

Where y_{n+1} is the expected solution of the difference equation while P_{n+1} is the computer output at amount by which the computed approximation P_{n+1} differ from the expected approximation y_{n+1} of the schemes (12) at point $x_{n+1.}$

Truncation error is the error introduced as a result of ignoring some of the higher terms of the power series (Taylor and Binomial series expansion) during the development of the new scheme.

Truncation error of scheme (12) can be defined as

$$T_{n+1} = y(x_{n+1}) - \frac{y(x_n)}{1 + y\ \ (x_n)\sum_{i=1}^{2} V_i H_i} \tag{39}$$

$$H_i = hg(x_n + d_i h, z_n + \sum_{j=1}^{2} b_{ij} H_j), \mathrm{i} = (1)2 \tag{40}$$

Discretization error is the error introduced as a result of transforming a differential equation (1) into difference equation (12). Mathematically the discretization error ℓ_{n+1} associated formula (12) is the difference between the exact solution $y(x_{n+1})$ and the numerical solution y_{n+1} generated by (12) at point $x_{n+1.}$ That is

$$e_{n+1} = y_{n+1} - y(x_{n+1}) \tag{41}$$

4.2 **Consistency**

A scheme is said to be consistent if the difference equation of the computation formulas exactly approximate the differential equation it intend to solve (Ademiluyi and Babatola (2001)).
To prove that the scheme is consistent. Recall that equation (12)

$$y_{n+1} = \frac{y_n}{1 + y_n \sum_{i=1}^{2} V_i H_i} \tag{42}$$

Subtracting y_n on both sides of equation (42), we get

$$y_{n+1} - y_n = \frac{y_n}{1 + y_n \sum_{i=1}^{2} V_i H_i} - y_n \tag{43}$$

Simplifying further yields

$$y_{n+1} - y_n = \frac{y_n{}^2 \sum_{i=1}^{2} V_i H_i}{1 + y_n \sum_{i=1}^{2} V_i H_i} \tag{44}$$

But

$$H_i = hg(x_n + d_i h,\ \mathrm{z}_n + \sum_{j=1}^{2} b_{ij} H_j) \tag{45}$$

Hence

$$y_{n+1} - y_n = \frac{-y_n^2 h \sum \mathrm{V_i}\, g\left(x_n + d_i h,\ \mathrm{z_n} + \sum_{i=1}^{2} b_{ij} H_j \right)}{1 + y_n \sum \mathrm{V_i}\, hg\left(x_n + d_i h,\ \mathrm{z_n} + \sum_{i=1}^{2} b_{ij} H_j \right)} \tag{46}$$

Dividing throughout by h and taking the limit as h tends to zero

$$\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = y_n^2 g(x_n, \mathrm{z_n}) \tag{47}$$

But $g(x_n, \mathrm{z_n}) = \frac{-1}{y_n^2} f(x_n, \mathrm{z_n})$

$$\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = -y_n^2 \left(- \frac{1}{y_n^2} f(x_n, \mathrm{y_n} \right) \tag{48}$$

$$y'(x_n) = f\left(x_n, \mathrm{y}_\mathrm{n}\right)$$

Hence the scheme is consistent.

4. 3 **Convergence**

A numerical scheme such as equation (10) is said to be convergence, if when applied to initial value problem (1), it generate a corresponding approximation y_n which tends to the exact solution $y(x_n)$ as n approaches infinity.

To show the convergence of the two-stage Implicit Hybrid R-K scheme.

Recall that

$$y_{n+1} = \frac{y_n}{1 + y_n \sum_{i=1}^{2} V_i H_i(y_n)} \qquad (49)$$

While the exact solution $y(x_{n+1})$ is seen to satisfy the difference equation of the form

$$y(x_{n+1)} = \frac{y(x_n)}{1 + y(x_n) \sum_{i=1}^{2} V_i H_i(y(x_n)\Big)} + T_{n+1} \qquad (50)$$

