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STATISTICAL MODEL FOR UNBALANCED INCOMPLETE PAIRED COMPARISONS TOURNAMENTS

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ABSTRACT: *A class of modification is proposed for calculating a score for each Player/team in Unbalanced Incomplete paired Comparisons Tournaments. Many papers dealing with Balanced Incomplete Paired Comparison Tournaments with at most one comparison per pair have appeared since 1950. However, little has been written about unbalanced situations in which the player /the team (object) (j) plays unequal number of games against the player/the team(m) in a tournament, and the results of all games can be summarized in a Win-Lose matrix* $Y = \{ Y_{jm} \}$, *where* Y_{jm} = *1,0,1/2, respectively , according as the player or the team (j) wins, losses or draws against the player or the team (m). Published papers by Ramanujacharyulu (1964), Cowden,D.J. (1975), and David,H. A. have concentrated on the problem of converting the results of unbalanced incomplete paired comparison tournaments into rank with little consideration of the main relative ability on each player or team. We seggest (modification) other ways of quantifying the outcomes of the games / tournaments , in particular , ratings on a scale, 0 to 5 , 0 to10,..ect. It is important to consider not only the vector* $V_j^{(d)}$ *or the vectors* S_j *, in scoring and ranking the k teams in such tournaments , but also the vector* Z_j*, where* $Z_j = S_j + S_j R_j$ *,to take into account the ratio of the relative ability of each team (* R_j *). The proposed modification helps to introduce these methods for use in comparisons/games (tournaments), where the player/team are quantified on a special scale. e.g.0-5, 1-10, ..etc. We conclude the following:- The scores stabilized to three decimal places at iteration 2 in Cowden's method* $V_j^{(d)}$ *.see table(1.4). The scores stabilized to three decimal places at iteration 2 in David's method* S_j *,and it's modification* Z_j*. The proposed modification* (Z_j) *has the advantage of removing ties from David's method* (S_j)*, and* ***hence*** *it is the best method.*

KEYWORDS: Unbalanced Incomplete design; paired Comparisons; Scores; Ranking; Rating on a scale.

INTRODUCTION

Many papers dealing with Balanced Incomplete Paired Comparison Tournaments with at most one comparison per pair have appeared since 1950. However, little has been written about unbalanced situations in which the player /the team (object) (j) plays unequal number of games against the player/the team(m) in a tournament, and the results of all games can be summarized in a Win-Lose matrix $Y = \{ Y_{jm} \}$, where Y_{jm} = 1,0,1/2, respectively , according as the player or the team

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(j) wins, losses or draws against the player or the team (m). Published papers have concentrated on the problem of converting the results of unbalanced incomplete paired comparison tournaments into rank with little consideration of the main relative ability on each player or team.

1- Methodology

Suppose that there are k teams $T_1, \ldots, T_k$ competing in a tournament, and the jth team plays i = 1, 2, . . . , n games. The number of games won by the jth team, in a Win-Lose matrix, is similar to the number of judges who prefer the jth object in the "preference table".

2.1-Generating Scores:

Let Y_{jm} be the number of times that team j wins against team m, and

Let $Y = \{ Y_{jm} \}$

If $W_j^{(1)}$ are the row sums of the Win-Lose matrix Y (the marginal total number of wins),

Then $W_j^{(1)} = \sum_{m=1}^{k} Y_{jm} = Y \, . \, \underline{1}$

Where 1 is the column vector of k ones.

Now let:

$$W_j^{(2)} = Y \, . \, W_j^{(1)} = Y^2 . \; \underline{1}$$

This process may be continued to generate:

$$W_j^{(d)} = Y \, . \, W_j^{(d-1)} = Y^d . \; \underline{1} \qquad (2.1)$$

This approach is based on generating scores by powering the matrix (Y) eq (2.1). With increasing d, rankings based on $W_j^{(d)}$ tend to stabilize.

2.2-Ramanujacharyulu's Method

Ramanujacharyulu (1964) called $W_j^{(d)}$, the iterated power of order d and introduced the corresponding iterated weakness $W_m^{(d)}$ i.e $w_m^{(1)}$ is the total number of looses for team j, which are the column sums of the Win-Lose matrix, such that

$$W_m^{(1)} = \sum_{j=1}^{k} Y_{jm} = Y^t \,.\, \underline{1} \qquad \ldots \qquad (2.2)$$

Where Y^t is the transpose of the matrix Y, as the matrix of losses.

