STATISTICAL INFERENCE FOR PARAMETERS OF GOMPERTZ DISTRIBUTION BASED ON GENERAL PROGRESSIVELY TYPE-II RIGHT CENSORED ORDER STATISTICS

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ABSTRACT: In this article, we concluded the Bayesian estimations for the parameters of the Gompertz distribution (GD) based on general progressively type-II right censored order statistics (GPTIICO). The squared error loss function (SEL) and linear-exponential loss function (LINEX) and the generalization of the entropy loss function (GE) and Al-Bayyati loss function (ABL) are considered for Bayesian estimation. Finally, a numerical example is established to clear the theoretical procedures.

KEYWORDS: Gompertz Distribution; General Progressively type-II Right Censored Order Statistics.

INTRODUCTION


In failure data analysis, it is common that some individuals cannot be observed for the full failure times. GPTICOS is a useful and more general scheme in which a specific fraction of individuals at risk may be removed from the study at each of several ordered failure times. Progressively censored samples have been considered, among others, as solved by Davis and Feldstein [7], Balakrishnan et al. [4], and Guilbaud [10]. This scheme of censoring was generalized by Balakrishnan and Sandhu [3] as follows: at time $X_0 = 0$, units are placed on test; the first $r$ failure times, $X_1, \ldots, X_r$, are not observed; at time $X_i = 0$, where $X_i$ is the $i^{th}$ ordered failure time ($i = r + 1, \ldots, m - 1$), $R_i$ units are removed from the test randomly, so prior to the $(i + 1)^{th}$ failure there are $n_i = n - i - \sum_{j=r}^{i} R_j$ units on test; finally, at the time of the $m^{th}$ failure, $X_m$, the experiment is terminated, i.e., the remaining $R_m$ units are removed from the test. The $R_i$'s, $m$ and $r$ are prespecified integers which must satisfy the conditions: $0 \leq r < m \leq n$, $0 \leq R_i \leq n_{i-1}$ for $i = r + 1, \ldots, m - 1$ with $n_r = n - r$ and $R_m = n_{m-1} - 1$.

If the failure times are based on an absolutely continuous distribution function (cdf) $F$ with probability density function (pdf) $f$, the joint probability density function based on GPTICOS failure times $X_{r+1, m; n}, X_{r+2, m; n}, \ldots, X_{m, m; n}$, is given by

$$f_{X_{r+1, m; n}, X_{r+2, m; n}, \ldots, X_{m, m; n}}(x_{r+1}, x_{r+2}, \ldots, x_m) = A_{(n, m-1)}[F(x_{r+1}, \theta)]^{r} \times \prod_{i=r+1}^{m} f(x_i, \theta)[1 - F(x_i, \theta)]^{R_i}, \quad x_{r+1} < x_{r+2} < \cdots < x_m, \quad (1.1)$$
where,

\[ A_{(n,m-1)} = \frac{n!}{r! (n-r)!} \left( \prod_{j=r}^{m-1} n_j \right), \quad n_i = n - i - \sum_{j=r+1}^{i} R_j, \quad i = r + 1, ..., m - 1. \]

Through this paper, we concluded the Bayesian estimations for the parameters of the GD based on GPTIIRCO. The SEL, ABL, LINEX and GE loss functions are considered for Bayesian estimation. Finally, a numerical example is established to clear the theoretical procedures are obtained. For more details of the SEL, ABL, LINEX and GE loss functions see Al-Aboud [1], Al-Bayyati [2], Bernardo and Smith [5], Calabria and Pulcini [6], and Figueiredo [8].

Let \( X_{(r+1}, \ldots, R_m) < X_{r+2, m} < \ldots < X_{m, m} \) be the \( m \) ordered observed failure times in a sample of size \((n-r)\) under GPTIICOS scheme from the GD with probability density function (pdf) given by

\[ f(x, \alpha, \beta) = \alpha e^{\beta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)}, \quad \alpha > 0, \quad \beta > 0, \quad x \geq 0 \]  \hspace{1cm} (1.2)

and the corresponding cumulative distribution function (cdf) is given by

\[ F(x, \alpha, \beta) = 1 - e^{\beta x - \frac{\alpha}{\beta} (e^{\beta x} - 1)}, \quad \alpha > 0, \quad \beta > 0, \quad x \geq 0. \]  \hspace{1cm} (1.3)

The hazard function at time \( t \) is given by

\[ H(t) = \frac{f(t)}{F(t)} = \alpha e^{\beta t}. \]  \hspace{1cm} (1.4)

The GD was introduced by Gompertz [9].

Bayes estimation

In this section, we have three types of loss functions, which are SEL, ABL, LINEX and GE loss functions. We apply these types in two cases, the first is known shape parameter \( \beta \) and the second is unknown shape \( \beta \) and scale parameter.

