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STOCHASTIC ANALYSIS OF MOTHER – TO – CHILD TRANSMISSION OF HIV/AIDS EPIDEMIC IN THE PRESENCE OF TREATMENT.

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ABSTRACT. *This study is concerned with the mathematical modeling for human immunodeficiency virus (HIV) transmission epidemics. The mathematical models are specified by stochastic differential equations that are solved using probability Generating Functions (PGF). Models based on Mother to child transmission (MTCT) were developed, the expectations and variances of Susceptible (S) persons, Infected (I) persons, Treatment (T) and AIDS (A) cases were found. Sensitivity analysis was carried out to investigate the influence of key parameters on the spread of the disease. The result are presented graphically*.

KEYWORDS: Probability Generating Function, HIV Transmission, Stochastic compartmental model, Mother to Child Transmission and Treatment.

INTRODUCTION

The human immunodeficiency virus (HIV) infection which often leads to acquired immunodeficiency syndrome (AIDS), has become a hazardous infectious disease in both the developed and developing nations. The disease break down the infected individual's immune system, leaving the victim vulnerable to a host of life threatening opportunistic infections, neurological disorders or unusual malignancies. It is a fatal disease and it has caused mortality of millions of people. Also, the threat of the disease has necessitated the expenditure of enormous amount of money in health care delivery and disease control (Naresh*et. al*.(2006)). The HIV infection in children is generally serious than in adult due to faster disease complications and progression. However, there are threemajour mechanism of Mother – To – Child transmission of HIV, Bashiru et- al (2009). These are, infection through placenta, known as utero infection, infection during birth known as intra – parturm infection and infection through breast – feeding known as post – partum.

Treatment is the process of offering the HIV positive individual with a life prolonging drugs/medicines known as antiretroviral (ARV) medicines or antiretroviral therapy (ART). ART drugs are the main types of treatment for HIV/AIDS. It is not a cure but it delays the onset of AIDS in the patients, thus enabling them to live longer than they would have done without the drugs. The therapy consists of drugs that have to be taken every day for the rest of patient's life. Treatment with anti-retroviral increases the life expectancy of people infected with HIV, even after HIV has progressed to diagnosable AIDS. The average survival time with antiretroviral therapy was estimated to be more than 5 years as at 2005 (Schneider *et al* (2005)).In the absence of treatment, the epidemic rate moved up to 25% in the Mother –To – Child TransmissionBrian et al (2011). However, where combination of antiretroviral drug treatment and Cesarean section are available, this risk can be reduced to as low as 1%. However there is no doubt that the treatment of pregnant women with their children with antiretroviral recombinants has reduced the transmission to some low levels in most developed countries.(Bashiru 2005). Olowofeso et al (2005) studied MTCT transmission of virus in a varying population they developed SIA model, treatment not inclusive. In this research we adopt Olowofeso et – al and incorporate treatment on the model for the mode of transmission.

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This is important because ARV treatment now plays a major role on the spread and progression of the disease.

THE MODEL

We intend to propose a simple HIV/AIDS model with treatment class for mother to child mode of transmission. In this model, the sexually mature population is divided into four compartments: the Susceptible, the Infectives (also assumed to be Infectious), the Treatment and the AIDS population whose numbers are denoted by S,I,T and A respectively. The number of total population is denoted by N(t), at any time t i.e N=S+I+T+A. In the model, we assume that the Susceptibles become HIV infected via sexual contacts with infectives which may also lead to birth of infected children. We also assumed that a fraction of newborns are infected at birth and hence are directly recruited into the Infective class with a rate $\gamma\varepsilon$ (Naresh et al (2006)). We do not consider direct recruitment of other infected persons except through mother to child transmission only. We also assume that some of the infectives move to join treated class with the rate σ_2 , while others with serious infection directly join the AIDS class with a rate σ_1. The proposed schematic diagram is presented in fig. 1.

Model Presentation.

