International Research Journal of Pure and Applied Physics
Vol.4, No.4, pp.7-21, December 2016
Published by European Centre for Research Training and Development UK (www.eajournals.org)

SOME NEW CHARACTERIZATIONS OF SPACE CURVES ACCORDING TO TYPE-1 BISHOP FRAME IN EUCLIDEAN 4- SPACE E_1^4

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ABSTRACT: *In this paper, the position vectors of space curve with constant curvatures was investigate according to type-1 Bishop Frame in the Euclidean 4- Space E_1^4. And some characterizations of the Space Curves with constant curvatures spanned by subspaces of E_1^4 are given.*
KEYWORDS: Euclidean Space E_1^4; type-1 Bishop Frame; *Space Curves Characterizations*.

INTRODUCTION

There are many authors have been studied the position vectors of space curves with an arbitrary constant curvatures with respect to Frenet frame in Euclidean 4-space an also they studied geometry of curves and his characterizations. In [5] Kazim was studied some characterizations of the curve that their position vector lied in the rectifying curves in the Minkowski 3-space. In [2] Authors was discus the position vectors space curve with constant curvatures in Euclidean 4-space. In [3] Fathi studied the position vectors of space curves in Euclidean 3-Space depending on the Type-1 Bishop Frame with constant Curvatures. The general helix position vector with respect to Frenet frame in Euclidean 3-Space studied by A. T. Ali In [1], in addition to that the natural representation of a general helix in terms of the curvature and torsion was deduced. The spacelike normal curves in Minkowski 4-Space E_1^4 , in according to some characterizations of spacelike normal curves with spacelike, and null principal normal was studied by K. Ilarslan [6]. In [4] Hüseyin Kocayiğit and others, they gave us some characterizations of timelike curves according to Bishop Frame in Minkowski 3-space $\boldsymbol{E_1^3}$. some characterizations for spacelike helices was given in Minkowski space-time $\boldsymbol{E_1^4}$ by authors in [8].

PRELIMINARIES

In this work E_1^4 denote to the Minkowski space together with a metric $\langle\,,\rangle$ of sign nature $(-,+,+,+)$. For each: $X=(x_1,x_2,x_3,x_4)\in E_1^4$,$Y=(y_1,y_2,y_3,y_4)\in E_1^4$

The standard scalar product in the Euclidean 4- space E_1^4 given by:

$$\langle X,Y\rangle = x_1y_1+x_2y_2+x_3y_3-x_4y_4 \qquad (2.1)$$

Since a vector X is said to be Timelike if $\langle X,X\rangle<0$, Spacelike if $\langle X,X\rangle>0$ and null(Lightlike) if $\langle X,X\rangle=0$ and $X\neq 0$. And can define the norm of a vector X as $\|X\|=\sqrt{|\langle X,X\rangle|}$

Take the arbitrary space curve in E_1^4 as $\alpha: I\subset R\to E_1^4$. Recall that the curve α is said to be of unit speed (or parameterized by arc length function s) if $\langle\alpha'(s),\alpha'(s)\rangle=1$

Let $\{t,n_1,n_2,b\}$ Denoted the Frenet moving frame along the unit Speed curve α. Where the vectors t, n_1, n_2 *and* b are mutually orthogonal vectors as in [7].

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

$$\left.\begin{aligned} \acute{t} &= \kappa n_1 \\ \acute{n}_1 &= -\kappa t + \tau n_2 \\ \acute{n}_2 &= -\tau n_1 + \sigma b \\ b &= -\sigma n_2 \end{aligned}\right\} \qquad (2.2)$$

Let $\{T, N_1, N_2, B\}$ Denoted the type-1 Bishop moving frame along the unit Speed curve α. Where T, N_1, N_2 and B are orthogonal vectors satisfying

$$\langle T, T\rangle = \langle N_1, N_1\rangle = \langle N_2, N_2\rangle = 1, \langle B, B\rangle = -1$$

Then as in [3] the type-1 Bishop formulas and satisfying α are:

$$\left.\begin{aligned} \acute{T} &= k_1 N_1 + k_2 N_2 + k_3 N_3 \\ \acute{N}_1 &= -k_1 T \\ \acute{N}_2 &= -k_2 T \\ B &= -k_3 T \end{aligned}\right\} \qquad (2.3)$$

Where k_1 , k_2 and k_3 are the first, second and third Bishop curvature.

SOME CHARACTERIZATIONS OF A SPACELIKE CURVES WITH CONSTANT CURVATURES ACCORDING TO TYPE-1 BISHOP FRAME IN EUCLIDEAN 4- SPACE E_1^4

In this section we will give some investigation to characterize the Space curves that is lie on some subspaces of E_1^4 . Let $\propto$ be a Spacelike curve in E_1^4 with the type-1 Bishop frame$\{T, N_1, N_2, B\}$.