The likelihood function is

\[ L(\alpha, \beta | x) = A_{(n,m-1)} \left[ 1 - e^{\frac{\alpha}{\beta} (e^{\beta x_{i+1}} - 1)} \right]^{R} \prod_{i=r+1}^{m} \alpha e^{\beta x_i - \frac{\alpha}{\beta} (e^{\beta x_i} - 1)} \left[ e^{\frac{\alpha}{\beta} (e^{\beta x_{i-1}} - 1)} \right]^{R_i}. \]  \hspace{1cm} (2.1)

The case of known shape parameter \( \beta \)

For Bayesian estimation, we use here the natural conjugate prior density function for \( \alpha \) and the known shape parameter \( \beta \) is given by

\[ \pi(\alpha | x, \beta) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b \alpha}, \quad a, b > 0, \]  \hspace{1cm} (2.2)

Combining the likelihood function given by Eq.(2.1) and prior function Eq.(2.2), the posterior density of \( \alpha \) given \( x \) is
Bayes estimator based on squared error loss function (SEL) The Bayes estimators $\hat{\alpha}_{\text{BS}}$ of $\alpha$ based on SEL loss function given by

$$\hat{\alpha}_{\text{BS}} = E(\alpha|\mathbf{x}, \beta) = \int_{0}^{\infty} \alpha \pi^*(\alpha|\mathbf{x}, \beta) d\alpha = \frac{a}{b} \int_{a}^{b} (a + 1, b) \frac{d\alpha}{l(a, b)} \quad (2.3)$$

The hazard function $H(t)$ at time under the SEL loss function given by

$$\tilde{H}_{\text{BS}}(t) = E(1|t)|\mathbf{x}, \beta) = e^{\tilde{\theta}} \hat{\alpha}_{\text{BS}} \quad (2.5)$$

Bayes estimator based on Al-Bayyati loss function (ABL) The Bayes estimators $\hat{\alpha}_{\text{BA}}$ of $\alpha$ based on ABL loss function given by

$$\hat{\alpha}_{\text{BA}} = E(\alpha^{q+1}|\mathbf{x}, \beta) = \frac{\int_{0}^{\infty} \alpha^{q+1} \pi^*(\alpha|\mathbf{x}, \beta) d\alpha}{\int_{0}^{\infty} \alpha^{q} \pi^*(\alpha|\mathbf{x}, \beta) d\alpha} = \frac{(a + q) l(a + q + 1, b)}{b l(a + q, b)} \quad (2.6)$$

Similarly, hazard function $H(t)$ at time $t$ under the ABL loss function given by

$$\tilde{H}_{\text{BA}}(t) = \frac{E(H^{q+1}(t)|\mathbf{x}, \beta)}{E(H^{q}(t)|\mathbf{x}, \beta)} = e^{\tilde{\theta}} \hat{\alpha}_{\text{BA}} \quad (2.7)$$

Bayes estimator based on linear-exponential loss function (LINEX)

The Bayes estimators $\hat{\alpha}_{\text{BL}}$ of $\alpha$ based on LINEX loss function given by

$$\hat{\alpha}_{\text{BL}} = \frac{-1}{p} \log(E(e^{-p\alpha}|\mathbf{x}, \beta)) = \frac{-1}{p} \log\left(\int_{0}^{\infty} e^{-p\alpha} \pi^*(\alpha|\mathbf{x}, \beta) d\alpha\right)$$
Similarly, hazard function $H(t)$ at time $t$ under the LINEX loss function given by

$$
\hat{H}_{\beta A}(t) = \frac{1}{p} \log \left( \mathcal{E}(e^{-p\theta(t)}|x, \beta) \right)
= \frac{1}{p} \log \left( \frac{b^\theta \Gamma(a, b + p \theta)}{(b + p)^a \Gamma(a, b)} \right).
$$

Bayes estimator based on generalization of the entropy loss function (GE)

The Bayes estimators $\hat{\theta}_{\beta A}$ of $\theta$ based on GE loss function given by

$$
\hat{\theta}_{\beta A} \equiv \left( \mathcal{E}(\theta^v|x, \beta) \right)^{-\frac{1}{v}}
= \left( \int_0^\infty \alpha^{-v} \mathcal{E}(\alpha|x, \beta) d\alpha \right)^{-\frac{1}{v}}
= \left( \frac{b^v \Gamma(a - v) \Gamma(a - v, b)}{\Gamma(a) \Gamma(a, b)} \right)^{-\frac{1}{v}}.
$$

The hazard function $H(t)$ at time $t$ under the GE loss function given by

$$
\hat{H}_{\beta A}(t) = \left( \mathcal{E}(H^{-\nu}(t)|x, \beta) \right)^{-\frac{1}{\nu}}
= e^{\nu \beta} \hat{H}_{\beta A}.
$$