Again $W_m^{(2)} = Y^t \,.\, W_m^{(1)} = (Y^t)^2 \,.\, \underline{1}$

And $W_m^{(d)} = Y^t \,.\, W_m^{(d-1)} = (Y^t)^{(d)} \,.\, \underline{1}$

2.3-Cowden's Method

Cowden, D.J. in 1975, proposed that, instead of multiplying each Y_{jm} by the number of games won by each opponent defeated by him, we my multiply by $V_j^{(d)}$, the weighted proportion of games won, at iteration d.

Let $V_j^{(1)}$ be the proportion of games won by the jth team .

Let $V_c^{(1)}$ be the proportion of games lost by the jth team.

So

$$V_j^{(1)} = W_j^{(1)} / \{ W_j^{(1)} + W_m^{(1)} \}$$

And $V_c^{(1)} = W_c^{(1)} / \{ W_j^{(1)} + W_m^{(1)} \}$

Let

$$U_1^{(1)} = Y \,.\, V_j^{(1)}$$
$$U_2^{(1)} = Y' \,.\, V_c^{(1)}$$
$$V_j^{(2)} = U_1^{(1)} / \{ U_1^{(1)} + U_2^{(1)} \}$$
$$V_c^{(2)} = U_2^{(1)} / \{ U_1^{(1)} + U_2^{(1)} \}$$

The process may be continued to generate

$$V_j^{(d)} = U_j^{(d-1)} / \{ U_j^{(d-1)} + U_c^{(d-1)} \} \qquad \ldots \qquad (2.3)$$

Where $0 < V_j^{(d)} < 1$, $(d = 1, 2, \ldots\ldots)$

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And $V_c^{(d)} = 1 - V_j^{(d)}$

It is convenient (as Cowden said) to use the values of $V_j^{(0)} = V_c^{(0)} = 0.5$, in which case $V_j^{(1)}$ is the weighted proportion of games won with iteration (1), and $V_c^{(1)}$ is the weighted proportion of games lost with iteration (1).

The values of $U_j^{(d)}$, $U_c^{(d)}$, $V_j^{(d)}$, and $V_c^{(d)}$, change with each iteration until the values of $V_j^{(d}$ stabilize. This method of ranking and scoring is not very sensitive to initial values of $V_j^{(0)}$ and $V_c^{(0)}$.

2.4-David's Method

David, H. A. (1987) proposed a simple and useful formula for scoring, and hence ranking, the kth team/player in an incomplete or unbalanced tournament, by using the scores vector S_j ,such that

$$S_j = W_j^{(2)} - W_m^{(2)} + W_j^{(1)} - W_m^{(1)} \quad \ldots \quad (2.4)$$

The idea of this approach is that the vector S_j reflects equally the strength of those teams defeated by T_j and the weakness of those teams by whom T_j was defeated. He considers only (0 ,1 ,1/2) outcomes.

Ranking according to the score vectors , S_j , eq(2.4) has one disadvantage, in that it produces ties amongst players/ teams / objects.

2.5- A Proposed Modification to David's Method :

We will investigate other ways of quantifying the outcomes of the games / tournaments , in particular , ratings on a scale, 0-5 , 0-10, instead of (0, 1) ..ect.

It is important to consider not only the kx1 vector $V_j^{(d)}$ (eq. 2.3) or the kx1 vectors S_j (eq. 2.4), in scoring and ranking the k teams or contestants in such tournaments , but also to take into account the ratio of the relative ability of each team(R_j) .

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To find this R_j ,we work as follows :

Let

K = number of teams.

H = number of characters scores (attributes) per team per game

N = number of games.

Then let

X_{ijmp} $\quad i = 1, 2, \ldots , n$

$j, m = 1, 2, \ldots, k \quad j \neq m$

$p = 1, 2, \ldots ,h$

Be the score (rating on a scale) of the object jth team playing against the mth team for characteristic score in the ith game.

Thus

$$M_{ijm} = \sum^{h}_{p=1} X_{ijmp} / h \qquad \ldots \qquad (2.5)$$

Is the average score per team per game.

Now, define M_{jm} such that

$$M_{jm} = \sum^{h}_{p=1} \sum^{n}_{i=1} X_{ijmp} / hn$$

Hence

$$M_{jm} = \sum^{n}_{i=1} M_{ijm} / n \qquad \ldots \qquad (2.6)$$

Are the means of averages.