Theorem 3.1: If we suppose that the curve which is lies on the subspace spanned by $\{T, N_1\}$ are: $\propto$: R → E_1^4 , then the position vector of Space curve $\propto$ can be written in the form:

$$\propto (s) = \frac{-1}{k_1} N_1(s) \qquad (3.1)$$

If $k_3 \neq 0$ and $\lambda = 0$
And can be written in the form:

$$\propto (s) = [c_1 cosk_1 s + c_2 sink_1 s]T(s) + \left[c_2 cosk_1 s - c_1 sink_1 s - \frac{sin(k_1 s)^2}{k_1}\right] N_1(s) \qquad (3.2)$$

If $k_2 = 0$, $k_3 = 0$ and $\lambda \neq 0$
Proof: We can write the position vector in the form:

$$\propto (s) = \lambda(s)T(s) + \mu(s)N_1(s) \qquad (3.3)$$

Where $\lambda = \lambda(s)$ and $\mu = \mu(s)$ are differentiable functions of s. The equation (3.3) after differentiation become:

$$\propto' = \lambda' T + \lambda T' + \mu' N_1 + \mu N_1{}' \qquad (3.4)$$

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

From the equations (2.3) and (3.4) we find:

$$\begin{cases} \lambda' - \mu k_1 = 1 \\ \lambda k_1 + \mu' = 0 \\ \lambda k_2 = 0 \\ \lambda k_3 = 0 \end{cases} \qquad (3.5)$$

If: $k_3 \neq 0 \rightarrow$ $\lambda = 0$ (3.6)

and $\mu(s) = \frac{-1}{k_1}$ (3.7)

We get the equation (3.1) after substituting equations (3.6) and (3.7) into (3.3)

And if $\lambda \neq 0 \rightarrow k_3 = 0$ *and* $k_2 = 0$

The solution of the first two equations of (3.5) give:

$$\mu = c_2 cosk_1 s - c_1 sink_1 s - sin(k_1 s)^2 \qquad (3.8)$$
$$\lambda = c_1 cosk_1 s + c_2 sink_1 s \qquad (3.9)$$

We get the equation (3.2) after substituting equations (3.8) and (3.9) into (3.3)

Theorem 3.2: If we suppose that the curve which is lies on the subspace spanned by $\{T, N_2\}$ *are*: $\propto$: R $\rightarrow$ E_1^4 , then the position vector of Space curve $\propto$ can be written in the form:

$$\propto (s) = [c_1 cosk_2 s + c_2 sink_2 s]T(s) + \left[c_2 cosk_2 s - c_1 sink_2 s - \frac{sin(k_2 s)^2}{k_2}\right] N_2(s) \qquad (3.10)$$

If $k_3 = 0$ *and* $k_1 = 0$

And can be written in the form:

$$\propto (s) = \frac{-1}{k_2} N_1(s) \qquad (3.11)$$

If: $k_3 \neq 0$ *and* $k_1 \neq 0$

Proof: We can write the position vector in the form:

$$\propto (s) = \lambda(s)T(s) + \mu(s)N_2(s) \qquad (3.12)$$

Where $\lambda = \lambda(s)$ *and* $\mu = \mu(s)$ are differentiable functions of s. The equation (3.12) after differentiation become:

$$\propto' = \lambda' T + \lambda T' + \mu' N_2 + \mu N_2' \qquad (3.13)$$

From the equations (2.3) and (3.13) we find:

$$\begin{cases} \lambda' - \mu k_2 = 1 \\ \lambda k_2 + \mu' = 0 \\ \lambda k_1 = 0 \\ \lambda k_3 = 0 \end{cases} \qquad (3.14)$$

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

If $\lambda \neq 0 \rightarrow k_3 = 0$ and $k_1 = 0$
The solution of the first two equations of (3.14) give:

$$\lambda = c_1 cosk_2 s + c_2 sink_2 s \qquad (3.15)$$

$$\mu = c_2 cosk_2 s - c_1 sink_2 s - sin(k_2 s)^2 \qquad (3.16)$$

We get the equation (3.10) after substituting equations (3.15) and (3.16) into (3.12)

And If: $k_3 \neq 0$, $k_1 \neq 0 \rightarrow \lambda(s) = 0$ (3.17)
and $\mu(s) = \frac{-1}{k_2}$ (3.18)

We get the equation (3.11) after substituting equations (3.17) and (3.18) into (3.12)

Theorem 3.3: If we suppose that the curve which is lies on the subspace spanned by $\{T, B\}$ are: $\propto$: R $\rightarrow$ E_1^4 , then there are two position vector of Space curve $\propto$ as :

(i) if $k_2 \neq 0$, $k_1 \neq 0$. Then the position vector can be written in the form:

$$\propto (s) = \frac{-1}{k_3} N_1(s) \qquad (3.19)$$

(ii) If $k_1 = 0$ and $k_2 = 0$. Then the position vector can be written in the form:

$$\propto (s) = [C_1 cosk_2 s + C_2 sink_2 s]T(s) + \left[C_2 Cos(k_2 s) - \frac{Cos(k_2 s)^2}{k_2} - C_1 Sin(k_2 s) - \frac{Sin(k_2 s)^2}{k_2}\right] B(s) \qquad (3.20)$$

Proof: We can write the position vector in the form:

$$\propto (s) = \lambda(s)T(s) + \mu(s)B(s) \qquad (3.21)$$

Where $\lambda = \lambda(s)$ and $\mu = \mu(s)$ are differentiable functions of s. The (3.21) after differentiation become :

$$\propto' = \lambda' T + \lambda T' + \mu' B + \mu B' \qquad (3.22)$$

From the equations (2.3) and (3.22) we find:

$$\begin{cases} \lambda' - \mu k_3 = 1 \\ \lambda k_3 + \mu' = 0 \\ \lambda k_1 = 0 \\ \lambda k_2 = 0 \end{cases} \qquad (3.23)$$