Subtracting equation (49) from (50), we have

$$y(x_{n+1}) - y_{n+1} = \frac{y(x_n)}{1 + y \sum_{i=1}^{2} V_i H_i y(x_n)} - \frac{y_n}{1 + y_n \sum_{i=1}^{2} \mathrm{V}_\mathrm{i} H_i(y_n)} \qquad (51)$$

$$e_{n+1} = \frac{y(x_n)\left[y_n \sum_{i=1}^{2} V_i H_i(y_n) - y_n \left[\sum_{i=1}^{2} V_i H_i y(x_n) \right] \right.}{\left[1 + y(x_n) \sum_{i=1}^{2} V_i H_i (y \sum_{i=1}^{2} V_i H_i y(x_n)) \right]} \qquad (52)$$

$$e_{n+1} = e_n \frac{\left[1 - y(x_n) y_n \sum_{i=1}^{2} V_i \frac{\partial H_i}{\partial y} \right]}{1 + y(x_n) \sum_{i=1}^{2} V_i H_i(y(x_n)) \left[1 + y_n (\sum_{i=1}^{2} V_i H_i(y_n) \right]} + T_{n+1} \qquad (53)$$

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Setting

$$P_n = 1 + y_n(x_n)\sum V_i H_i(y_n(x))$$

$$Q_n = 1 + y_n y(x_n)\sum_{i=1}^{2} V_i H_i(y)$$

$$R_n = \left[1 + y_n y(x_n)\sum_{i=1}^{2} V_i H_i \frac{\partial H_i}{\partial y}\right.$$

Then equation (53) becomes

$$e_{n+1} = \frac{R_n}{P_n Q_n} e_n + T_{n+1.} \qquad (54)$$

Let P = $\max_{0\le\infty}(P_n)$, $\mathrm{Q} = \max_{0\le n\le\infty}(Q_n)$

R = $\max_{0\le n\le\infty}(R_n)$

Then

$$e_{n+1} = \frac{\mathrm{R}}{\mathrm{PQ}} e_n + T_{n+1} \qquad (55)$$

Set $\frac{R}{PQ}$ = K and

T = $\max_{0\le n\le\infty} \mathrm{T}_{n+1}$

E_n = $\max_{0\le n\le\infty} e_n$

Then

$$E_{n+1} = KE_n + T \qquad (56)$$

$$E_1 \le KE_o + T \qquad (56a)$$

$$E_2 \le KE_1 + T \qquad (56b)$$

Substituting equation (56a) into equation (56b)

$$E_2 \leq K\,(KE_o + T) + T$$
$$K^2E_o + KT + T$$

$$E_3 \leq KE_2 + \mathrm{T} \qquad (57)$$

$$K^3E_o + K^2T + \mathrm{T}$$

Therefore

$$E_{n+1} = K^{n+1}E_o + \sum_{i=1}^{2} K^{m}T + T \qquad (58)$$

Since $\frac{R}{PQ}$ = K < 1 (59)

It is easy to see that as n $\rightarrow \infty$, E$\rightarrow$ 0. This proves that this particular case, the schemes converges.

4.4 Stability properties

Since a Consistent and Convergent one-step scheme is stable, the scheme is stable. However, to ensure that the scheme is A-stable and P-stable and able to solve initial value problem. It is adopted for solution of the A-stability test model equation

$$y' = \lambda y,\ \mathrm{y(x_o)} = y_o \qquad (60)$$

Applying the scheme for the numerical solution of equation (60) we obtain a recurrent equation:

$$y_{n+1} = P(z)\, y_n$$

where

$$P(z) = \frac{1 - 1\tfrac{1}{2}z - \tfrac{1}{12}z^2}{1 - \tfrac{1}{2}z + \tfrac{1}{12}z^2} \qquad (61)$$

This scheme is A-stable since interval of absolute stable is (-∞, 0) and also P-stable since the region of absolute stability is (-∞, -∞).

5.0 <u>Numerical Experiment</u>

In order to confirm the applicability and suitability of the scheme for solution of Second order ODEs, the scheme was computerized.

Problem 1: In another development, the approximate solution of the system of 2nd order ODEs initial value problem of the form

(i) $$x' = 5x + 3y$$ $$y' = 3x \qquad (62)$$

with initial condition x(o) = 1, y(o)=2 was attempted. Table 1 below show the numerical results of the problem (62).

Problem 2: Consider the Second order ODEs of the form

$$x'' + 8x' + 16x = 0 \qquad (63)$$

with initial condition x(o) =1, $x'(o) = 3.$

With exact solution

$$x(t) = (1 - t)e^{-4t} \qquad (64)$$

Table (2) below shows the numerical results of the problem.