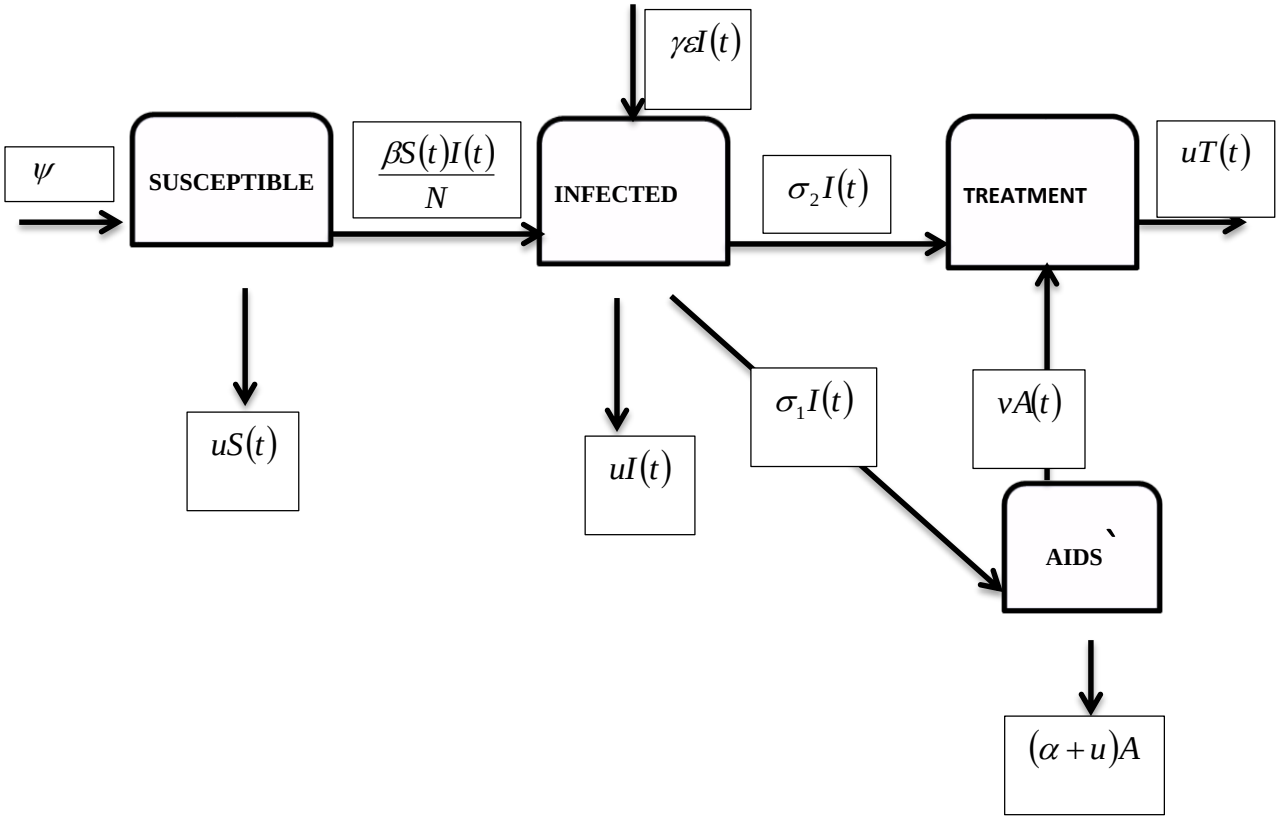

Fig. :1. Proposed schematic diagram for Mother – to – Child Transmission of HIV/AIDS using Birth – Death process.

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2.1 Notation for the models.

α - Disease – induced death rate due to AIDS.

σ_2 - Rate of movement from infectious class to treatment class.

σ_1 - Rate of movement from infectious class to AIDS class.

u - Natural mortality rate.

v - Rate at which AIDS group get treatment

β - The contact rate of the epidemic.

ψ - The rate of recruitment into susceptible population.

γ - Birth rate of infected newborn

ε - The fraction of infected newborn.

$S(t)$ - Number of Susceptible individuals at time t. ,

$I(t)$ - Number of Infected individuals at time t.

$T(t)$ - Number of individuals under ARV Treatment at time t.

$A(t)$ - Number of individual with full – blown AIDS at time t.

$\lambda_n(t)$ - Birth (immigration) rate.

$\mu_n(t)$ - Death (emigration) rate.

3. Method of Solution and Model Analysis.

let

$p_n(t) =$ probability that exactly "n" population are in the system at time (t)

$p_{n+1}(t) =$ probability that exactly "$n+1$" population are in the system at time (t)

$p_{n-1}(t) =$ probability that exactly "$n-1$" population are in the system at time (t)

$p_n(t+\Delta t) =$ probability that exactly "n" population are in the system at time $(t+\Delta t)$

Probability of no birth $= 1-(n\lambda\Delta t + o\Delta t)$

Probability of no death $= 1-(n\mu\Delta t + o\Delta t)$

Probability of birth $= n\lambda\Delta t + o\Delta t$

Probability of death $= n\mu\Delta t + o\Delta t$

Using the probability rule , we arrive at

$$p_n(t+\Delta t) = (1-n\lambda\Delta t + o\Delta t)(1-n\mu\Delta t + o\Delta t)p_n + p_{n+1}(1-n\lambda\Delta t + o\Delta t)(n\mu\Delta t + o\Delta t) + p_{n-1}(1-n\mu\Delta t + o\Delta t)(n\lambda\Delta t + o\Delta t) + p_n(n\mu\Delta t + o\Delta t)(n\lambda\Delta t + o\Delta t) \quad \ldots 3.1$$

