International Journal of Sociology and Anthropology Research
Vol.6, No.2, pp.13-25, October 2018
Published by European Centre for Research Training and Development UK (www.eajournals.org)

STABILITY ANALYSIS OF PREDATOR-PREY MODEL WITH STAGE STRUCTURE FOR THE PREY: A REVIEW

Ahmed Buseri Ashine

Department of Mathematics, College of Natural and Computational Sciences, Madda Walabu University, Ethiopia; P.O.Box: 247, Bale Robe, Ethiopia, East Africa.

ABSTRACT: *In this paper, a food web model consisting of two predator-one stage structured prey involving Lotka-Voltera type of functional response is proposed and analyzed. It is assumed that the prey growth logistically in the absence of predator. The existence, uniqueness and boundedness of the solution are studied. The existence and the stability analysis of all possible equilibrium points are studied. Suitable Lyapunov functions are used to study the global dynamics of the proposed model.*

KEYWORDS: Food Web, Lyapunovfunction, Stability Analysis, Stage-Structure.

INTRODUCTION

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Over the past decades, Mathematics has made a considerable impact as a tool to model and understand biological phenomena. In return, biologists have confronted the mathematics with variety of challenging problems, which have simulated developments in the theory of nonlinear differential equations. Such differential equations have long played important role in the field of theoretical population dynamics, and they will, no doubt, continue to serve as indispensable tools in future investigations. Differential equation models for interactions between species are one of the classical applications of mathematics to biology. The development and use of analytical techniques and the growth of computer power have progressively improved our understanding of these types of models. Although the predator-prey theory has been much progress, many long standing mathematical and ecological problems remain open[1]. Food chains and food webs depict the network of feeding relationship within ecological communities. During the last few decades, a large number of food-chain and food-web systems have been proposed to describe the food transition patterns and processes [2−4] .Living organisms enter into a variety of relationships, such as Prey-Predator, Competition, Mutualism, Commensalism and so on, among themselves according to the needs of individuals as well as those of species groups. Food webs are one example of interactions that go beyond feeding relationships. Recently, number of researchers have been proposed and studied the dynamics of food webs involving some types of these relationships, for example see [5−12] and the references their in. The study of the consequences of hiding behavior of prey on the dynamics of predator prey interactions can be recognized as a major issue in applied mathematics and theoretical ecology [13, 14].On the other hand, it is well known that, the age factor is importance for the dynamics and evolution of many mammals. The rate of survival, growth and reproduction almost depend on age or development stage and it has been noticed that the life history of many species is composed of at least two stages, (immature and mature). Recently, several of the prey-predator models with stage-structure of species with or without time delays are proposed and analyzed [15−19]. In this paper the food

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

web prey-predator model involving stage-structured of prey is proposed and analyzed, so that the prey growing logistically in the absence of predators. The effect of prey stage-structure on the dynamical behavior of the food web model is investigated theoretically as well as analytically.

The mathematical model

Consider the food web model consisting of two predators-stage structure prey in which the prey species growth logistically in the absence of predation, while the predators decay exponentially in the absence of prey species. It is assumed that the prey population divides into two compartments: immature prey population $P_1(t)$ that represents the population size at time t and mature prey population $P_2(t)$ which denotes to population size at time t. Furthermore the population size of the first predator at time t is denoted by H_1 (t), while H_2 (t) represents the population size of second predator at time t. Now in order to formulate the dynamics of such system the following assumptions are considered:

1) The immature prey depends completely in its feeding on the mature prey that growth logistically with intrinsic growth rate r >0 and carrying capacity K>0. The immature prey individuals grown up and become mature prey individuals with grown up rate β>0. However the mature prey facing death with natural death rate c_1>0.

2) The first and second predators consumed the mature prey individuals only according to the Lotka-Voltera type of functional response with predation rates d_1>0 *and*c_2>0 respectively and contribute a portion of such food with conversion rates $0<e_1<1$ *and* $0<e_2<1$ respectively. Moreover, there is an enter-specific competition between these two predators with competition force rate d_3>0 *and*d_4>0 respectively. Finally in the absence of food the first and second predators facing death with natural death rate c_2>0 *and*c_3>0.