The case of unknown shape and scale parameters $\alpha$ and $\beta$

For Bayesian estimation, we use here the natural conjugate prior density function for $(\alpha, \beta)$ given by

$$
\pi(\alpha, \beta|x) = \pi_1(\beta) \pi_2(\alpha|\beta)
= \frac{b_1^a b_2^a}{\Gamma(a_1) \Gamma(a_2)} \alpha^{a_2-1} \beta^{a_1-1} e^{-\frac{b_1 a}{\beta} - b_2 \alpha}, \quad a_1, a_2, b_1, b_2 > 0,
$$

where

$$
\pi_1(\beta) = \frac{b_1^a}{\Gamma(a_1)} \beta^{a_1-1} e^{-b_1 \beta} \quad \text{and} \quad \pi_2(\alpha|\beta) = \frac{1}{\Gamma(a_2)} \alpha^{a_2-1} e^{-\frac{b_2 a}{\beta}}.
$$

Combining the likelihood function given by Eq. (2.1) and prior function Eq. (2.12), the posterior density of $\alpha$ and $\beta$ given $x$ is

\[
\pi^* (\alpha, \beta|x) = \frac{L(\alpha, \beta|x) \pi(\alpha, \beta|x)}{\int_0^\infty \int_0^\infty L(\alpha, \beta|x) \pi(\alpha, \beta|x) d\alpha d\beta}.
\]
Bayes estimator based on squared error loss function (SEL)

The Bayes estimators $\hat{\alpha}$ and $\hat{\beta}$ based on SEL loss function are given respectively, by

$$\hat{\alpha}_{BS} = E(\alpha|x) = \int_0^\infty \int_0^\alpha \alpha \pi^*(\alpha, \beta|x) d\alpha d\beta$$

and

$$\hat{\beta}_{BS} = E(\beta|x) = \frac{a_1 a_2 b_1 + a_2+1, b_2}{b_1 b_2 l(a_1, b_1, a_2, b_2)},$$

(2.14)

The hazard function $H(t)$ at time $t$ under the SEL loss function given by

$$\hat{H}_{BS}(t) = E(\|l(t)|x) = \frac{a_1 a_2 b_1 t^{a_1} l(a_1 + 1, b_1 - t, a_2 + 1, b_2)}{b_1 b_2 l(a_1, b_1, a_2, b_2)}. $$

(2.16)

Bayes estimator based on Al-Bayyati loss function (ABL)

The Bayes estimators $\hat{\alpha}$ and $\hat{\beta}$ based on ABL loss function given by

$$\hat{\alpha}_{BA} = \frac{E(\alpha^{t+1}|x)}{E(\alpha^t|x)}$$

$$\hat{\beta}_{BA} = \frac{a_1 l(a_1 + 1, b_1, a_2, b_2)}{b_1 l(a_1, b_1, a_2, b_2)},$$

(2.17)

and

Similarly, hazard function $H(t)$ at time $t$ under the ABL loss function given by

$$\hat{H}_{BA}(t) = \frac{E(H^{t+1}(t)|x)}{E(H^t(t)|x)} = e^{t\beta} \hat{a}_{BA}. $$

(2.19)
Bayes estimator based on linear-exponential loss function (LINEX)

The Bayes estimators $\tilde{\alpha}_{BL}$ and $\tilde{\beta}_{BL}$ based on LINEX loss function given by

$$\tilde{\alpha}_{BL} = \frac{-1}{p} \log \left( E(e^{-p\alpha|X}) \right)$$

$$= \frac{-1}{p} \log \left( \int_0^{\alpha_1} \int_0^{\alpha_2} e^{-p\alpha_1 \alpha_2} \frac{\alpha_1}{(a_1 + l)^{a_1}} \frac{\alpha_2}{(a_2 + l)^{a_2}} \ d\alpha_1 d\alpha_2 \right).$$

(2.20)

and

$$\tilde{\beta}_{BL} = \frac{-1}{p} \log \left( E(e^{-p\beta|X}) \right)$$

$$= \frac{-1}{p} \log \left( \frac{b_1^{a_1} \Gamma(a_1 + l) \Gamma(a_2 + l) \Gamma(a_1 + l, b_1 + p, a_2, b_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(l + 1) \Gamma(l + 1, b_1 + p, a_2, b_2)} \right).$$

(2.21)

Similarly, hazard function $H(t)$ at time $t$ under the LINEX loss function given by

$$\tilde{H}_{BL}(t) = \frac{-1}{p} \log \left( E(e^{-pH(t)|X}) \right)$$

$$= \frac{-1}{p} \log \left( \sum_{l=0}^{\infty} \frac{(-p)^l b_1^{a_1} \Gamma(a_1 + l) \Gamma(a_2 + l) \Gamma(a_1 + l, b_1 + t, a_2 + l, b_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(l + 1) \Gamma(l + 1, b_1 + t, a_2 + l, b_2)} \right).$$