Simplifying and neglecting all $(o\Delta t)$ we obtained, divide through by Δt and limit $\Delta t \to 0$, we obtained

$$p_n' = -p_n(n\lambda + n\mu) + p_{n+1}(n\mu) + p_{n-1}(n\lambda)$$

Therefore ,

$$\frac{dp_n(t)}{dt} = -(n\lambda + n\mu)p_n(t) + (n\mu)p_{n+1}(t) + (n\lambda)p_{n-1}(t) \quad \ldots 3.2$$

SUSCEPTIBLE $S(t)$

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The change in population size during the time interval (t , t + Δt) is governed by the following conditional probabilities.

$$pr[S(t,t+\Delta t)=n+1/S(t)]=\psi+o\Delta(t)$$
$$pr[S(t,t+\Delta t)=n-1/S(t)]=(\beta I+u)S+o\Delta(t)$$
$$pr[S(t,t+\Delta t)=n/S(t)]=1-\psi-(\beta I+u)S-o\Delta(t)$$

Considering probability rules, we obtained the following stochastic model with the aid Kolmogorov differential equation,

$$\frac{dp_n(t)}{dt}=-(\psi+(\beta I+u)S)np_n(t)-(n+1)(\beta I+u)Sp_{n+1}(t)+(n-1)\psi p_{n-1}(t) \quad \ldots 3.1$$

Solving equation (3.1) using probability generating function approach, by multiplying (3.1) by z^n and sum over n, we obtained,

$$\frac{\partial G(z,t)}{\partial t}=-(\psi+(\beta I+u)S)\sum_{n=1}^{\infty}np_n(t)z^n-(\beta I+u)S\sum_{n=1}^{\infty}(n+1)p_{n+1}(t)z^n+\psi\sum_{n=1}^{\infty}(n-1)p_{n-1}(t)z^n \quad \ldots 3.2$$

Simplifying we obtained

$$\frac{\partial G(z,t)}{\partial t}=[-\psi z(1-z)+(\beta I+u)s(1-z)]\frac{\partial G(z,t)}{\partial z} \quad \ldots 3.3$$

Equation (3.3) is a quadratic probability differential equation with auxiliary equation

$$\frac{dt}{1}=\frac{\partial G(z,t)}{\{(1-z)[z\psi-(\beta I+u)S]\}}=\frac{dG}{0} \quad \ldots 3.4$$

The two solution to the auxiliary equation (3.4) is

$$\frac{dt}{1}=\frac{dG}{0} \quad \ldots 3.5$$

and

$$-\frac{dt}{1}=\frac{\partial G(z,t)}{\{(1-z)[z\psi-(\beta I+u)S]\}} \quad \ldots 3.6$$

Considering 3.5

$$\int dG(z,t)=\int 0dt \quad \ldots 3.7$$
$$G(z,t)=k$$

Also , considering

$$-\frac{dt}{1}=\frac{\partial G(z,t)}{\{(1-z)[z\psi-(\beta I+u)S]\}} \quad \ldots 3.8$$

Solving (3.8) by separation of variables, using partial fraction and simplifying, we obtained the distribution of the model as

$$G(z,t)=\left(\frac{(\beta I+u)S-(\beta I+u)Se^{[\psi-(\beta I+u)S]t}-z(\psi-(\beta I+u)Se^{[\psi-(\beta I+u)S]t})}{(\beta I+u)S-\psi e^{[\psi-(\beta I+u)S]t}-z\psi(1-e^{[\psi-(\beta I+u)S]t})}\right)^m \quad \ldots 3.9$$

Considering T.J Bailey (1975), differentiating (3.9) with respect to z and let z $\uparrow 1$, we obtained expectation as

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$E(G(z,t)) = me^{(\psi-(\beta I+u)S)t} \quad \ldots 3.10$$

Also, Variance as

$$V(G(z,t)) = m\left(\frac{\psi+(\beta I+u)s}{\psi-(\beta I+u)s}\right)\ell^{(\psi-(\beta I+u)s)t}\left(\ell^{(\psi-(\beta I+u)s)t}-1\right) \quad \ldots 3.11$$

INFECTED CASE

The change in population size of this class during the time interval (t, t + Δt) is governed by the following conditional probabilities.

$$pr[I(t,t+\Delta t) = n+1/I(t)] = \beta SI + \gamma\varepsilon + o\Delta(t)$$

$$pr[I(t,t+\Delta t) = n-1/I(t)] = (u+\sigma_2+\sigma_1)I + o\Delta(t)$$

$$pr[I(t,t+\Delta t) = n/I(t)] = 1-(\beta SI+\gamma\varepsilon)-(u+\sigma_1+\sigma_2)I - o\Delta(t)$$

Following the steps above, we obtained stochastic model with aid of Kolmogorov differential equation as,

$$\frac{dp_n(t)}{\partial t} = -n(\beta SI+\gamma\varepsilon+(u+\sigma_2+\sigma_1)I)p_n(t)+n+1((u+\sigma_2+\sigma_1)I)p_{n+1}(t)+(n-1)[\beta SI+\gamma\varepsilon]p_{n-1}(t) \quad \ldots 3.12$$