Therefore the dynamics of this model can be represented by the set of first order nonlinear differential equations:

$$\begin{cases} \frac{dP_1}{dt} = rP_1\left(1-\frac{P_2}{K}\right) - \beta P_1 \\ \frac{dP_2}{dt} = \beta P_1 - c_1 P_2 - d_1 P_2 H_1 - d_2 P_2 H_2 \\ \frac{dH_1}{dt} = -c_2 H_1 + e_1 d_1 P_2 H_1 - d_3 H_1 H_2 \\ \frac{dH_2}{dt} = -c_3 H_2 + e_2 d_2 P_2 H_2 - d_4 H_1 H_2 \end{cases} \qquad (1)$$

with initial conditions $P_i(0) \geq 0$. Note that the above proposed model has twelve parameters in all which make the analysis difficult. So in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters:

$t = rT$, $a_1 = \frac{\beta}{r}$, $a_2 = \frac{c_1}{r}$, $a_3 = \frac{c_3}{r}$, $a_4 = \frac{e_1 d_1 K}{r}$, $a_5 = \frac{d_3}{d_2}$, $a_6 = \frac{c_3}{r}$,

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

$a_7 = \frac{e_2 d_2 K}{r}$, $a_8 = \frac{d_4}{d_1}$ $x = \frac{P_1}{K}$, $y = \frac{P_2}{K}$, $z = \frac{d_1 H_1}{r}$, $w = \frac{d_2 H_2}{r}$

Then the non-dimensional form of system (1) can be written as:

$$\begin{aligned}
\frac{dx}{dt} &= \left(\frac{y(1-y)}{x} - a_1\right)x && \cong F_1(x,y,z,w)_1 \\
\frac{dy}{dt} &= \left(\frac{a_1 x}{y} - a_2 - z - w\right)y && \cong F_2(x,y,z,w) \\
\frac{dz}{dt} &= (-a_3 + a_4 - a_5 w)z && \cong F_3(x,y,z,w) \\
\frac{dw}{dt} &= (-a_6 + a_7 y - a_8 z)w && \cong F_4(x,y,z,w)
\end{aligned} \tag{2}$$

with $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$ $and w(0) \geq 0$. It is observed that the number of parameters have been reduced from twelve in the system (1) to seven in the system (2).

The positivity and boundedness

Theorem 3.1. *All solutions of system (2) initiating R_+^4 are positive and uniformly bounded.*

Proof. Let (x(t),y(t),z(t),w(t)) be one of the solution of system (2), then from non-negative initial conditions, the interaction functions of the system are continuous and have continuous partial derivatives on the following positive four dimensional space,

$R_+^4 = \{(x, y, z, w) \in R^4 : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}$.

Hence all solutions starting in R_+^4 remain in R_+^4 for all $t \geq 0$. That is it is Lipschitzian on R_+^4

Next, we will prove the boundedness of the solutions.

Let (x(t),y(t),z(t),w(t)) be any solution of the system (2) with non-negative initial condition $(x_0, y_0, z_0, w_0) \in R_+^4$. Now according to the first equation of system (2) we have :

$\frac{dx}{dt} = y(1-y) - a_1 x$. So, by using the comparison theorem on the above differential inequality with the initial point $x(0) = x_0$ we get: $x(t) \leq \frac{1}{4a_1} + \left(x_0 - \frac{1}{4a_1}\right)e^{-a_1 t}$. Thus, $\lim_{t\to\infty} x(t) \leq \frac{1}{4a_1}$ and hence $\sup x(t) \leq \frac{1}{4a_1}$,∀t>0 ,∀a_1>0 . Now define the function: $V(t) = x(t) + y(t) + \frac{1}{a_1} z(t) + \frac{1}{a_7} w(t)$ and then taken the time derivative of V(t) along the solution of the system (2) we get: $\frac{dV}{dt} \leq \frac{1}{4} + x - bV$, where $b = \min\{1, a_2, a_3, a_6\}$. Then

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

$\frac{dV}{dt} + bV \le M$ where $M = \frac{1}{4} + \frac{1}{4a_1}$. Again by solving this differential inequality for the initial value $V(0) = V_0$, we get: $V(t) \le \frac{M}{b} + \left(V_0 - \frac{M}{b}\right)e^{-bt}$. Then, $\lim_{t\to\infty} V(t) \le \frac{M}{b}$, $\forall t>0$ .Hence all the solutions of system (2) are uniformly bounded , hence the theorem.

The existence of equilibrium points

All equilibrium points of system (2) can be obtained by solving the following equations:

$$\left(\frac{y(1-y)}{x} - a_1\right)x = 0$$
$$\left(\frac{a_1 x}{y} - a_2 - z - w\right)y = 0$$
$$(-a_3 + a_4 - a_5 w)z = 0$$
$$(-a_6 + a_7 y - a_8 z)w = 0$$

These points are as follows:

(1) The trivial equilibrium point E_0 (0,0,0,0),

(2) The first equilibrium point $E_1(\bar{x}, \bar{y}, 0, 0)$,

(3) the first three species equilibrium point $E_2(\tilde{x}, \tilde{y}, \tilde{z}, 0)$,

(4) the first three species equilibrium point $E_3(\hat{x}, \hat{y}, 0, \hat{w})$,