If : $k_1 \neq 0$, $k_2 \neq 0 \rightarrow \lambda = 0$ (3.24)
and $\mu(s) = \frac{-1}{k_3}$ (3.25)

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

We get the equation (3.19) after substituting equations (3.24) and (3.25) into equation (3.21)

And if $\lambda \neq 0 \rightarrow k_2 = 0 \ and \ k_1 = 0$
The solution of the first two equations of (3.23) give:

$$\lambda(s) = C_1 cos(k_2 s) + C_2 sin(k_2 s) \qquad (3.26)$$

$$\mu(s) = C_2 \mathrm{Cos}(k_2 s) - \frac{\mathrm{Cos}(k_2 s)^2}{k_2} - C_1 \mathrm{Sin}(k_2 s) - \frac{\mathrm{Sin}(k_2 s)^2}{k_2} \qquad (3.27)$$

We get the equation (3.20) after substituting equations (3.26) and (3.27) into (3.21)

Theorem 3.4: If we suppose that the curve which is lies on the subspace spanned by $\{N_1, N_2\}$ are: $\propto$: R $\rightarrow$ E_1^4 , then the position vector of Space curve $\propto$ can be written in the form:

$$\propto (s) = \left[\frac{-(1 + ck_2)}{k_1}\right] N_1(s) + cN_2(s) \qquad (3.28)$$

Proof: We can write the position vector in the form:

$$\propto (s) = \lambda(s) N_1(s) + \mu(s) N_2(s) \qquad (3.29)$$

Where $\lambda = \lambda(s) \ and \ \mu = \mu(s)$ are differentiable functions of s. The equation (3.29) after differentiation become :

$$\propto' = \lambda' N_1 + \lambda \, N_1{}' + \mu' N_2 + \mu \, N_2{}' \qquad (3.30)$$

From the equations (2.3) and (3.30) we find:

$$\begin{cases} \lambda k_1 + \mu k_2 = -1 \\ \mu' = 0 \\ \lambda' = 0 \end{cases} \qquad (3.31)$$

The solution of the equations of (3.31) give:

$$\mu = c \qquad (3.32)$$

$$\lambda = \frac{-(1 + ck_2)}{k_1} \qquad (3.33)$$

So $$\propto (s) = \left[\frac{-(1+ck_2)}{k_1}\right] N_1(s) + cN_2(s) \qquad (3.34)$$

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

Theorem 3.5: If we suppose that the curve which is lies on the subspace spanned by $\{N_1, B\}$ are: $\propto$: R $\rightarrow$ E_1^4 , then the position vector of Space curve $\propto$ can be written in the form:

$$\propto (s) = \left[\frac{-(1+ck_3)}{k_1}\right] N_1(s) + cN_2(s) \qquad (3.35)$$

Proof: We can write the position vector in the form:

$$\propto (s) = \lambda(s)N_1(s) + \mu(s)B \qquad (3.36)$$

Where $\lambda = \lambda(s)$ and $\mu = \mu(s)$ are differentiable functions of s. The equation (3.36) after differentiation become :

$$\propto' = \lambda' N_1 + \lambda\, N_1{}' + \mu' B + \mu B' \qquad (3.37)$$

From the equations (2.3) and (3.37) we find:

$$\begin{cases} \lambda k_1 + \mu k_3 = -1 \\ \mu' = 0 \\ \lambda' = 0 \end{cases} \qquad (3.38)$$

The solution of the equations of (3.38) give:

$$\mu = c \qquad (3.39)$$

$$\lambda = \frac{-(1+ck_3)}{k_1} \qquad (3.40)$$

So $$\propto (s) = \left[\frac{-(1+ck_3)}{k_1}\right] N_1(s) + cN_2(s) \qquad (3.41)$$

Theorem 3.6: If we suppose that the curve which is lies on the subspace spanned by $\{N_2, \mathrm{B}\}$ are: $\propto$: R $\rightarrow$ E_1^4 , then the position vector of Space curve $\propto$ can be written in the form:

$$\propto (s) = \left[\frac{-(1+ck_3)}{k_2}\right] N_1(s) + cN_2(s) \qquad (3.42)$$

Proof: We can write the position vector in the form:

$$\propto (s) = \lambda(s)N_2(s) + \mu(s)B \qquad (3.43)$$

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

Where $\lambda = \lambda(s)$ *and* $\mu = \mu(s)$ are differentiable functions of s. The equation (3.43) after differentiation become :

$$\propto' = \lambda' N_2 + \lambda\, N_2{}' + \mu' B + \mu B' \qquad (3.44)$$

From the equations (2.3) and (3.44) we find:

$$\begin{cases} \lambda k_2 + \mu k_3 = -1 \\ \mu' = 0 \\ \lambda' = 0 \end{cases} \qquad (3.45)$$

