International Journal of Mathematics and Statistics Studies

Vol.1, No.3, pp.9-19, September 2013

Published by European Centre for Research Training and Development UK (www.ea-journals.org)

THE DETERMINATION OF PARADOXICAL PAIRS IN A LINEAR TRANSPORTATION PROBLEM

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ABSTRACT: *The transportation paradox is related to the classical transportation problem. For particular reasons of this problem, an increase in the quantity of goods to be transported may lead to a decrease in the optimal total transportation cost. In this paper, an efficient algorithm for solving a linear programming problem was discussed, and it was concluded that paradox exists. The North-West Corner method was used to obtain the optimal solution using the TORA Statistical Software Package. The method however gives a step by step development of the solution procedure for finding all the paradoxical pair*.

KEYWORDS: Transportation Paradox; Transportation Problem; Paradoxical Range of Flow; Linear Programming

INTRODUCTION

The term Paradox arises when a transportation problem admits of a total cost which is lower than the optimum and is attainable by shipping larger quantities of goods over the same routes that were previously designated as optimal. This unusual phenomenon however, was noted by Szwarc (1971). The classical transportation problem is the name of a mathematical model which has a special mathematical structure. The mathematical formulation of a large number of problems conforms to this special structure. Hitchcock (1941) originally developed the basic linear transportation problem. Charnes et al (1953) developed the stepping stone methods which provide an alternative way of determining the simplex method information. Dantzig (1963) used the simplex method to the transportation problem as the primal simplex transportation method.

Appa (1973) also developed the solution procedure for solving the transportation problem and its variants. Klingman and Russel (1974 and 1975) introduced a specialized method for solving a transportation problem with several additional linear constraints. Hadley (1987) gave the detailed solution procedure for solving linear transportation problem. Till date, several researchers studied extensively to solve cost minimizing transportation problem in various ways.

In some situations, if we obtain more flow with lesser cost than the flow corresponding to the optimum cost then we say paradox occurs. Charnes and Klingman (1971), Szwarc (1973), Adlakha and Kowalski (1998) and Storoy (2007) considered the paradoxical transportation problem. Gupta et al (1993) considered a paradox in linear fractional transportation problem with mixed constraints. Joshi and Gupta (2010) studied paradox in linear plus fractional transportation problem. In the early day of linear programming problem some of the pioneers observed paradox but by whom no one knows.

In this paper we present a method for solving transportation problem with linear constraints. Thereby, we state the sufficient condition of existence of paradox, paradoxical range of flow and paradoxical flow for a specified flow in such type of linear transportation problem. Having known that paradox does not exit regularly in so many linear transportation problems, the rationale behind this research work is to unveil a numerical practical example that will suit the algorithm discussed in this paper. We also justify the theory by illustrating a numerical example, and at the same time reviewing past researches.

The purpose of this research paper is to obtain the best paradoxical pair from the optimal basic feasible solution of the transportation problem, and at the same time, to obtain the paradoxical range of flow.

DEFINITIONS OF TERMS

1. Paradox in a transportation problem: in a transportation problem if we can obtain more flow (F^1) with lesser cost (Z^1) than the optimum flow (F^0) corresponding to the optimum cost (Z^0) i.e. $F^1 > F^0$ and $Z^1 < Z^0$, then we say that a paradox occurs in a transportation problem.
2. Cost-flow pair: if the value of the objective function is Z^0 and the flow is F^0 corresponding to the feasible solution X^0 of a transportation problem, then the pair (Z^0, F^0) is called the cost-flow pair corresponding to the feasible solution X^0.
3. Paradoxical pair: A cost-flow pair, (Z, F) of an objective function is called paradoxical pair if $Z < Z^0$ and $F > F^0$ where Z^0 is the optimum cost and F^0 is the optimum flow of the transportation problem.
4. Best paradoxical pair: The paradoxical pair (Z*, F*) is called the best paradoxical pair of a transportation problem if for all paradoxical pair (Z, F), either $Z^* < Z$ or $Z^* = Z$ but $F^* > F$.
5. Paradoxical range of flow: if F^0 be the optimum flow and F* be the flow corresponding to the best paradoxical pair of a transportation problem then [F^0, F*] is called paradoxical range of flow.