(5) the positive equilibrium point $E_4(x^*, y^*, z^*, w^*)$,

Where

$$\bar{x} = \frac{a_2}{a_1}(1-a_2) \; and \; \bar{y} = 1 - a_2$$

$$\tilde{x} = \frac{a_3}{a_1 a_4}\left(\frac{a_4 - a_3}{a_4}\right), \quad \tilde{y} = \frac{a_3}{a_4} \; and \; \tilde{z} = \frac{a_4(1-a_2) - a_3}{a_4}$$

$$\hat{x} = \frac{a_6}{a_1 a_7}\left(\frac{a_7 - a_6}{a_7}\right), \quad \hat{y} = \frac{a_6}{a_7} \; and \; \hat{w} = \frac{a_7(1-a_2) - a_6}{a_7}$$

$$x^* = \frac{y^*}{a_1}(1-y^*), \; y^* = \frac{1 - a_2 + \frac{a_3}{a_5} + \frac{a_6}{a_8}}{1 + \frac{a_4}{a_5} + \frac{a_7}{a_8}}, \; z^* = \frac{a_7}{a_8}y^* - \frac{a_6}{a_8} \; and \; w^* = \frac{a_4}{a_5}y^* - \frac{a_3}{a_5}$$

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

Remark 4.1. The equilibrium point $E_0 = (0, 0, 0,0)$, which is known as the vanishing point is always exist.

Remark 4.2. The necessary and sufficient condition for the existence of the first equilibrium point $E_1(\bar{x}, \bar{y},0,0)$, in Int. R_+^2 of $xy-plane$ is $a_2 < 1$

Remark 4.3. The first three species equilibrium point $E_2(\tilde{x}, \tilde{y}, \tilde{z},0)$ exists uniquely in Int. R_+^3 of

$xyz-space$ under the following necessary and sufficient condition :

$a_3 < \min\{a_4, a_4(1-a_2)\}$.

Remark 4.4. The second three species equilibrium point $E_3(\hat{x}, \hat{y},0, \hat{w})$ exists uniquely in Int. R_+^3 of $xyw-space$ under the following necessary and sufficient condition:

$a_6 < \min\{a_7, a_7(1-a_2)\}$

Remark 5.5. Finally, the positive equilibrium point $E_4(x^*, y^*, z^*, w^*)$ exists in Int. R_+^4 under the following necessary and sufficient conditions:

i) $1+\frac{a_4}{a_5}+\frac{a_7}{a_8} > 1-a_2+\frac{a_3}{a_5}+\frac{a_6}{a_8}$

ii) $a_7\left(1-a_2+\frac{a_3}{a_5}+\frac{a_6}{a_8}\right) > a_6\left(1+\frac{a_4}{a_5}+\frac{a_7}{a_8}\right)$

iii) $a_4\left(1-a_2+\frac{a_3}{a_5}+\frac{a_6}{a_8}\right) > a_3\left(1+\frac{a_4}{a_5}+\frac{a_7}{a_8}\right)$

The stability Property

Local stability Analysis

In this section, we will study the local stability of the equilibrium points of system (2).The Jacobian matrix J(x, y, z, w) of system (2) is given by:

$$J(x,y,z,w) = \begin{pmatrix} -a_1 & 1-y & 0 & 0 \\ a_1 & -a_2-z-w & -y & -y \\ 0 & a_4 z & -a_3+a_4-a_5 w & -a_5 z \\ 0 & a_7 w & -a_8 w & -a_6+a_7 y-a_8 z \end{pmatrix}$$

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

Proposition 5.1.1. If $a_2 < 1$, then E_0 is locally asymptotically stable.

*Proof.*The Jacobian matrix of system (2) at E_0 can be written as

$$J(E_0) = \begin{pmatrix} -a_1 & 1 & 0 & 0 \\ a_1 & -a_2 & 0 & 0 \\ 0 & 0 & -a_3 & 0 \\ 0 & 0 & 0 & -a_6 \end{pmatrix}$$

Then the characteristic equation of (E_0)is given by: $(\lambda^2 + A\lambda + B)(-a_3 - \lambda)(-a_6 - \lambda) = 0$

Where $A = a_1 + a_2$, $B = a_1(a_2 - 1)$, Here, either $(\lambda^2 + A\lambda + B) = 0$ or $(-a_3 - \lambda)(-a_6 - \lambda) = 0$

The eigenvalues are $\lambda_{0z} = -a_3 1$, $\lambda_{0w} = -a_6$, $\lambda_{0x} = \frac{A}{2} + \frac{\sqrt{A^2 - 4B}}{2}$, and $\lambda_{0y} = \frac{A}{2} - \frac{\sqrt{A^2 - 4B}}{2}$

Clearly, the trivial equilibrium pointE_0 (0, 0, 0, 0) is locally asymptotical stable in the R_+^4 and saddle point otherwise.

Proposition 5.1.2.E_1 is locally asymptotically stable in the R_+^4 if $\frac{1-a_2}{2} < \bar{y} < \min\left\{\frac{a_3}{a_4}, \frac{a_6}{a_7}\right\}$

Proof. The proof is the same in the same in proposition 5.1.