The solution of the equations of (3.45) give:

$$\lambda = \frac{-(1 + ck_3)}{k_2} \qquad (3.46)$$

$$\mu = c \qquad (3.47)$$

So $$\propto (s) = \left[\frac{-(1+ck_3)}{k_2}\right] N_1(s) + cN_2(s) \qquad (3.48)$$

Theorem 3.7: If we suppose that the curve which is lies on the subspace spanned by $\{T, N_1, N_2\}$ *are*: $\propto : \mathrm{R} \to \mathrm{E}_1^4$, then the position vector of Space curve $\propto$ can be written in the form:

$$\propto (s) = \left[\frac{ck_2 - 1}{k_1}\right] N_1(s) + cN_2(s) \qquad (3.49)$$

If $k_3 \neq 0$
And if $k_3 = 0$ can be written as:

$$\propto (s) = \lambda(s)T(s) + \mu(s)N_1(s) + \nu(s)N_2(s) \qquad (3.50)$$

Where $M = \sqrt{k_1{}^2 + k_2{}^2}$ and

$$\lambda(s) = \left[-\frac{k_1{}^2 \mathrm{e}^{-2\mathrm{M}s}(-1 + \mathrm{e}^{2\mathrm{M}s})(1 + \mathrm{e}^{2\mathrm{M}s})}{4\mathrm{M}^3} - \frac{k_1{}^2 \mathrm{e}^{-2\mathrm{M}s}(-1 + \mathrm{e}^{2\mathrm{M}s})(1 + \mathrm{e}^{2\mathrm{M}s})}{4\mathrm{M}^3} + \frac{\mathrm{e}^{-2\mathrm{M}s}(-1 + \mathrm{e}^{2\mathrm{M}s})(1 + \mathrm{e}^{2\mathrm{M}s})}{4\mathrm{M}} + \frac{C_1}{2}\mathrm{e}^{-\mathrm{M}s}(1 + \mathrm{e}^{2\mathrm{M}s}) + \frac{k_1 C_2 \mathrm{e}^{-\mathrm{M}s}(-1 + \mathrm{e}^{2\mathrm{M}s})}{2\mathrm{M}} - \frac{k_1 C_3 \mathrm{e}^{-\mathrm{M}s}(-1 + \mathrm{e}^{2Ms})}{2\mathrm{M}}\right] \qquad (3.51)$$

$$\mu(s) = \left[-\frac{k_1 k_1{}^2 \mathrm{e}^{-2\mathrm{M}s}(-1 + \mathrm{e}^{\mathrm{M}s})^2(1 + \mathrm{e}^{2\mathrm{M}s})}{4\mathrm{M}^4} + \frac{k_1 \mathrm{e}^{-2\mathrm{M}s}(-1 + \mathrm{e}^{2\mathrm{M}s})^2}{4\mathrm{M}^2} - \frac{k_1 \mathrm{e}^{-2\mathrm{M}s}(1 + \mathrm{e}^{2\mathrm{M}s})(k_1{}^2 + \mathrm{e}^{2\mathrm{M}s} k_1{}^2 + 2\mathrm{e}^{\mathrm{M}s} k_1{}^2)}{4\mathrm{M}^4} + \frac{C_1 k_1 \mathrm{e}^{-\mathrm{M}s}(-1 + \mathrm{e}^{2\mathrm{M}s})}{2\mathrm{M}} + \frac{C_2 \mathrm{e}^{-\mathrm{M}s}(k_1{}^2 + k_1{}^2 \mathrm{e}^{2\mathrm{M}s} + 2k_1{}^2 \mathrm{e}^{\mathrm{M}s})}{2M^2} - \frac{k_1 k_1 C_3 \mathrm{e}^{-\mathrm{M}s}(-1 + \mathrm{e}^{Ms})^2}{2\mathrm{M}^2}\right] \qquad (3.52)$$

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

$$\nu(s) = \left[\frac{{k_1}^2 k_2 e^{-2Ms}(-1+e^{Ms})^2(1+e^{2Ms})}{4M^4} - \frac{k_1 e^{-2Ms}(-1+e^{2Ms})^2}{4M^2} + \frac{k_1 e^{-2Ms}(1+e^{2Ms})(2e^{Ms}{k_1}^2 + {k_1}^2 + e^{2Ms}{k_1}^2)}{4M^4} - \frac{k_2 C_1 e^{-Ms}(-1+e^{2Ms})}{2M} - \frac{k_1 k_2 C_2 e^{-Ms}(-1+e^{Ms})^2}{2M^2} + \frac{C_3 e^{-Ms}(2{k_1}^2 e^{Ms} + {k_2}^2 + {k_2}^2 e^{2Ms})}{2M^2}\right] \quad (3.53)$$

Proof: We can write the position vector in the form:

$$\propto (s) = \lambda(s)T(s) + \mu(s)N_1(s) + \nu(s)N_2(s) \qquad (3.54)$$

Where $\lambda = \lambda(s)$ *and* $\mu = \mu(s)$ are differentiable functions of s. The equation (3.54) after differentiation become :

$$\propto' = \lambda' T + \lambda T' + \mu' N_1 + \mu {N_1}' + \nu' N_2 + \nu {N_2}' \qquad (3.55)$$