The resulting distribution of (3.12) is (3.13) using probability generating function approach.

$$G(z,t) = \left(\frac{z(\beta SI+\gamma\varepsilon)-(u+\sigma_2+\sigma_1)I+(u+\sigma_2+\sigma_1)I(1-z)e^{((\beta SI+\gamma\varepsilon)-(u+\sigma_2+\sigma_1)I)t}}{z(\beta SI+\gamma\varepsilon)-(u+\sigma_2+\sigma_1)I+(\beta SI+\gamma\varepsilon)(1-z)e^{((\beta SI+\gamma\varepsilon)-(u+\sigma_2+\sigma_1)I)t}}\right)^m \quad \ldots 3.13$$

Differentiating (3.13) with respect to z and let z $\uparrow$1, we obtain expectation as

$$E(G(z,t)) = me^{[(\beta SI+\gamma\varepsilon)-(u+\sigma_2+\sigma_1)I]t} \quad \ldots 3.14$$

And Variance as

$$V(G(z,t)) = m\left(\frac{(\beta SI+\gamma\varepsilon)-(u+\sigma_2+\sigma_1)I}{(\beta SI-\gamma\varepsilon)+(u+\sigma_2+\sigma_1)I}\right)\ell^{((\beta SI+\gamma\varepsilon)-(u+\sigma_2+\sigma_1)I)t}\left(\ell^{((\beta SI+\gamma\varepsilon)-(u+\sigma_2+\sigma_1)I)t}-1\right) \quad \ldots 3.15$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

TREATMENT CASE.

The change in population size during the time interval (t, t + Δt) is governed by the following conditional probabilities.

$$pr(T(t,t+\Delta t)=n+1/T(t))=\sigma_2 I+\nu A+0\Delta(t)$$

$$pr(T(t,t+\Delta t)=n-1/T(t))=uT+0\Delta(t)$$

$$pr(T(t,t+\Delta t)=n/T(t))=1-(\sigma_2 I+\nu A)-uT-0\Delta(t)$$

Following the steps above, we obtained stochastic model with aid of Kolmogorov differential equation as,

$$\frac{\partial G(z,t)}{\partial t}=-n(\sigma_2 I+\nu A+uT)p_n(t)+n+1(uT)p_{n+1}(t)+n-1(\sigma_2 I+\nu A)p_{n-1}(t) \quad \ldots 3.16$$

Using probability generating function approach , we obtained (3.17)

$$G(z,t)=\left(\frac{z(\sigma_2 I+\nu A)-uT+uT(1-z)e^{((\sigma_2 I+\nu A)-uT)t}}{z(\sigma_2 I+\nu A)-uT+(\sigma_2 I+\nu A)(1-z)e^{((\sigma_2 I+\nu A)-uT)t}}\right)^m \quad \ldots 3.17$$

Differentiating (3.17) with respect to z and let z $\uparrow 1$ and simplifying, we obtain expectation as

$$E(G(z,t))=me^{[(\sigma_2 I+\nu A)-uT]t} \quad \ldots 3.18$$

And Variance as

$$V(G(z,t))=m\left(\frac{(\sigma_2 I+\nu A)-uT}{(\sigma_2 I-\nu A)+uT}\right)\ell^{((\sigma_2 I+\nu A)-uT)t}\left(\ell^{((\sigma_2 I+\nu A)-uT)t}-1\right) \quad \ldots 3.19$$

AIDS CASE

The change in population size of this class during the time interval (t, t+Δt) is governed by the following conditional probabilities.

$$pr(A(t,t+\Delta t)=n+1/A(t))=\sigma_1 I+0\Delta(t)$$

$$pr(A(t,t+\Delta t)=n-1/A(t))=(\alpha+\nu+u)A+0\Delta(t)$$

ISSN 2055-0154(Print), ISSN 2055-0162(Online)

$$pr\left(A(t,t+\Delta t)=n/A(t)\right)=1-(\sigma_1 I)-(\alpha+\nu+u)A-0\Delta(t)$$

Following the steps above, we obtained stochastic model with aid of Kolmogorov differential equation as,

$$\frac{dp_n(t)}{dt}=-n(\sigma_1 I+(\alpha+\nu+u)A)p_n(t)+n+1((\alpha+\nu+u)A)p_{n+1}(t)+n-1(\sigma_1 I)p_{n-1}(t) \quad \ldots 3.20$$

Using probability generating function approach , we have (3.21) as the distribution

$$G(z,t)=\left(\frac{z(\sigma_1 I)-(\alpha+\nu+u)A+(\alpha+\nu+u)A(1-z)e^{((\sigma_1 I)-(\alpha+\nu+u)A)t}}{z(\sigma_1 I)-(\alpha+\nu+u)A+(\sigma_1 I)(1-z)e^{((\sigma_1 I)-(\alpha+\nu+u)A)t}}\right)^m \quad \ldots 3.21$$