Proposition 5.1.3. $E_2$is locally asymptotically stable if and only if conditions

i) $a_7\tilde{y} < a_6 + a_8\tilde{z}$, and

ii) $a_4[a_2a_3 + a_1(1 + a_1) + a_4(1 - a_2)] + a_3^{\,2} > a_3(a_1 + 2a_4)$ are hold.

*Proof.*The Jacobian matrix of system (2) at E_2 can be written as:

$$J(E_2) = \begin{pmatrix} -a_1 & 1-\tilde{y} & 0 & 0 \\ a_1 & -a_2 - \tilde{z} & -\tilde{y} & -\tilde{y} \\ 0 & a_4\tilde{y} & -a_3 + a_4\tilde{y} & -a_5\tilde{z} \\ 0 & 0 & 0 & -a_6 + a_7\tilde{y} - a_8\tilde{z} \end{pmatrix}$$

Then the characteristic equation of $J(E_2)$ is given by:

$$[\lambda^3 + \tilde{A}\lambda^2 + \tilde{B}\lambda + \tilde{C}](-a_6 + u_7\tilde{y} - a_8\tilde{z} - \lambda) = 0$$

Where $\tilde{A} = 1 + a_1 - \frac{a_3}{a_4}$,

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

$$\tilde{B} = \frac{a_1[a_4(a_2 - 1) + 2a_3]}{a_4} + \frac{(a_1 + a_3)[a_4(1 - a_2) - a_3]}{a_4}$$

$$\tilde{C} = \frac{a_1 a_3[a_4(1 - a_2) - a_3]}{a_4}$$

So, either $\lambda^3 + \tilde{A}\lambda^2 + \tilde{B}\lambda + C = 0$ or $-a_6 + u_7\tilde{y} - a_8\tilde{z} - \lambda = 0$

From the second equation we obtain that: $\lambda_{2w} = -a_6 + a_7\tilde{y} - a_8\tilde{z}$, which is negative if $a_7\tilde{y} < a_6 + a_8\tilde{z}$. On the other hand by using Routh-Hawirtiz criterion the first equation has roots(eigenvalues) with negative real parts if and only if $\tilde{A} > 0$, $\tilde{C} > 0$ and $\Delta = \tilde{A}\tilde{B} - \tilde{C} > 0$. Now a direct computation gives that:

$\Delta = \dfrac{a_3[a_4[a_2 a_3 + a_1(1 + a_1) + a_4(1 - a_2)] + a_3{}^2]}{a_4{}^2} - \dfrac{a_3{}^2[a_1 + 2a_4]}{a_4{}^2}$. So, $\Delta > 0$. Under the condition $a_4[a_2 a_3 + a_1(1 + a_1) + a_4(1 - a_2)] + a_3{}^2 > a_3(a_1 + 2a_4)$. And one can easily verify that $\tilde{A} > 0$, $\tilde{C} > 0$ from remark 4.3. Then all the eigenvalues λ_{2x}, λ_{2y} and λ_{2z} of the second equation have negative real part.So, E_2 is locally asymptotically stable if and only if the above conditions are hold. However , it is a saddle point otherwise.

Proposition5.1.4. E_3 is locally asymptotically stable if and only if conditions

i) $a_4\hat{y} < a_3 + a_5\hat{w}$, and

ii) $a_7[a_1(1 + a_1 + a_6) + a_7(1 - a_2)] > a_6[a_1 + a_7(2 + a_1 - a_2)]$ are hold.

Proof. The Jacobian matrix of system (2) at E_3 can be written as:

$$J(E_3) = \begin{pmatrix} -a_1 & 1 - 2\hat{y} & 0 & 0 \\ a_1 & -a_2 - \hat{w} & -\hat{y} & -\hat{y} \\ 0 & 0 & -a_3 + a_4\hat{y} - a_5\hat{w} & 0 \\ 0 & a_7\hat{w} & -a_8\hat{w} & -a_6 + a_7\hat{y} \end{pmatrix}$$

Then the characteristic equation of $J(E_3)$ is given by:

$$[\lambda^3 + \hat{A}\lambda^2 + \hat{B}\lambda + \hat{C}](-a_3 + u_4\hat{y} - a_5\tilde{w} - \lambda) = 0$$

Where $\hat{A} = 1 + a_1 - \dfrac{a_6}{a_7}$,

$$\hat{B} = \frac{a_1[a_7(a_2 - 1) + 2a_2]}{a_7} + \frac{(a_6 - a_1)[a_7(1 - a_2) - a_6]}{a_7}$$

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

$$\widehat{C}=\frac{a_1a_6[a_7(1-a_2)-a_6]}{a_7}-\frac{a_1(1+a_2)[a_7-2a_6]}{a_7}$$