The marginal means of the matrix M_{jm} represents the means of the average scores per game for the jth and the mth teams.

i.e

$$M_{j.} = \sum^{k}_{m=1} M_{jm} / k \text{ [mean of the average scores for player j].}$$

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and $M_{.m} = \sum^{k}_{j=1} M_{jm} / k$ [mean of the average scores for player j's opponents].

Hence, our ratio R_j becomes

$$R_j = M_{j.} / [M_j + M_{.m}] \quad \ldots \quad (2.7)$$

As a ratio of the relative ability of team j to team m.

Cowden (1975) has ranked the players/teams according to the score vector $V_j^{(d)}$, eq(2.3).

David (1987) has ranked the players / teams to the score vector , S_j , eq(2.4). But S_j uses simply the information in to the Win-Lose matrix. If the information relating to the actual scores in any game/tournament is available, one could use a simple combination of S_j and R_j.

e.g. $Z_j = S_j \, . R_j + S_j$

$$Z_j = S_j \ (R_j \ +1) \quad \ldots \quad (2.8)$$

This particular combination of S_j and R_j seems a responsible way for incorporating all the information. It should reduce the number of ties produced by David's method.

3. Application of the Proposed Modification.

ROUND BRITAIN QUIZ

This is a seasonal BBC Radio 4 program, broadcast weekly. Several teams from different regions of Britain compete overall. Each week, two compete against each other. Points are scored in answers to questions of variable length and difficulty, and there are variable numbers of questions in each game. Both teams receive the same number of questions per game. The maximum number of points awarded per question is 6.

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Suppose the maximum number of questions per team per game is 6. There are likely to be an unequal number of games per team. Typical data for this tournament is given in Table (3.1).

Table (3.1)
Paired comparison tournament for round Iraq quiz
A seasonal BBC Radio 4 program

Week	Team	Question's score X_{ijmh}		No. of questions per team per game (n)	Average score M_{ijm}	
1	T1vT2	65566	23455	5	5.6	3.8
2	T1vT3	534555	443346	6	4.5	4.0
3	T1vT4	55456	54342	5	5.0	3.6
4	T1vT6	465533	544533	6	4.33	4.00
5	T1vT7	5645	4543	4	5.00	4.00
6	T2vT1	53344	34333	5	3.80	3.20
7	T2vT3	5445	4633	4	4.50	4.00
8	T2vT7	33333	44343	5	3.00	3.60
9	T3vT5	334333	554534	6	3.17	4.33
10	T3vT6	223344	554424	6	3.00	4.00
11	T3vT7	55455	44334	5	4.80	3.60
12	T4vT5	55555	55244	5	5.00	4.00
13	T4vT6	524445	333333	6	4.00	3.00
14	T4vT7	3434	5445	4	3.50	4.50
15	T5vT6	55444	34423	5	4.40	3.20
16	T5vT7	24333	44433	5	3.00	3.60
17	T1vT2	544333	664332	6	3.67	4.00
18	T1vT2	33433	44334	5	3.20	3.60
*rated on 1-6 scale (1= poor, ... , 6 = excellent)						

J, m = 1, 2, ... , 7. j = m.
** for example, $M_{112} = (6+5+5+6+6) / 5 = 5.6$

(i. e: the average score of the first team who played against the second team in the second team in the first game between these two teams),and

$$M_{121} = (2+3+4+5+5) / 5 = 3.8$$

(i.e: the average score of the second team who played against the first team in the first game between these two teams).

We can see the following points from Table (3.1) above:

1- Each game involves two teams.
2- Each team does not play against every other one.

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3- Some of the pairs play each other on more than one occasion.

The results of all games can be summarized in an average score matrix, as shown in Table (3.2):

Table (3.2)

Average Scores of Team per game

Team	T_1	T_2	T_3	T_4	T_5	T_6	T_7	$M_{j.}$	R_J
T1		3.92	4.50	5.00		4.33	5.00	4.55	0.54
T2	3.80		4.50				3.00	3.77	0.49
T3	4.00	4.00			3.17	3.00	4.80	3.79	0.48
T4	3.60				5.00	4.00	3.50	4.03	0.49
T5			4.33	4.00		4.40	3.00	3.93	0.51
T6	4.00		4.00	3.00	3.20			3.55	0.47
T7	4.00	3.60	3.60	4.50	3.60			3.86	0.50
$M_{.m}$	3.88	3.84	4.186	4.125	3.743	3.93	3.86		

The results of all the games can be summarized in a Win-Lose matrix Y = { y_{jm} } , as shown in Table(3.3):