Differentiating (3.21) with respect to z and let z $\uparrow 1$, we obtained expectation as

$$E(G(z,t))=me^{[(\sigma_1 I)-(\alpha+\nu+u)A]t} \quad \ldots 3.22$$

And Variance as

$$V(G(z,t))=m\left(\frac{(\sigma_1 I)+(\alpha+\nu+u)A}{(\sigma_1 I)-(\alpha+\nu+u)A}\right)\ell^{((\sigma_1 I)-(\alpha+\nu+u)A)t}\left(\ell^{((\sigma_1 I)-(\alpha+\nu+u)A)t}-1\right) \quad \ldots 3.23$$

1. RESULT AND DISCUSSION.

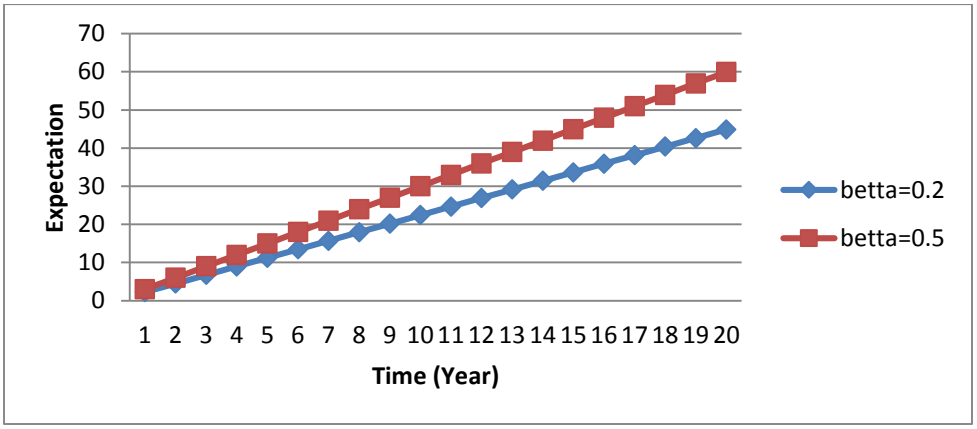

Fig.3.1: Expectation of Susceptible with varying value of β *when* $\sigma_1=0.004, \nu=0.7, u=0.03$

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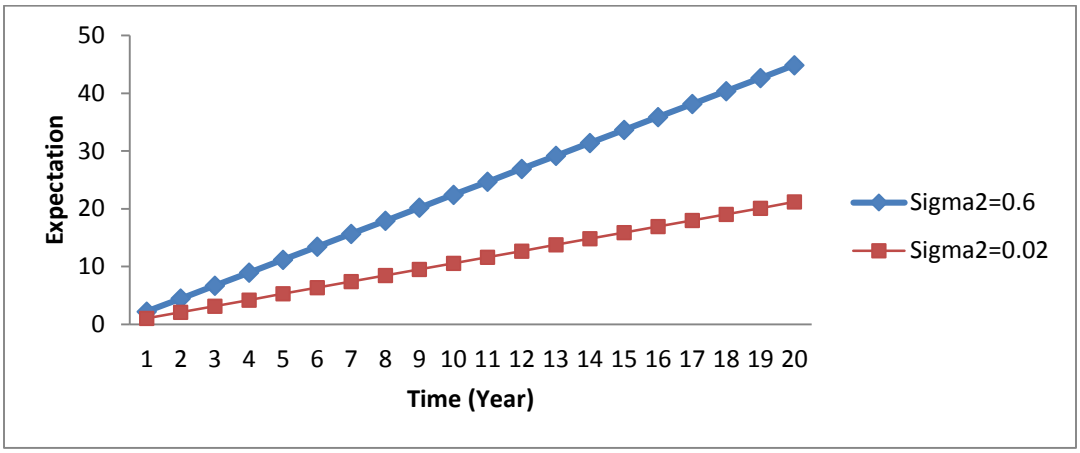


Fig.3.2: Expectation of Treatment with varying value of σ_2 *when* $I(0)=1.793,\ \sigma_1=0.004, v=0.3, u=0.03$