So, either $\lambda^3+\widehat{A}\lambda^2+\widehat{B}\lambda+\widehat{C}=0$ or $-a_3+u_4\widehat{y}-a_5\widetilde{w}-\lambda=0$. From the second equation we obtain that: $\lambda_{3z}=-a_3+a_4\widehat{y}-a_5\widehat{w}$, which is negative provided that $a_4\widehat{y}<a_3+a_5\widehat{w}$. On the other hand it is easy to verify that $\widehat{A}>0$, $\widehat{C}>0$ by remark 4.4. While $\Delta=\widehat{A}\widehat{B}-\widehat{C}>0$ under the condition: $a_7[a_1(1+a_1+a_6)+a_7(1-a_2)]>a_6[a_1+a_7(2+a_1-a_2)]$.Then all the eigenvalues λ_{3x}, λ_{3y}, and λ_{3w} of the second equation have negative real parts. So, E_3 is locally asymptotically stable if and only if the above two conditions are hold. However, it is a saddle point otherwise.

Theorem 5.1.5. Assume that the positive equilibrium point $E_4(x^*,y^*,z^*,w^*)$ of system (2) exists

in $Int\,R_+^4$. Then it is locally asymptotically stable provided that the following conditions hold:

i) $\tau_{11}{}^2<\dfrac{4\tau_{11}\tau_{22}}{3}$

ii) $\tau_{23}{}^2<\dfrac{2\tau_{22}\tau_{33}}{3}$

iii) $\tau_{24}{}^2<\dfrac{2\tau_{22}\tau_{44}}{3}$

iv) $\tau_{33}\tau_{44}<\tau_{34}{}^2$

Where, $\tau_{11}=a_1$, $\tau_{12}=a_2+1-2y^*$, $\tau_{22}=a_2+z^*+w^*$, $\tau_{23}=a_4z^*-y^*$, $\tau_{24}=a_7w^*-y^*$,

$\tau_{33}=a_3-a_4y^*+a_5w^*$, $\tau_{34}=-a_5z^{.*}-a_8w^*$, $\tau_{44}=a_6-a_7y^*+a_8z^*$

*Proof.*The Jacobian matrix of system (2) at E3 can be written as:

$$J(E_4)=\begin{pmatrix}-a_1 & 1-2y^* & 0 & 0\\ a_1 & -a_2-y^*-w^* & -y^* & -y^*\\ 0 & a_4y^* & -a_3+a_4y^*-a_5w^* & -a_5z^*\\ 0 & a_7w^* & -a_8w^* & -a_6+a_7y^*-a_8z^*\end{pmatrix}$$

One can easily verify that, the linearized system (2) can be written as:

$\dfrac{dX}{dt}=\dfrac{dS}{dt}=J(E_4)S$ Here, $X=(x,y,z,w)$ and $S=(s_1,s_2,s_3,s_4)^t$,

Where, $s_1=x_1-x_1{}^*$, $s_2=x_2-x_2{}^*$, $s_3=x_3-x_3{}^*$ and $s_4=x_4-x_4{}^*$ .Now

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

Consider the following positive definite function:

$V = \frac{s_1^2}{2} + \frac{s_2^2}{2} + \frac{s_3^2}{2} + \frac{s_4^2}{2}$.Clearly, $V : R_+^4 \to S$ and a C^1 positive definite function. Now by differentiating V with respect to time t and doing some algebraic manipulation give that:

$$\frac{dV}{dt} = -\tau_{11}s_1^2 + \tau_{12}s_1s_2 - \tau_{22}s_2^2 + \tau_{23}s_2s_3 + \tau_{24}s_2s_4 - \tau_{33}s_3^2 + \tau_{34}s_3s_4 - \tau_{44}s_4^2$$

Now it is easy to verify that the above set of condition guarantee the quadratic term given below:

$$\frac{dV}{dt} = -\left[\sqrt{\tau_{11}}s_1 - \frac{\sqrt{\tau_{22}}}{\sqrt{3}}s_2\right]^2 - \left[\frac{\sqrt{\tau_{22}}}{\sqrt{3}}s_2 - \frac{\sqrt{\tau_{33}}}{\sqrt{2}}s_3\right]^2 - \left[\frac{\sqrt{\tau_{22}}}{\sqrt{3}}s_2 - \frac{\sqrt{\tau_{44}}}{\sqrt{2}}s_4\right]^2 - \left[\frac{\sqrt{\tau_{33}}}{\sqrt{2}}s_3 + \frac{\sqrt{\tau_{44}}}{\sqrt{2}}s_4\right]^2$$

So, $\frac{dV}{dt}$ is negative definite, hence V is a Lyapunov function. Thus,E_4 is locally asymptotically stable and the proof is complete.

Global Stability Analysis

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable, of system (2) is studied analytically with the help of Lyapunov method as shown in the following theorems.

Theorem 5.2.1. Assume that the equilibrium point E_0 (0, 0, 0,0) of system (2) is locally asymptotically stable in the R_+^4. Then the equilibrium point E_0 of system (2) is globally asymptotically stable.