From the equations (2.3) and (3.55) we find:

$$\begin{cases} \lambda' - \mu k_1 + \nu k_2 = 1 \\ \lambda k_1 - \mu' = 0 \\ \lambda k_2 + \nu' = 0 \\ \lambda k_3 = 0 \end{cases} \qquad (3.56)$$

If: $k_3 \neq 0 \quad \rightarrow \qquad \lambda = 0 \qquad (3.57)$

$$\nu = c \qquad (3.58)$$

And $\qquad \mu = \frac{ck_2 - 1}{k_1} \qquad (3.59)$

We get the equation (3.49) after substituting equations(3.57), (3.58) and (3.59) into (3.54)

And if: $k_3 = 0 \quad \rightarrow \quad \lambda \neq 0$

$$\lambda = -\frac{{k_1}^2 e^{-2Ms}(-1+e^{2Ms})(1+e^{2Ms})}{4M^3} - \frac{{k_1}^2 e^{-2Ms}(-1+e^{2Ms})(1+e^{2Ms})}{4M^3} + \frac{e^{-2Ms}(-1+e^{2Ms})(1+e^{2Ms})}{4M} + \frac{C_1}{2} e^{-Ms}(1+e^{2Ms}) + \frac{k_1 C_2 e^{-Ms}(-1+e^{2Ms})}{2M} - \frac{k_1 C_3 e^{-Ms}(-1+e^{2Ms})}{2M} \quad (3.60)$$

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

$$\mu = -\frac{k_1 {k_1}^2 \mathrm{e}^{-2\mathrm{M}s}(-1+\mathrm{e}^{\mathrm{M}s})^2(1+\mathrm{e}^{2\mathrm{M}s})}{4\mathrm{M}^4} + \frac{k_1 \mathrm{e}^{-2\mathrm{M}s}(-1+\mathrm{e}^{2\mathrm{M}s})^2}{4\mathrm{M}^2} - \frac{k_1 \mathrm{e}^{-2\mathrm{M}s}(1+\mathrm{e}^{2\mathrm{M}s})\left({k_1}^2 + \mathrm{e}^{2\mathrm{M}s}{k_1}^2 + 2\mathrm{e}^{\mathrm{M}s}{k_1}^2\right)}{4\mathrm{M}^4} + \frac{C_1 k_1 \mathrm{e}^{-\mathrm{M}s}(-1+\mathrm{e}^{2\mathrm{M}s})}{2\mathrm{M}} + \frac{C_2 \mathrm{e}^{-\mathrm{M}s}\left({k_1}^2 + {k_1}^2\mathrm{e}^{2\mathrm{M}s} + 2{k_1}^2\mathrm{e}^{\mathrm{M}s}\right)}{2M^2} - \frac{k_1 k_1 C_3 \mathrm{e}^{-\mathrm{M}s}(-1+\mathrm{e}^{Ms})^2}{2\mathrm{M}^2} \quad (3.61)$$

$$\nu = \frac{{k_1}^2 k_2 \mathrm{e}^{-2\mathrm{M}s}(-1+\mathrm{e}^{\mathrm{M}s})^2(1+\mathrm{e}^{2\mathrm{M}s})}{4M^4} - \frac{k_1 \mathrm{e}^{-2\mathrm{M}s}(-1+\mathrm{e}^{2\mathrm{M}s})^2}{4M^2} + \frac{k_1 \mathrm{e}^{-2\mathrm{M}s}(1+\mathrm{e}^{2\mathrm{M}s})\left(2\mathrm{e}^{\mathrm{M}s}{k_1}^2 + {k_1}^2 + \mathrm{e}^{2\mathrm{M}s}{k_1}^2\right)}{4M^4} - \frac{k_2 C_1 \mathrm{e}^{-\mathrm{M}s}(-1+\mathrm{e}^{2Ms})}{2M} - \frac{k_1 k_2 C_2 \mathrm{e}^{-\mathrm{M}s}(-1+\mathrm{e}^{\mathrm{M}s})^2}{2M^2} + \frac{C_3 \mathrm{e}^{-\mathrm{M}s}\left(2{k_1}^2\mathrm{e}^{\mathrm{M}s} + {k_2}^2 + {k_2}^2\mathrm{e}^{2\mathrm{M}s}\right)}{2M^2} \quad (3.62)$$

Where: $M = \sqrt{{k_1}^2 + {k_2}^2}$

We get the equation (3.50) after substituting equations(3.60), (3.61) and (3.62) into (3.51)

Theorem 3.8: If we suppose that the curve which is lies on the subspace spanned by $\{T, N_1, \mathrm{B}\}$ are: $\propto: \mathrm{R} \to \mathrm{E}_1^4$, then the position vector of Space curve $\propto$ can be written in the form:

$$\propto\ (s) = \frac{1 - ck_3}{k_1} N_1(s) + cB(s) \qquad (3.63)$$

If: $k_2 \neq 0$

And if $k_2 = 0$ we find that:

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

$$\propto(s)=\left[-\frac{{k_1}^2 e^{-2Hs}(-1+e^{2Hs})(1+e^{2Hs})}{4H^3}-\frac{k_2k_3e^{-2Hs}(-1+e^{2Hs})(1+e^{2Hs})}{4H^3}\right.$$
$$+\frac{e^{-2Hs}(-1+e^{2Hs})(1+e^{2Hs})}{4H}+\frac{C_1}{2}e^{-Hs}(1+e^{2Hs})+\frac{C_2k_1e^{-Hs}(-1+e^{2Hs})}{2H}$$
$$\left.-\frac{C_3k_3e^{-Hs}(-1+e^{2Hs})}{2H}\right]T(s)$$
$$+\left[-\frac{k_1k_2k_3e^{-2Hs}(-1+e^{Hs})^2(1+e^{2Hs})}{4H^4}+\frac{k_1e^{-2Hs}(-1+e^{2Hs})^2}{4H^2}\right.$$
$$-\frac{k_1e^{-2Hs}(1+e^{2Hs})({k_1}^2+{k_1}^2e^{2Hs}+2k_2k_3e^{Hs})}{4H^4}+\frac{C_1k_1e^{-Hs}(-1+e^{2Hs})}{2H}$$
$$\left.+\frac{C_2e^{-Hs}({k_1}^2+e^{2Hs}{k_1}^2+2e^{Hs}k_2k_3)}{2H^2}-\frac{k_1k_3C_3e^{-Hs}(-1+e^{Hs})^2}{2H^2}\right]N_1(s)$$
$$+\left[\frac{e^{-2Hs}(-1+e^{Hs})^2(1+e^{2Hs}){k_1}^2k_2}{4H^4}-\frac{k_2e^{-2Hs}(-1+e^{2Hs})^2}{4H^2}\right.$$
$$+\frac{k_2e^{-2Hs}(1+e^{2Hs})(2e^{Hs}{k_1}^2+k_2k_3+k_2k_3e^{2Hs})}{4H^4}-\frac{C_1k_2e^{-Hs}(-1+e^{2Hs})}{2H}$$
$$\left.-\frac{C_2e^{-Hs}(-1+e^{Hs})^2k_1k_2}{2H^2}+\frac{C_3e^{-Hs}(2e^{Hs}{k_1}^2+k_2k_3+k_2k_3e^{2Hs})}{2H^2}\right]B(s) \qquad (3.64)$$

Where $H=\sqrt{k_1^2+k_2k_3}$

Proof: We can write the position vector in the form:

$$\propto(s)=\lambda(s)T(s)+\mu(s)N_1(s)+\nu(s)B(s) \qquad (3.65)$$

Where $\lambda=\lambda(s)$ *and* $\mu=\mu(s)$ are differentiable functions of s. The equation (3.65) after differentiation become:

$$\propto'=\lambda' T+\lambda T'+\mu' N_1+\mu {N_1}'+\nu' B+\nu B' \qquad (3.66)$$

From the equations (2.3) and (3.66) we find:

$$\begin{cases}\lambda'-\mu k_1+\nu k_3=1\\ \lambda k_1+\mu'=0\\ \lambda k_3+\nu'=0\\ \lambda k_2=0\end{cases} \qquad (3.67)$$

If: $k_2\neq 0 \quad\rightarrow$ $\qquad \lambda=0 \qquad (3.68)$

$\nu=c \qquad (3.69)$

And $\qquad \mu=\frac{1-ck_3}{k_1} \qquad (3.70)$

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So we get $\propto(s) = \frac{1-ck_3}{k_1} N_1(s) + cN_2(s)$ (3.71)

And if : $k_2 = 0 \rightarrow \lambda \neq 0$

$$\lambda = -\frac{{k_1}^2 e^{-2Hs}(-1+e^{2Hs})(1+e^{2Hs})}{4H^3} - \frac{k_2 k_3 e^{-2Hs}(-1+e^{2Hs})(1+e^{2Hs})}{4H^3} + \frac{e^{-2Hs}(-1+e^{2Hs})(1+e^{2Hs})}{4H} + \frac{C_1}{2} e^{-Hs}(1+e^{2Hs}) + \frac{C_2 k_1 e^{-Hs}(-1+e^{2Hs})}{2H} - \frac{C_3 k_3 e^{-Hs}(-1+e^{2Hs})}{2H} \quad (3.72)$$

$$\mu = -\frac{k_1 k_2 k_3 e^{-2Hs}(-1+e^{Hs})^2(1+e^{2Hs})}{4H^4} + \frac{k_1 e^{-2Hs}(-1+e^{2Hs})^2}{4H^2} - \frac{k_1 e^{-2Hs}(1+e^{2Hs})({k_1}^2 + {k_1}^2 e^{2Hs} + 2k_2 k_3 e^{Hs})}{4H^4} + \frac{C_1 k_1 e^{-Hs}(-1+e^{2Hs})}{2H} + \frac{C_2 e^{-Hs}({k_1}^2 + e^{2Hs}{k_1}^2 + 2e^{Hs}k_2 k_3)}{2H^2} - \frac{k_1 k_3 C_3 e^{-Hs}(-1+e^{Hs})^2}{2H^2} \quad (3.73)$$

$$\nu = \frac{e^{-2Hs}(-1+e^{Hs})^2(1+e^{2Hs}){k_1}^2 k_2}{4H^4} - \frac{k_2 e^{-2Hs}(-1+e^{2Hs})^2}{4H^2} + \frac{k_2 e^{-2Hs}(1+e^{2Hs})(2e^{Hs}{k_1}^2 + k_2 k_3 + k_2 k_3 e^{2Hs})}{4H^4} - \frac{C_1 k_2 e^{-Hs}(-1+e^{2Hs})}{2H} - \frac{C_2 e^{-Hs}(-1+e^{Hs})^2 k_1 k_2}{2H^2} + \frac{C_3 e^{-Hs}(2e^{Hs}{k_1}^2 + k_2 k_3 + k_2 k_3 e^{2Hs})}{2H^2} \quad (3.74)$$