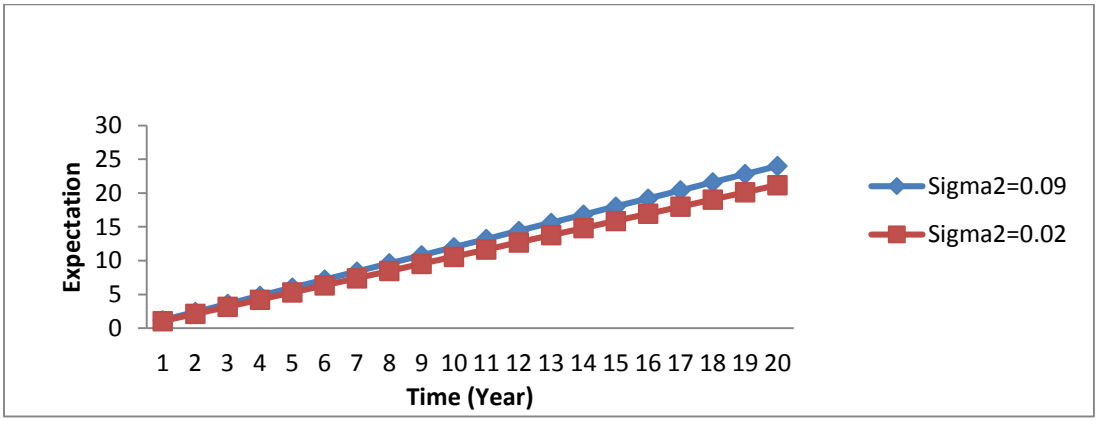
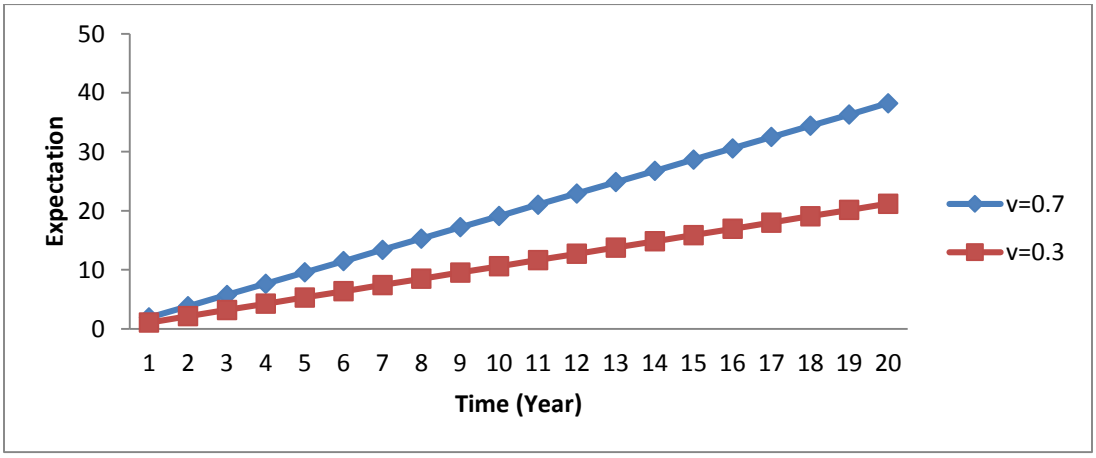


Fig.3.3: Expectation of Treatment with varying value of σ_2 *when* $I(0)=1.793,\ \sigma_1=0.004, v=0.3, u=0.03$



ISSN 2055-0154(Print), ISSN 2055-0162(Online)

Fig.3.4: Expectation of Treatment with varying value of
v *when* $I(0)=1.793,\ \sigma_1=0.004,\ u=0.03$

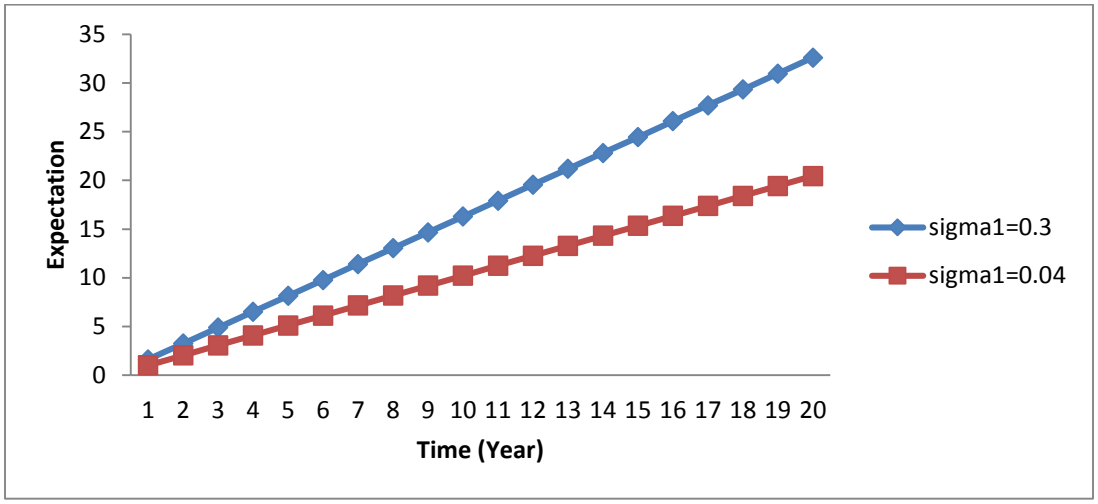


Fig.3.5: Expectation of AIDS with varying value of
σ_1 *when* $I(0)=1.793, \alpha=0.1, v=0.3, u=0.03$