Proof. Consider the following function

$V_1 = k_1x + k_2y + k_3z + k_4w$, where $k_1, k_2, k_3,$ and k_4 are positive constants to be determined.

Clearly, $V_1 : R_+^4 \to S$ and a C^1 positive definite function. Now by differentiating V_1 with respect to time t and doing some algebraic manipulation, give that:

$$\frac{dV_1}{dt} = k_1y(1-y) + a_1(k_2 - k_1)x - k_2a_2y + (k_3a_4 - k_2)yz - k_2a_2y +$$
$$(k_4a_7 - k_2)yw - k_3a_3z - k_4a_6w - (k_3a_5 + k_4a_8)wz$$

By choosing $k_1 = k_2 = 1$, $k_3 = \frac{1}{a_4}$, $k_4 = \frac{1}{a_7}$ .We get: $\frac{dV_1}{dt} \le -(a_2 - 1)y$. Then we obtain that $\frac{dV_1}{dt}$ is negative definite and hence V_1 is a Lyapunov function. Thus E_0 is globally asymptotically stable and the proof is complete.

Theorem 5.2.2. Assume that the equilibrium point $E_1(\bar{x}, \bar{y}, 0, 0)$ of system (2) is locally asymptotically stable in the $IntR_+^2$, Then E_1 is globally asymptotically stable on any region $\Omega_1 \subset IntR_+^2$ that satisfies the following three conditions:

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

$$\frac{1}{x}-\frac{y+\bar{y}}{\bar{x}}+\frac{a_1}{y\bar{y}}\leq 2\sqrt{\frac{a_1}{xy\bar{y}}}$$

$$\frac{y+\bar{y}}{\bar{x}}<\frac{1}{\bar{x}}+\frac{a_1}{y\bar{y}}$$

$$\frac{y^2(x-\bar{x})^2}{x\bar{x}}<\left\{\sqrt{\frac{(1-y)y}{x\bar{x}}}(x-\bar{x})-\sqrt{\frac{a_1\bar{x}}{y\bar{y}^2}}(y-\bar{y})\right\}^2$$

Proof. Consider

$$V_2(x,y,z,w)=k_1\left(x-\bar{x}-\bar{x}\ln\frac{x}{\bar{x}}\right)+k_2\left(y-\bar{y}-\bar{y}\ln\frac{y}{\bar{y}}\right)+k_3z+k_4w$$

where $k_1, k_2, k_3,$ and k_4 are positive constants to be determined.Clearly, $V_1 : R_+^4 \to S$ and a C^1 positive definite function. Now by differentiating V_2 with respect to time t and doing some algebraic manipulation, give that:

$$\frac{dV_2}{dt}=-k_1\frac{(x-\bar{x})^2\bar{y}}{x\bar{x}}+\left[\frac{k_1}{x}-\frac{k_1(y+\bar{y})}{\bar{x}}+\frac{k_2a_1}{y}\right](x-\bar{x})(y-\bar{y})-k_1\frac{\bar{y}}{x\bar{x}}(x-\bar{x})^2-k_2\frac{a_1\bar{x}}{y\bar{y}}(y-\bar{y})^2$$

$$+k_2\frac{a_1}{\bar{}}(x-\bar{x})(y-\bar{y})-k_2(y-\bar{y})z-k_2(y-\bar{y})w-k_3a_3z+k_3a_4y$$

$$-k_4a_6w+k_5a_7yw-(k_3a_5+k_4a_8)wz$$

By choosing $k_1=1$, $k_2=\frac{1}{\bar{y}}$ $k_3=\frac{1}{a_3}$, $k_4=\frac{1}{a_6}$. We get:

$$\frac{dV_2}{dt}\leq-\left[\sqrt{\frac{\bar{y}}{x\bar{x}}}(x-\bar{x})-\sqrt{\frac{a_1\bar{x}}{y\bar{y}^2}}(y-\bar{y})\right]^2+\frac{y^2(x-\bar{x})^2}{x\bar{x}}-\left[\frac{(a_3-a_4)\bar{y}}{a_3\bar{y}}\right]yz--\left[\frac{(a_6-a_7)\bar{y}}{a_6\bar{y}}\right]yw$$

So, according to proposition 5.2.1, we obtain that:

$$\frac{dV_2}{dt}\leq-\left[\sqrt{\frac{\bar{y}}{x\bar{x}}}(x-\bar{x})-\sqrt{\frac{a_1\bar{x}}{y\bar{y}^2}}(y-\bar{y})\right]^2+\frac{y^2(x-\bar{x})^2}{x\bar{x}}.$$ However, the first two conditions guarantee the completeness of the quadratic term between x . So, if the third condition holds then we obtain that $\frac{dV_2}{dt}$ is negative definite on the region Ω_1 and hence V_2 is a Lyapunov function defined on the region Ω_1. Thus E_1 is globally asymptotically stable on the region Ω_1 and the proof is complete.