Where $H = \sqrt{k_1^2 + k_2 k_3}$

We get the equation (3.64) after substituting equations (3.72), (3.73) and (3.74) into (3.65)

Theorem 3.9: If we suppose that the curve which is lies on the subspace spanned by $\{T, N_2, B\}$ are: $\propto: R \rightarrow E_1^4$, then the position vector of Space curve $\propto$ can be written in the form:

$$\propto(s) = cN_2(s) + \left[\frac{-(1+ck_2)}{k_3}\right] B(s) \quad (3.75)$$

If: $k_1 \neq 0$
And if $k_1 = 0$ it can be written in the form:

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$$\propto(s)=\left[\frac{e^{-2Fs}(-1+e^{2Fs})(1+e^{2Fs})}{4F}-\frac{k_3{}^2e^{-2Fs}(-1+e^{2Fs})(1+e^{2Fs})}{4F(k_2{}^2+k_3{}^2)}\right.$$
$$-\frac{k_2{}^2e^{-2Fs}(-1+e^{2Fs})(1+e^{2Fs})}{4F(k_2{}^2+k_3{}^2)}+\frac{C_1}{2}e^{-Fs}(1+e^{2Fs})-\frac{C_2k_2Fe^{-Fs}(-1+e^{2Fs})}{2(k_2{}^2+k_3{}^2)}$$
$$\left.-\frac{C_3k_3Fe^{-Fs}(-1+e^{2Fs})}{2(k_2{}^2+k_3{}^2)}\right]T(s)$$
$$+\left[\frac{k_2e^{-2Fs}(-1+e^{2Fs})^2}{4(k_2{}^2+k_3{}^2)}+\frac{k_2k_3{}^3e^{-2Fs}(-1+e^{Fs})^2(1+e^{2Fs})}{4F^2(k_2{}^2+k_3{}^2)}\right.$$
$$+\frac{k_2e^{-2Fs}(1+e^{2Fs})(2e^{Fs}k_3{}^2+k_2{}^2+e^{2Fs}k_2{}^2)}{4F^2(k_2{}^2+k_3{}^2)}+\frac{C_1k_2Fe^{-Fs}(-1+e^{2Fs})}{2(k_2{}^2+k_3{}^2)}$$
$$\left.+\frac{C_2e^{-Fs}(2e^{Fs}k_3{}^2+k_2{}^2+e^{2Fs}k_2{}^2)}{2(k_2{}^2+k_3{}^2)}+\frac{C_3k_2k_3e^{-Fs}(-1+e^{Fs})^2}{2(k_2{}^2+k_3{}^2)}\right]N_2(s)$$
$$+\left[\frac{e^{-2Fs}(-1+e^{2Fs})^2k_3}{4F^2}+\frac{e^{-2Fs}(-1+e^{Fs})^2(1+e^{2Fs})k_3k_2{}^2}{4F^2(k_2{}^2+k_3{}^2)}\right.$$
$$+\frac{k_3e^{-2Fs}(1+e^{2Fs})(k_3{}^2+e^{2Fs}k_3{}^2+2e^{Fs}k_2{}^2)}{4F^2(k_2{}^2+k_3{}^2)}+\frac{C_1e^{-Fs}(-1+e^{2Fs})LF}{2(k_2{}^2+k_3{}^2)}$$
$$\left.+\frac{C_2e^{-Fs}(-1+e^{Fs})^2k_2k_3}{2(k_2{}^2+k_3{}^2)}+\frac{C_3e^{-Fs}(k_3{}^2+e^{2Fs}k_3{}^2+2e^{Fs}k_2{}^2)}{2(k_2{}^2+k_3{}^2)}\right]B(s) \qquad (3.76)$$

Where: $F=\sqrt{-k_2{}^2-k_3{}^2}$

Proof: We can write the position vector in the form:

$$\propto(s)=\lambda(s)T(s)+\mu(s)N_2(s)+\nu(s)B(s) \qquad (3.77)$$

Where $\lambda=\lambda(s)$ *and* $\mu=\mu(s)$ are differentiable functions of s. The equation (3.77) after differentiation become :

$$\propto'=\lambda'\,T+\lambda\,T'+\mu'N_2+\mu\,N_2{}'+\nu'B+\nu\,B' \qquad (3.78)$$

From the equations (2.3) and (3.78) we find:

$$\begin{cases}\lambda'-\mu k_2-\nu k_3=1\\ \lambda k_2+\mu'=0\\ \lambda k_3+\nu'=0\\ \lambda k_1=0\end{cases} \qquad (3.79)$$