LITERATURE REVIEW

The transportation paradox is, however, hardly mentioned at all in any of the great number of textbooks and teaching materials where the transportation problem is treated. The simulation research reported by Finke (1998) indicates, however, that the paradox may occur quite frequently.
Apparently, several researchers have discovered the paradox independently from each other. But most papers on the subject refer to the paper by Charines and Klingman (1971) and Szware (1973) as the initial papers. Charines and Klingman (1971) name it the more-for-less paradox and they write: "*The paradox was first observed in the early days of linear programming history (by whom no one knows)* and has been part of the folklore known to some (e.g. A. Charnes and Cooper) but unknown to the great majority of workers in the field of linear programming.

According to Appa (1973), the transportation paradox is known as Doigs paradox at the London School of Economics, named after Alison Doig who used it in exams etc around 1959 (Doig did not publish any paper on it).

Since the transportation paradox seems not to be known to the majority of those who are working with (or teaching) the transportation problem, one may be tempted to believe that this phenomenon is only an academic curiosity which will most probably not occur in any practical situation. But that seems not to be true. Experiments done by Finke (1978), with randomly generated instances of the transportation problem of size 100×100 and allowing additional shipments (post-optimal) show that the paradoxical properties. More precisely, the average cost reductions achieved are reported to be 18.6% with total additional shipments of 20.5%.

In a recent paper, Deineko et al (2003) develop necessary and sufficient conditions for a cost matrix C to be immune against the transportation paradox.

Arora and Ahuja (2010) carried out a research work in paradox on a fixed charge transportation problem. In their findings, a paradox arises when the fixed charge transportation problem admits of a total cost which is lower than the optimum cost, by transporting larger quantities of goods over the same routes. A sufficient condition for the existence of a paradox is established. Paradoxical Range of flow is obtained for any given flow in which the corresponding objective function value is less than the optimum value of the fixed charge transportation problem.

Manjusri et al (2012) in their research paper "The Algorithm of Finding all Paradoxical Pairs in a Linear Transportation problem" established a sufficient condition for the presence of paradox in a linear programming problem, obtained the paradoxical pairs and finally obtained the paradoxical range of flow.

Vishwas and Nilama (2010) in their research titled "On a Paradox in Linear Fractional Transportation Problem" discovered that a paradoxical situation arises in a linear plus linear fractional transportation problem (LPLFTP), when value of the objective function falls below the optimal value and this lower value is attainable by transporting larger amount of quantity. In their research paper, a new heuristic is proposed for finding initial basic feasible solution for LPLFTP and a sufficient condition for the existence of a paradoxical solution is established in LPLFTP.

PROBLEM FORMULATION

In this paper, we consider the following transportation problem:
Let the transportation problem consists of m sources and n destinations, where
x_{ij} = the amount of product transported from the ith source to the jth destination,
c_{ij} = the cost involved in transporting per unit product from the ith source to the jth destination,
a_i = the number of units available at the ith source,
b_j = the number of units required at the jth destination.

In this paper, we consider the cost minimizing linear transportation problem as:

$$P_1 : \text{Min} Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i ;\ \forall i \in I = (1, 2, \ldots, m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j ;\ \forall j \in J = (1, 2, \ldots, n) \text{ and}$$

$$x_{ij} \geq 0\ \forall (i, j) \in I \times J.$$

Let $X^0 = \{x_{ij}^0 \mid (i, j) \in I \times J\}$ be a basic feasible solution corresponding to the basis B of the problem P_1 and the value of the objective function Z^0 corresponding to the basic feasible solution X^0 is

$$Z^0 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}^0 \text{(say)}$$

Let F^0 be the corresponding flow.
Then $F^0 = \sum_{i \in I} a_i = \sum_{j \in I} b_j$

Now we consider the dual variables u_i for $i \in I$ and $j \in I$ such that $u_i + v_j = c_{ij}$ corresponding to the basis B.
Also $\forall (I, j) \notin B$, let

$$c'_{ij} = (u_i + v_j) - c_{ij}$$

If $c'_{ij} < 0 \forall (i, j) \notin B$ then the solution is optimum.

Theorem: the sufficient condition for the existence of paradoxical solution of (P_1) is that if $\exists$ at least one cell (r, s) $\notin$ B in the optimum table of (P_1) where a_r and b_s are replaced by $a_r + l$ and $b_s + l$ respectively ($l > 0$) then $(u_r + v_s) < 0$.