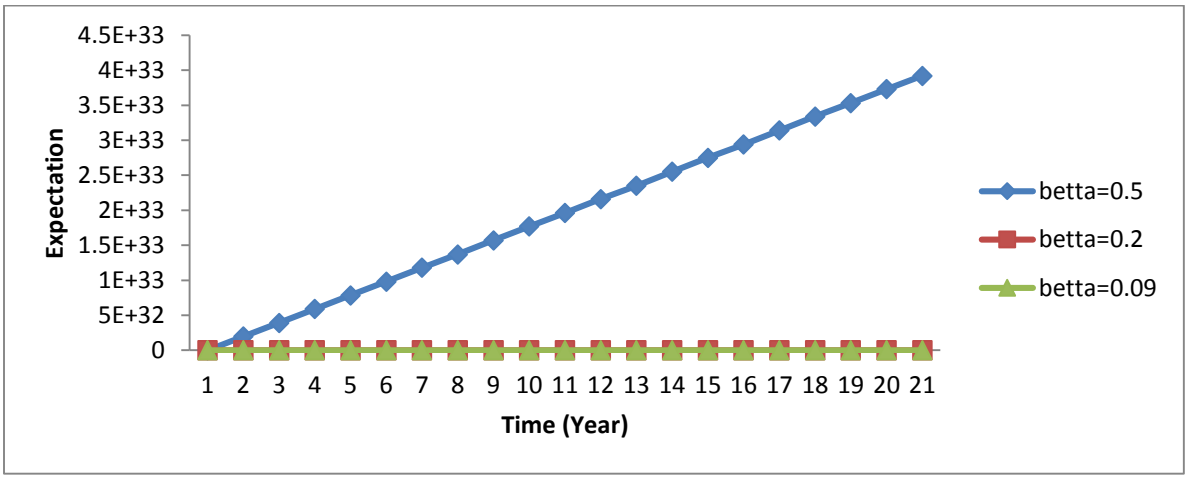


Fig.3.6: Expectation of Infective with varying value of

2. β *when* $\sigma_2=0.04,\ \sigma_1=0.004, v=0.7, u=0.03$

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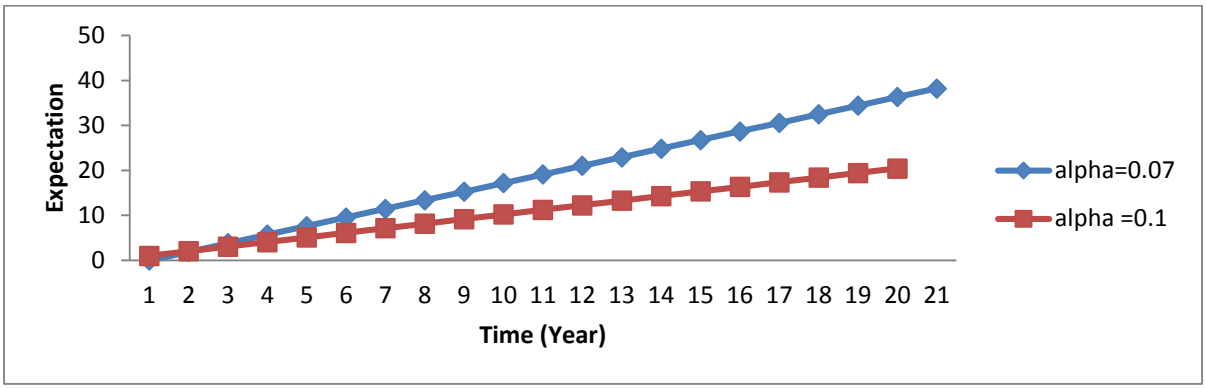

Fig.3.7: Expectation of AIDS with varying value of
α, *when* $I(0)=1.793, \alpha=0.1, v=0.3, u=0.03, \sigma_1=0.004, \beta=0.5$