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

Theorem 5.2.3. Assume that the equilibrium point $E_2(\tilde{x},\tilde{y},\tilde{z},0)$ of system (2) is locally asymptotically stable in $IntR_+^3$. Then E_2 is globally asymptotically stable on any region $\Omega_2 \subset IntR_+^3$ that satisfied the following three conditions:

$$\frac{1}{x}+\frac{a_1}{y}-\frac{(y-\tilde{y})}{\tilde{x}} \le 2\sqrt{\frac{a_1}{xy}}$$

$$\frac{y+\tilde{y}}{\tilde{x}} < \frac{1}{x}+\frac{a_1}{y}$$

$$\frac{y^2(x-\tilde{x})^2}{\tilde{x}} < \left[\sqrt{\frac{\tilde{y}}{x\tilde{x}}}(x-\tilde{x})-\sqrt{\frac{a_1\tilde{x}}{y\tilde{y}}}(y-\tilde{y})\right]^2$$

In addition to these $\frac{a_5\tilde{z}}{a_4}+\tilde{y} < \frac{a_6}{a_7}$ also true.

Proof. The same as the proof of theorem 5.2.1, above.

Theorem 5.2.4. Assume that the equilibrium point $E_3(\hat{x},\hat{y},0,\hat{w})$ of system (2) is locally asymptotically stable in $IntR_+^3$ .Then E_3 is globally asymptotically stable on any region $\Omega_3 \subset IntR_+^3$ that satisfy the following three conditions:

$$\frac{1}{x}-\frac{(y-\hat{y})}{\hat{x}}+\frac{a_1}{y} \le 2\sqrt{\frac{a_1}{xy}}$$

$$\frac{y+\hat{y}}{\hat{x}} < \frac{1}{x}+\frac{a_1}{y}$$

$$\frac{y^2(x-\hat{x})^2}{x\hat{x}} < \left[\sqrt{\frac{\hat{y}}{x\hat{x}}}(x-\hat{x})-\sqrt{\frac{a_1\hat{x}}{y\hat{y}^2}}(y-\hat{y})\right]^2$$

In addition to these conditions, $\frac{a_8}{a_7}\hat{w}+\hat{y} < \frac{a_3}{a_4}$ also holds.

Theorem 5.2.5. Assume that the equilibrium point $E_4(x^*,y^*,z^*,w^*)$ of system (2) is locally asymptotically stable in $IntR_+^4$ .Then E_4 is globally asymptotically stable on any region $\Omega_4 \subset IntR_+^4$ that satisfy the following three conditions:

$$\frac{1}{x}-\frac{(y-y^*)}{x^*}+\frac{a_1}{y} \le 2\sqrt{\frac{a_1}{xy}}$$

$$\frac{y+y^*}{x^*} < \frac{1}{x}+\frac{a_1}{y}$$

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

$$\frac{y^2}{xx^*}(x-x^*)^2+\left(1+\frac{a_8}{a_7}\right)(wz^*w^*z)<\left[\sqrt{\frac{y^*}{xx^*}}(x-x^*)-\sqrt{\frac{a_1x^*}{yy^*}}(y-y^*)\right]^2$$

Proof. Consider the following functions

$$V_3(x,y,z,w)=k_1\left(x-x_4-x_4\ln\frac{x}{x_4}\right)+k_2\left(y-y_4-y_4\ln\frac{y}{y_4}\right)+k_3\left(z-z_4-z_4\ln\frac{z}{z_4}\right)$$
$$+k_4\left(w-w_4-w_4\ln\frac{w}{w_4}\right)$$

where $k_1, k_2, k_3,$ and k_4 are positive constants to be determined.

Clearly, $V_1: R_+^4 \to R$ and a C^1 positive definite function. Now by differentiating V_3 with respect to time t and doing some algebraic manipulation, give that:

$$\frac{dV_3}{dt}=-k_1\frac{y^*}{xx^*}(x-x^*)^2+\left[\frac{k_1}{x}-\frac{k_1(y+y^*)}{x^*}+\frac{k_2a_1}{y}\right](x-x^*)(y-y^*)-k_2\frac{a_1x^*}{yy^*}(y-y^*)^2$$
$$+\frac{k_1y^2}{xx^*}(x-x^*)(y-y^*)-(k_2a_8+k_3a_4)(w-w^*)(z-z^*)+(k_4a_7-k_2)(y-y^*)(w-w^*)$$
$$+(k_3a_4-k_2)(y-y^*)(z-z^*)$$

By choosing $k_1=k_2=1,\ k_3=\frac{1}{a_4}\ k_4=\frac{1}{a_7}$, We get:

$$\frac{dV_3}{dt}\le-\left[\sqrt{\frac{y^*}{xx^*}}(x-x^*)-\sqrt{\frac{a_1x^*}{yy^*}}(y-y^*)\right]^2+\frac{y^2}{xx^*}(x-x^*)^2+\frac{1}{a_7}(a_7+a_8)(wz^*+w^*z)$$

However, the first two conditions guarantee the completeness of the quadratic term between xa . So, if the third condition holds then we obtain that $\frac{dV_3}{dt}$ is negative definite on the region Ω_3 and hence V_3 is a Lyapunov function defined on the region Ω_3. Thus E_4 is globally asymptotically stable on the region $\Omega_4$and the proof is complete.