(2.22)

Bayes estimator based on generalization of the entropy loss function (GE)

The Bayes estimators $\tilde{\alpha}_{GE}$ and $\tilde{\beta}_{GE}$ based on GE loss function given by

$$\tilde{\alpha}_{GE} = \left[ E(\alpha^{-v}|X) \right]^{-\frac{1}{v}}$$

$$= \left( \int_0^{\alpha_1} \int_0^{\alpha_2} \alpha^{-v \pi^* (\alpha|X)} \ d\alpha \ d\beta \right)^{-\frac{1}{v}}$$

$$= \left( \frac{(b_1 b_2)^v \Gamma(a_1 - v) \Gamma(a_2 - v) \Gamma(a_1 - v, b_1, a_2 - v, b_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(a_1 + l, b_1 + p, a_2, b_2)} \right)^{-\frac{1}{v}},$$

(2.23)

and

$$\tilde{\beta}_{GE} = \left( E(\beta^{-v}|X) \right)^{-\frac{1}{v}}$$

$$= \left( \frac{(b_1)^v \Gamma(a_1 - v) \Gamma(a_1 - v, b_1, a_2, b_2)}{\Gamma(a_1) \Gamma(a_1 + l, b_1, a_2, b_2)} \right)^{-\frac{1}{v}}.$$

(2.24)

The hazard function $H(t)$ at time $t$ under the GE loss function given by
Numerical results

In this section, we obtained some numerical results to compare the Bayes estimation under SEL, ABL (with $\theta = 0.5$ and $\alpha = -0.5$), LINEX (with $\alpha = 0.5$) and GE (with $\alpha = -0.5$) loss functions according to the following steps:

Algorithm

1. Specify the values of $n, m, r, l, q, p$ and $v$.
2. Specify the values of the parameters $\alpha, \beta$.
3. For given values of the prior parameters.
4. Generate a simple random sample of size $m$ from $U(0,1)$ distribution, $(U_r, \ldots, U_m)$.
5. Determine the values of the censoring scheme $R_i$, $i = r + 1, \ldots, m$.

6- Set $E_i = U_i^{1/m-i+1/R_i}$, $i = r + 1, \ldots, m$.

7. Obtain the General Progressively type-II censored sample $(U^*_{r+1,m:n}, \ldots, U^*_{m, m:n})$, where $U^*_{i,m:n} = 1 - [\prod_{j=m-i+1}^{m} E_i]$, $i = r + 1, \ldots, m$.

8. From steps (4)-(7), the order observations $y_{r+1,m:n}, y_{r+2,m:n}, \ldots, y_{n,m:n}, y_{n+1,m:n}, \ldots, y_{m,m:n}$ are calculated as follows

$$ y_{i,m:n} = \frac{1}{\beta} \log \left[ \frac{\beta}{\alpha} \log(1 - U_{i,m:n}) \right], \quad i = r + 1, \ldots, m. $$

9. We perform Monte Carlo simulation to compare the performances of the different estimators for different sampling schemes. Monte Carlo simulations were performed utilizing 1000 general progressively type-II censored samples for each simulation. The mean squared error (MSE) is used to compare the estimators, where $MSE = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta} - \theta)^2$ and $\hat{\theta}$ is the estimator of $\theta$.

10. Compute the BEs of the model parameters relative to SEL, ABL, LINEXL and GE loss functions.
The following is the generated sample from Gompertz distribution pdf in (1).

<table>
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<tr>
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<th>R</th>
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The case of known parameter $\beta$

Table 1: The values of BIAS and MSE of the Bayesian estimates of $\alpha$ based on SEL, ABL, LINEX and GE loss functions when $\beta = 0.3$ and $0.5$.

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<th>$\beta$</th>
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Table 2: The values of BIAS and MSE of the Bayesian estimates of $H(t)$ when $t = 3$ based on SEL, ABL, LINEX and GE loss functions when $\beta = 0.3$ and $0.5$.

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The case of unknown parameters $\alpha$ and $\beta$

Table 3: The values of BIAS and MSE of the Bayesian estimates of $\alpha$ based on SEL, ABL, LINEX and GE loss functions

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Table 4: The values of BIAS and MSE of the Bayesian estimates of $\beta$ based on SEL, ABL, LINEX and GE loss functions

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Table 5: The values of BIAS and MSE of the Bayesian estimates of $H(t)$ when $t = 3$ based on SEL, ABL, LINEX and GE loss functions

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CONCLUSION

The Bayes estimators under both symmetric and asymmetric loss functions are obtained. When we compare the loss functions, we find that the ABL loss function is better than SEL, LINEX and GE loss functions in tables 1,3,4, and 5, while find that the ABL and LINEX loss functions are better than SEL and GE loss functions in table 2.
REFERENCES