Table (3.3)

Unbalanced and incomplete paired comparison tournament
Data-Matrix Y = Win-Lose Matrix

Team	T_1	T_2	T_3	T_4	T_5	T_6	T_7	$W_j^{(1)}$
T1		1	1	1		1	1	5
T2	3		1				0	4
T3	0	0			0	0	1	1
T4	0			0	1	1	0	2
T5			1	0		1	0	2
T6	0		1		0			1
T7	0	1	0		1			3
$(W_m)^{(1)}$	3	2	4	2	2	3	2	

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Because of the unbalanced and incomplete nature of the data as shown in the Table (3.3), we can use either Cowden's method Eq.(2.3), David's method Eq.(2.4) or the proposed method Eq.(2.8).Thus

1- Use Eq(2.7) , and data in Table (3.2) to find R_j ,the relative ability of each team.
2- Use Eq(2.3), and data in Table (3.3) to find $V_j^{(d)}$,the scores vector. (Cowden's method , Table (3.4))

Table (3.4)
Cowden's Method

Team	T_1	T_2	T_3	T_4	T_5	T_6	T_7	$W_j^{(1)}$	$V_j^{(2)}$	Rank
T1		1	1	1		1	1	5	0.689	2
T2	3		1				0	4	0.728	1
T3	0	0			0	0	1	1	0.234	6
T4	0			0	1	1	0	2	0.491	4
T5			1	0		1	0	2	0.333	5
T6	0		1		0			1	0.127	7
T7	0	1	0		1			3	0.586	3
$(W_m)^{(1)}$	3	2	4	2	2	3	2			

3- Use Eq.(2.4), and data in Table (3.3) to find (S_j,) the scores vector (David's method, in Table(3.5)) :

Table (3.5)
David's method

Team	T_1	T_2	T_3	T_4	T_5	T_6	T_7	$W_j^{(1)}$	$W_j^{(2)}$	S_j	Rank
T1		1	1	1		1	1	5	11	7	2
T2	3		1				0	4	16	13	1
T3	0	0			0	0	1	1	3	-10	6
T4	0			0	1	1	0	2	3	-2	4
T5			1	0		1	0	2	2	-2	5
T6	0		1		0			1	1	-8	7
T7	0	1	0		1			3	8	2	3
$(W_m)^{(1)}$	3	2	4	2	2	3	2				
$(W_m)^{(2)}$	6	5	10	5	4	7	7				

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Then weight the score S_j by R_j [Eq.(2.8) = Z_j] . see Table (3.6):

Table (3.6)
Modifying David's Method, using Z_j

Team	R_j	S_j	Z_j	Rank
T_1	0.539	7	10.773	2
T_2	0.495	13	19.435	1
T_3	0.475	-10	-14.750	7
T_4	0.494	-2	-2.988	4
T_5	0.512	-2	-3.024	5
T_6	0.474	-8	-11.792	6
T_7	0.500	2	3.000	3

Then rank the result to see the difference between the three vectors, $V_j^{(d)}$, S_j,Z_j . see Table (3.7):

Table (3.7)
Ranks for three scores vectors

Team	$V_j^{(d)}$	S_j	Z_j
T_1	2	2	2
T_2	1	1	1
T_3	6	7	7
T_4	4	4.5	4
T_5	5	4.5	5
T_6	7	6	6
T_7	3	3	3

CONCLUSION

The two methods, Cowden's method and David's method, are interested in calculating scores and then ranks for each player/team (object) from the "preference matrix" Win-Lose matrix as in Table (2.3). the proposed modification helps to introduce these methods for use in comparisons/games (tournaments), where the player/team are quantified on a special scale.e.g.0-5,1-10,instead(0,1)data...etc.The scores stabilized to three decimal places at iteration 2 in Cowden's method $V_j^{(d)}$. See Table(2.4).

The scores stabilized to three decimal places at iteration 2 in David's method, S_j ,and it's modification, Z_j. [See Tables (3.5) and (3.6)]

Table (3.7) shows the ranks for three scores vectors $V_j^{(d)}$,S_j and Z_j at the second iteration. There is a difference between $V_j^{(d)}$ and Z_j , where team no.3 becomes the 7^{th} instead of the 6^{th}. [Team no.3 won one game out of five games whereas team no.6 won one game out of four games].

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Important Note: The proposed modification (Z_j) for David's method, (S_j), **has the advantage of removing ties from David's method,** and hence it is the best method

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