If $k_1\neq 0 \quad\rightarrow\quad \lambda=0$ (3.80)

$$\mu=c \qquad (3.81)$$

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$$\nu = \frac{-(1+ck_2)}{k_3} \qquad (3.82)$$

Then

$$\propto\ (s) = cN_2(s) + \left[\frac{-(1+ck_2)}{k_3}\right]B(s) \qquad (3.83)$$

And if $k_1 = 0 \quad \rightarrow \lambda \neq 0$

$$\lambda = \frac{e^{-2Fs}(-1+e^{2Fs})(1+e^{2Fs})}{4F} - \frac{{k_3}^2 e^{-2Fs}(-1+e^{2Fs})(1+e^{2Fs})}{4F({k_2}^2+{k_3}^2)} - \frac{{k_2}^2 e^{-2Fs}(-1+e^{2Fs})(1+e^{2Fs})}{4F({k_2}^2+{k_3}^2)} + \frac{C_1}{2}e^{-Fs}(1+e^{2Fs}) - \frac{C_2 k_2 F e^{-Fs}(-1+e^{2Fs})}{2({k_2}^2+{k_3}^2)} - \frac{C_3 k_3 F e^{-Fs}(-1+e^{2Fs})}{2({k_2}^2+{k_3}^2)} \quad (3.84)$$

$$\mu = \frac{k_2 e^{-2Fs}(-1+e^{2Fs})^2}{4({k_2}^2+{k_3}^2)} + \frac{k_2 {k_3}^3 e^{-2Fs}(-1+e^{Fs})^2(1+e^{2Fs})}{4F^2({k_2}^2+{k_3}^2)} + \frac{k_2 e^{-2Fs}(1+e^{2Fs})(2e^{Fs}{k_3}^2+{k_2}^2+e^{2Fs}{k_2}^2)}{4F^2({k_2}^2+{k_3}^2)} + \frac{C_1 k_2 F e^{-Fs}(-1+e^{2Fs})}{2({k_2}^2+{k_3}^2)} + \frac{C_2 e^{-Fs}(2e^{Fs}{k_3}^2+{k_2}^2+e^{2Fs}{k_2}^2)}{2({k_2}^2+{k_3}^2)} + \frac{C_3 k_2 k_3 e^{-Fs}(-1+e^{Fs})^2}{2({k_2}^2+{k_3}^2)} \quad (3.85)$$

$$\nu = \frac{e^{-2Fs}(-1+e^{2Fs})^2 k_3}{4F^2} + \frac{e^{-2Fs}(-1+e^{Fs})^2(1+e^{2Fs})k_3{k_2}^2}{4F^2({k_2}^2+{k_3}^2)} + \frac{k_3 e^{-2Fs}(1+e^{2Fs})({k_3}^2+e^{2Fs}{k_3}^2+2e^{Fs}{k_2}^2)}{4F^2({k_2}^2+{k_3}^2)} + \frac{C_1 e^{-Fs}(-1+e^{2Fs})LF}{2({k_2}^2+{k_3}^2)} + \frac{C_2 e^{-Fs}(-1+e^{Fs})^2 k_2 k_3}{2({k_2}^2+{k_3}^2)} + \frac{C_3 e^{-Fs}({k_3}^2+e^{2Fs}{k_3}^2+2e^{Fs}{k_2}^2)}{2({k_2}^2+{k_3}^2)} \quad (3.86)$$

Where: $F = \sqrt{-{k_2}^2 - {k_3}^2}$

We get the equation (3.76) after substituting equations (3.84), (3.85) and (3.86) into (3.77)

Theorem 3.10: If we suppose that the curve which is lies on the subspace spanned by $\{N_1, N_2, B\}$ are: $\propto : \mathrm{R} \rightarrow \mathrm{E}_1^4$, then the position vector of Space curve $\propto$ can be written in the form:

ISSN 2055-009X(Print), ISSN 2055-0103(Online)

$$\propto (s) = cT(s) + c_1 N_2(s) + \left[\frac{-(1 + c_1 k_2 + c k_1)}{k_3}\right] B(s) \qquad (3.87)$$

Proof: We can write the position vector in the form:

$$\propto (s) = \lambda(s) N_1(s) + \mu(s) N_2(s) + \nu(s) \mathrm{B}(s) \qquad (3.88)$$

Where $\lambda = \lambda(s)$ *and* $\mu = \mu(s)$ are differentiable functions of s. The equation (3.88) after differentiation become :

$$\propto' = \lambda' N_1 + \lambda N_1{}' + \mu' N_2 + \mu N_2{}' + \nu' \mathrm{B} + \nu B' \qquad (3.89)$$

From the equations (2.3) and (3.89) we find:

$$\begin{cases} \mu k_2 + \lambda k_1 + \nu k_3 = -1 \\ \mu' = 0 \\ \lambda' = 0 \\ \nu' = 0 \end{cases} \qquad (3.90)$$

$$\lambda = c \qquad (3.91)$$

$$\mu = c_1 \qquad (3.92)$$

$$\nu = \frac{-(1 + c_1 k_2 + c k_1)}{k_3} \qquad (3.93)$$

Then

$$\propto (s) = cT(s) + c_1 N_2(s) + \left[\frac{-(1 + c_1 k_2 + c k_1)}{k_3}\right] B(s) \qquad (3.94)$$

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