Proof: Let Z^0 be the value of the objective function and F^0 be the optimum flow corresponding to the optimum solution (X^0) of problem (P_1). The dual variables u_i and v_j are given by

$$u_i + v_j = c_{ij}, \qquad \forall (i, j) \in B$$

Then

$$Z^0 = \sum_i \sum_j c_{ij} x_{ij}^0$$

$$= \sum_i \sum_j (u_i + v_{ij}) x_{ij}^0$$

$$= \sum_i \left(\sum_j x_{ij}^0 \right) u_i + \sum_j \left(\sum_i x_{ij}^0 \right) v_j$$

$$= \sum_i a_i u_i + \sum_j b_j v_j \text{ and}$$

$$F^0 = \sum_i a_i = \sum_j b_j$$

Now, let $\exists$ at least one cell $(r, s) \notin B$, where a_r are replaced by $a_r + l$ and $b_s + l$, respectively ($l > 0$) in such a way that the optimum basis remains same, then the value of the objective function $\hat{Z}$ is given by

$$\hat{Z} = \left[\sum_{i \neq r} a_i u_i + \sum_{j \neq s} b_j v_j + u_r (a_r + l) + v_s (b_s + l) \right] = \left[Z^0 + l(u_r + v_s) \right]$$

The new flow $\hat{F}$ is given by

$$\hat{F} = \sum_i a_i + l = \sum_j b_j + l = F^0 + l$$

$$\hat{F} - F^0 = l > 0$$

Therefore for the existence of paradox we must have $\hat{Z} - Z^0 < 0$. Hence the sufficient condition for the existence of paradox is that $\exists$ at least one cell $(r, s) \notin B$ in the optimum table of such that if a_r and b_s are replaced by $a_r + l$ and $b_s + l$ ($l > 0$) then $l(u_r + u_s) < 0$, i.e. $(u_r + v_s) < 0$.

Now we state the following algorithm to find all the paradoxical pair of the problem (P_1).

Algorithm:

Step 1: Find the cost-flow pair (Z^0, F^0) for the optimum solution X^0.

Step 2: i = 1

Step 3: Find all cells $(r, s) \notin B$ such that $(u_r + v_s) < 0$ if it exists otherwise go to step 8.

Step 4: Among all cells $(r, s) \notin B$ satisfying step 3 find min flow for $l = 1$ which enter into the existing basis whose corresponding cost is minimum. Let (Z^i, F^i) be the new cost flow pair corresponding to the optimum solution X^i.

Step 5: Write (Z^i, F^i).

Step 6: i = i + 1.

Step 7: go to step 3

Step 8: We write the best paradoxical pair $(Z^*, F^*) = (Z^i, F^i)$ for the optimum solution $X^* = X^i$.

Step 9: End.

International Journal of Mathematics and Statistics Studies
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Published by European Centre for Research Training and Development UK (www.ea-journals.org)

This algorithm gives all the paradoxical pairs. From these pairs we can find the paradoxical pair for a specified flow ($\bar{F}$) also.

DATA ANALYSIS

The data used for this research was extracted from Opara J. (2009), Introduction to Operation Research, Exercises 3 page 28. The estimated supply capacities of the five warehouses, the demand requirements at the five markets and the transportation cost of each product are given in Table I below:

Table I

(i, j)	M_1	M_2	M_3	M_4	M_5	S_i
W_1	7	3	6	9	5	55
W_2	4	3	1	4	4	34
W_3	6	2	4	5	3	22
W_4	3	7	3	8	4	14
W_5	2	5	3	4	9	11
dj	44	35	25	20	12	136

Solving the above problem using the Least-Cost method through TORA Statistical Software Package, the optimal transportation table is presented in Table II.

Table II

(i, j)	M_1	M_2	M_3	M_4	M_5	S_i	V_i
W_1	7 10	3 35	6	9	5 10	55	0
W_2	4 9	3	1 25	4	4	34	-3
W_3	6	2	4	5 20	3 2	22	-2
W_4	3 14	7	3	8	4	14	-4
W_5	2 11	5	3	4	9	12	-5
dj	44	35	25	20	12	136	
V_j	7	3	4	7	5		

The total cost is 456. We then check the sign of $(U_r + V_s)$, where $(r, s) \notin B$ in Table II, we observe that $U_4 + V_2 = -1 < 0$, $U_5 + V_2 = -2 < 0$, and $U_5 + V_3 = -1 < 0$. So Paradoxical pair of the problem (P_1) applying the algorithm discussed in this paper.