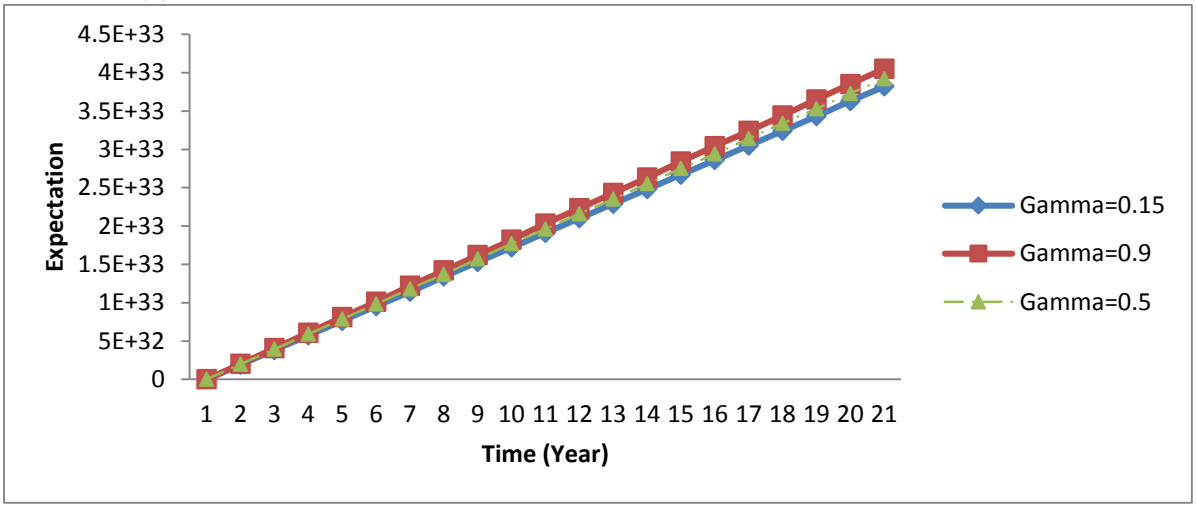

Fig.3.8: Expectation of Infective with varying value of
γ, *when* $T(0)=1.7449, v=0.3, u=0.03, \sigma_1=0.004, \beta=0.5, \sigma_2=0.04$

Table 1: Assumed values of parameters used in the data simulation.

Parameters Notation	Parameters	Assumed Values
β	The contact rate of the epidemic	0.5
ε	The fraction of infected newborn	0.02
γ	Rate of infected newborn baby	0.15
σ_1	Rate of movement from infected class to AIDS	0.04
σ_2	Rate of movement from infected class to Treatment	0.6
δ	Probability of infection from a sexual contact with an infected.	0.4
α	Disease – induced death rate	0.02

It was observed from figure 3.1 that when β is decreased from 0.5 to 0.2, the expected suspected population is increasing, when it is 0.2 the expected susceptible population at time (year) 10 is

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22.43 while it is 29.96 when β is 0.5. This also may be as a result of awareness of preventive measure, counseling and support to women living with HIV which enable them to make informed decisions about their reproductive lives, Figure 3.2 and 3.3 explained the expected population in treated class. From figure 3.2, when σ_2 is 0.6 at year 13, the expected treatment population is 29.15 while the expected treated population is 13.77 when σ_2 is 0.09, the expected population is 15.61 at the same point. With varying value of σ_2, it was observed that σ_2 is increasing the expected population of treated class is also increasing, which may likely result to decrease in expected population in the infected class. This is as a result of making ARVs more available to the society. So, this can serve as a control measure for mother-to-child transmission.

From fig (3.4) and fig 3.5, we discovered that as the rate at which in the AIDS class get treatment increases the expected population of treated class increases in fig.3.4 which is as a result of increase in public awareness of HIV/AIDS and also about half of the population of the country have known that a woman living with HIV can take drugs during pregnancy to reduce the risk of transmission. That is as a result of introduction of drugs for prevention of Mother – to – child transmission (PMTCT).Also from fig. 3.5 we can see that as σ_1 is increases the expectation of population of AIDS class is increase, when $\sigma_1 = 0.3$ the expected population is 16.29 and also at 0.04 the expected population is 10.23. this may be as a result of nonchalant attitude or fear of stigmatization that make the infected people not to go for HIV screening and advise on treatment on time before it get to chronic stage.

From fig (3.6), we can see that the expectation of population in the infected class was at equilibrium when β is 0.09 and 0.02 but as it increases to 0.5 the population of infected class is increasing. This show that the epidemic is still in the community. Many factors may have contributed to the decline in infective for varies β. Among these could be the impact of intervention program all over the country.We can see from fig. 3.7 that as the value of disease induce death (α) increasing the population of AIDS class is decreasing, from the graph it was observed that at year 2 there is equilibrium point for the two value of α but at the later years the population was decreasing as the α is increasing. Fig 3.8 shows that a little increments in γ turn to increment in expected population of infective class. When $\gamma = 0.5$ the expected population of infective is 1.77E+33 at year 10 while the expectation of infected population is 1.8E+33 as the $\gamma = 0.5$. The decrease in infective population seems to be due to cumulative effect of decrease in birth of infected newborns.

CONCLUSION

In this research, stochastic model for the HIV/AIDS was presented compartmentally for Mother – to – Child . The expectation and the variance of the models were then obtained. Sensitivity analysis was carried out to investigate the potential impact of treatment on disease progression, The analyses show that an increase in the treatment rate results in an expected decline of new disease incidences.

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