(63)

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TABLE 1: **NUMERICAL RESULT OF HYBRID RUNGE-KUTTA FOR SOLVING PROBLEM (52)**

T	**X**	Y	**E_1**	**E_2**
0.3000000000D-02	0.1007999657D-01	0.1998001998D+01	0.3238292541D-04	0.000000000D+00
0.9299997000D-01	0.1247991412D+01	0.1938062889D+01	0.2605580892D-04	0.000000000D+00
0.1830001000D+00	0.1487985833D+01	0.1878125689D+01	0.2179681180D-04	0.000000000D+00
0.30300010000D+00	0.1807980698D+01	0.1798183968D+01	0.1789630619D-04	0.1109145273D-05
0.4229997000D+00	0.2127977105D+01	0.1718184042D+01	0.1517983194D-04	0.1160637680D-05
0.51299960000D+01	0.2367975046D+01	0.1658184102D+01	0.1362833113D-04	0.1202507756D-05
0.60300020000D+00	0.2607973363D+01	0.1598184167D+01	0.1236456325D-04	0.1247511805D-05
0.6930009000D+00	0.2847971962D+01	0.1538184236D+01	0.1131528043D-04	0.1296015373D-05
0.7830015000D+00	0.3087970776D+01	0.1478184310D+01	0.1043015177D-04	0.1348443152D-05
0.8730021000D+00	0.3327969759D+01	0.1418184390D+01	0.9673451145D-05	0.1405291455D-05
0.9630027000D+00	0.3567968877D+01	0.1358184477D+01	0.9019118196D-05	0.1467143966D-05

TABLE 2: NUMERICAL RESULTS OF HYBRID RUNGE-KUTTA METHOD FOR SOLVING PROBLEM (63) OF DIFFERENTIAL EQUATIONS

T	STEP SIZE	X1	X2	E1	E2
0.3000000000D-02	0.300000000D-03	0.100900000002D+01	0.2984000304D+01	0.3982218057D-04	0.4232473964D-04
0.1815003000D+00	0.3000000000D-03	0.1544493921D+01	0.2032024089D+01	0.2626040814D-06	0.6286750724D-06
0.2115008000D+00	0.3000000000D-03	0.1634493893D+01	0.1872024177D+01	0.2481244317D-06	0.6822924885D-06
0.2445013000D+00	0.3000000000D-03	0.1733493864D+01	0.1696024289D+01	0.2339356311D-06	0.7529284456D-06
0.2745006000D+00	0.3000000000D-03	0.1823493840D+01	0.1536024409D+01	0.2223753053D-06	0.8311530348D-06
0.3044997000D+00	0.3000000000D-03	0.1913493817D+01	0.1376024554D+01	0.2119037228D-06	0.9275161248D-06
0.3344987000D+00	0.3000000000D-03	0.2003493796D+01	0.1216024732D+01	0.2023739949D-06	0.1049153894D-05
0.3644977000D+00	0.3000000000D-03	0.2093493775D+01	0.1056024960D+01	0.1936645195D-06	0.1207510929D-05
0.3944967000D+00	0.3000000000D-03	0.2183493756D+01	0.8960252635D+00	0.1856737635D-06	0.142216940D-05
0.4244957000D+00	0.3000000000D-03	0.2273493737D+01	0.7360256920D+00	0.1783162875D-06	0.1729647033D-05
0.4544947000D+00	0.3000000000D-03	0.2363493719D+01	0.5760263494D+00	0.1715196793D-06	0.2206749465D-05
0.4844938000D+00	0.3000000000D-03	0.24534937022D+01	0.4160274989D+00	0.1652221591D-06	0.3101922545D-05
0.5144928000D+00	0.3000000000D-03	0.2543493685D+01	0.2560276172D+00	0.1593707006D-06	0.5024412474D-05
0.5444918000D+00	0.3000000000D-03	0.2633493669D+01	0.9602769530D-01	0.1539195331D-06	0.1352023568D-04

Discussion

From the above Table of Results, we can see that the hybrid Implicit R-K is very effective and accurate in solving ODEs of any order. Provided it can be transformed into first order system of equations.

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