Applying Step 1: The cost-flow pair is $(Z^o, F^o) = (456, 136)$ corresponding to the optimum solution X = { $x_{11} = 10$, $x_{12} = 35$, $x_{15} = 10$, $x_{21} = 9$, $x_{23} = 25$, $x_{34} = 20$, $x_{35} = 2$, $x_{41} = 14$, $x_{51} = 11$}.
Applying step 2: set i=1
Applying step 3: Now we check the sign of $(U_r + V_s)$ and we obtain for the non-basic cells (4,2), (5,2) and (5,3), the sign that is negative.
Applying step 4: For $l = 1$
For the cell (4, 2)

Table III

(i, j)	M_1	M_2	M_3	M_4	M_5	S_i	V_i
W_1	7 9	3 36	6	9	5 10	55	0
W_2	4 9	3	1 25	4	4	34	-3
W_3	6	2	4	5 20	3 2	22	-2
W_4	3 15	7	3	8	4	15	-4
W_5	2 11	5	3	4	9	11	-5
dj	44	36	25	20	12	137	
V_j	7	3	4	7	5		

The total cost of this transportation problem is 455.

For cell (5, 2), the transportation table is presented in Table IV

Table IV

(i, j)	M_1	M_2	M_3	M_4	M_5	S_i	V_i
W_1	7 9	3 36	6	9	5 10	55	0
W_2	4 9	3	1 25	4	4	34	-3
W_3	6	2	4	5 20	3 2	22	-2
W_4	3 14	7	3	8	4	14	-4
W_5	2 12	5	3	4	9	12	-5
dj	44	36	25	20	12	137	
V_j	7	3	4	7	5		

The total cost of this transportation problem is 454.

For the cell (5, 3), the transportation table is presented in Table V.

Table V

(i, j)	M_1	M_2	M_3	M_4	M_5	S_i	V_i
W_1	7 10	3 35	6	9	5 10	55	0
W_2	4 8	3	1 26	4	4	34	-3
W_3	6	2	4	5 20	3 2	22	-2
W_4	3 14	7	3	8	4	14	-4
W_5	2 12	5	3	4	9	12	-5
dj	44	35	26	20	12	137	
V_j	7	3	4	7	5		

The total cost is 455.

The min cost = min{456, 454, 455} = 454.

Hence l = 1 enters in the optimum basis from the cell (5, 2) and corresponding table is Table IV, the corresponding paradoxical pair (Z′, F′) = (454, 137).

Employing steps 6 and 7. Then repeating this process, the next table is

Table VI

(i, j)	M_1	M_2	M_3	M_4	M_5	S_i	V_i
W_1	[7] 8	[3] 37	[6]	[9]	[5] 10	55	0
W_2	[4] 9	[3]	[1] 25	[4]	[4]	34	-3
W_3	[6]	[2]	[4]	[5] 20	[3] 2	22	-2
W_4	[3] 14	[7]	[3]	[8]	[4]	14	-4
W_5	[2] 13	[5]	[3]	[4]	[9]	13	-5
dj	44	37	25	20	12	138	
V_j	7	3	4	7	5		

Now repeating this process, the final table is presented in Table VII

Table VII

(i, j)	M_1	M_2	M_3	M_4	M_5	S_i	V_i
W_1	[7] 0	[3] 45	[6]	[9]	[5] 10	55	0
W_2	[4] 9	[3]	[1] 25	[4]	[4]	34	-3
W_3	[6]	[2]	[4]	[5] 20	[3] 2	22	-2
W_4	[3] 14	[7]	[3]	[8]	[4]	14	-4
W_5	[2] 21	[5]	[3]	[4]	[9]	21	-5
dj	44	45	25	20	12	146	
V_j	7	3	4	7	5		

Hence, from the above table, the corresponding paradoxical pair (Z′, F′) = (436, 146).

Applying step 8: The best paradoxical pair is (Z*, F*) = (436, 146) corresponding to the optimum solution is X* = {x_{11} = 0; x_{12} = 45, x_{15} = 10, x_{21} = 9, x_{23} = 25, x_{34} = 20, x_{35} = 2, x_{41} = 14, x_{51} = 21} and the paradoxical range of flow is [F^0, F*] = [136, 146]. Thus, all the paradoxical pair are {(454, 137), (452, 138), (450, 139), (448, 140), (446, 141), (444, 142), (442, 143), (440, 144), (438, 145) and (436, 146).

This research paper has really unveiled the application of the algorithms of paradoxical pairs in a linear transportation problem. This paper will however go a long way to assist researchers who may wish to embark on a similar research topic.

CONCLUSION

In this paper, we have been able to discuss an efficient statistical algorithm for computing paradox in a linear transportation problem if paradox does exist. The algorithm gives step by step development of the solution procedure for finding all the paradoxical pair, well understanding. The TORA statistical software package was used to obtain the optimal solution before implementing the algorithm of paradoxical pairs.

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