CONCLUSION AND FUTURE WORK

In this paper, we have considered a prey-predator system incorporating a stage structure of prey. It is assumed that the predator species prey upon the prey according to Lotka-Voltera type of functional response. The existence, uniqueness and boundedness of the solution of the system are discussed. The existence of all possible equilibrium points is studied. The local and global dynamical behaviors of the system are studied analytically. For the future, a numerical simulation has to be done for system (2) for different sets of parameters and different set of initial points to confirm the obtained analytical results.

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)

REFERENCES

[1] T. K. Kar and Ashim Batabyal, Persistence and stability of a two prey one predator, International Jornal of Engineering, Science and Technology, Vol.2, No.2, 2010, pp.174-190.
[2] Weiming Wang, Hailing Wan and Zhenqing Li, Chaotic behavior of a three-species beddington-type system with impulsive perturbations, Chaos, Solution and Fractals 37, 2008, 438-443.
[3] Sunita Gakkar, Brahampal Singh and Raid Kamel Naji, Dynamical behavior of two predators competing over a single prey, BioSystems, 2007, 90.808-817.
[4] Jawdat Alebraheem and YahYa Abu-Hasan, Presistence of predators in a two predators-one prey model with non-preiodic solution, App1.Math.Scie.,Vol.6, No.19, 2012, 943-956.
[5] Madhusudanan V., Viya S. and Gunasekaran M, Impact of harvesting in three species food web model with two distinct functional response, International Journal of Innovative Research in Science Engineering and Technology,Vol.3, Issue 2,2014, p.9505-9513.
[6] G. Iovane. Two-prey one-predator model, Chaos, Solitons & Fractals (2007),doi:10.1016/j.chaos.2007.06.058.
[7] Ahmed Buseri Ashine. Predator-prey interactions with disease in predator incorporating harvesting. Global Journals. Inc.(US)Vol. 17, Iissue 2, 2017, p. 23 – 28.
[8] T. K. Kar. A Dynamic Reaction Model of a Prey-predator System with Stage-structure for Predator. Vol 4. No. 5, 2010,p. 183 – 195.
[9] R. K. Naji and I.H. Kasim, The dynamics of food web model with defensive switching property, Nonlinear Analysis: Modelling and Control, Vol.13, No.2, 2008 , 225-240.
[10] B-Dubey and R.K. Upadhyay, Persistence and extinction of one-prey and two-predators system, Nonlinear Analysis: Modeling and Control, Vol.9, No.4, 2ew004 , 307-329.
[11] Ahmed Buseri Ashine. On prey-predator model with Holling type II and Leslie-Gower schemes. Trans stellar Research Publishers and consultancy. Vol. 3 Issue 1, 2016, p. 27 – 34.
[12] Sze-Bi Hsu, Shigui Ruan and Ting-Hui Yang ,Analysis of three species Lotka-Voltera food web models with omnivory, J. Math. Anal. Applic. 426,2015 ,659-687.
[13] Maynard Smith, Models in ecology Cambridge University Cambridge 1974.
[14] M. P. Hassell , The dynamics of arthropod predator-prey systems Princeton University Princeton NJ 1978.
[15] Kai Hong and Peixuam Weng, Stability and traveling waves of diffusive predator-prey model with age-structure and nonlocal effect, Jornal of Applied Analysis and Computation, Vol.2, No.2, 2012 , p.173-192.
[16] Rui Xu, Global stability and Hopf bifurcation of a predator-prey model with stage structure and delayed predator response, Nonlinear Dynamics, Vol.67, Issue 2, 2012 , pp.1683-1693.
[17] O. P. Misra, Poonam Sinha and Chhatrapal Singh, Stability and bifurcation analysis of a prey-predator model with age based predation, Applied Mathematical Modelling 37, 2013 , 6519-6529.
[18] Ye Kaili and Song XinYu, Predator-prey system with stage structure and delay, Appl. Math. J. Chinese Univ. Ser. B, 18(2) ,2003, 143-150.
[19] Paul-Georgescu and Ying-Hen Hsieh, Global dynamics of a predator-prey model with stage structure for the predator, SIAMJ.App1.Math.Vol.67, No.5, 2007 , pp.1379-1395.

ISSN: 2053-4108(Print), ISSN: 2053-4116(Online)