SOME NEW CHARACTERIZATIONS OF SPACE CURVES ACCORDING TO TYPE-1 BISHOP FRAME IN EUCLIDEAN 4- SPACE $E_1^4$

Fathi M. D. Elzaki
Mathematics Department, College of Science and Technology, Omdurman Islamic University, SUDAN & Mathematics Department, College of Sciences and Arts, Ranyah Branch, Taif University, KSA

ABSTRACT: In this paper, the position vectors of space curve with constant curvatures was investigate according to type-1 Bishop Frame in the Euclidean 4- Space $E_1^4$. And some characterizations of the Space Curves with constant curvatures spanned by subspaces of $E_1^4$ are given.

KEYWORDS: Euclidean Space $E_1^4$; type-1 Bishop Frame; Space Curves Characterizations.

INTRODUCTION

There are many authors have been studied the position vectors of space curves with an arbitrary constant curvatures with respect to Frenet frame in Euclidean 4-space and also they studied geometry of curves and his characterizations. In [5] Kazim was studied some characterizations of the curve that their position vector lied in the rectifying curves in the Minkowski 3-space. In [2] Authors was discus the position vectors space curve with constant curvatures in Euclidean 3-space depending on the Type-1 Bishop Frame with constant Curvatures. The general helix position vector with respect to Frenet frame in Euclidean 3-Space studied by A. T. Ali In [1], in addition to that the natural representation of a general helix in terms of the curvature and torsion was deduced. The spacelike normal curves in Minkowski 4-Space $E_1^4$, in accordance to some characterizations of spacelike normal curves with spacelike, and null principal normal was studied by K. Ilarslan [6]. In [4] Hüseyin Kocayiğit and others, they gave us some characterizations of timelike curves according to Bishop Frame in Minkowski 3-space $E_1^3$, some characterizations for spacelike helices was given in Minkowski space-time $E_1^4$ by authors in [8].

PRELIMINARIES

In this work $E_1^4$ denote to the Minkowski space together with a metric $\langle , \rangle$ of sign nature $(-,+,+,+)$. For each: $X=(x_1, x_2, x_3, x_4)\in E_1^4$, $Y=(y_1, y_2, y_3, y_4)\in E_1^4$
The standard scalar product in the Euclidean 4-space $E_1^4$ given by:

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 \quad (2.1)$$

Since a vector $X$ is said to be Timelike if $\langle X, X \rangle < 0$, Spacelike if $\langle X, X \rangle > 0$ and null(Lightlike) if $\langle X, X \rangle = 0$ and $X \neq 0$. And can define the norm of a vector $X$ as $\|X\| = \sqrt{\langle X, X \rangle}$

Take the arbitrary space curve in $E_1^4$ as $\alpha: I \subset R \to E_1^4$. Recall that the curve $\alpha$ is said to be of unit speed (or parameterized by arc length function $s$) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$

Let $\{t, n_1, n_2, b\}$ Denoted the Frenet moving frame along the unit Speed curve $\alpha$. Where the vectors $t$, $n_1$, $n_2$ and $b$ are mutually orthogonal vectors as in [7].
\[ \begin{align*}
\dot{t} &= \kappa n_1 \\
\dot{n}_1 &= -\kappa t + \tau n_2 \\
\dot{n}_2 &= -\tau n_1 + \sigma b \\
b &= -\sigma n_2 
\end{align*} \] (2.2)

Let \( \{T, N_1, N_2, B\} \) denote the type-1 Bishop moving frame along the unit speed curve \( \alpha \). Where \( T, N_1, N_2 \) and \( B \) are orthogonal vectors satisfying
\( \langle T, T \rangle = \langle N_1, N_1 \rangle = \langle N_2, N_2 \rangle = 1, \langle B, B \rangle = -1 \)

Then as in [3] the type-1 Bishop formulas and satisfying \( \alpha \) are:
\[ \begin{align*}
\dot{T} &= k_1 N_1 + k_2 N_2 + k_3 N_3 \\
\dot{N}_1 &= -k_1 T \\
\dot{N}_2 &= -k_2 T \\
B &= -k_3 T 
\end{align*} \] (2.3)

Where \( k_1, k_2 \) and \( k_3 \) are the first, second and third Bishop curvature.

**Some Characterizations of a Spacelike Curves with Constant Curvatures According to Type-1 Bishop Frame in Euclidean 4-Space** \( E_4^1 \)

In this section we will give some investigation to characterize the space curves that lie on some subspaces of \( E_4^1 \). Let \( \alpha \) be a Spacelike curve in \( E_4^1 \) with the type-1 Bishop frame \( \{T, N_1, N_2, B\} \).

**Theorem 3.1:** If we suppose that the curve which is lies on the subspace spanned by \( \{T, N_1\} \) are: \( \alpha: \mathbb{R} \rightarrow E_4^1 \), then the position vector of Space curve \( \alpha \) can be written in the form:
\[ \alpha (s) = \frac{-1}{k_1} N_1(s) \] (3.1)

If \( k_3 \neq 0 \) and \( \lambda = 0 \)

And can be written in the form:
\[ \alpha (s) = [c_1 \cos k_1 s + c_2 \sin k_1 s] T(s) + \left[ c_2 \cos k_1 s - c_1 \sin k_1 s - \frac{\sin(k_1 s)^2}{k_1} \right] N_1(s) \] (3.2)

If \( k_2 = 0 \), \( k_3 = 0 \) and \( \lambda \neq 0 \)

**Proof:** We can write the position vector in the form:
\[ \alpha (s) = \lambda(s) T(s) + \mu(s) N_1(s) \] (3.3)

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.3) after differentiation become:
\[ \alpha' = \lambda' T + \lambda T' + \mu' N_1 + \mu N_1' \] (3.4)
From the equations (2.3) and (3.4) we find:

\[
\begin{align*}
\lambda' - \mu k_1 &= 1 \\
\lambda k_1 + \mu' &= 0 \\
\lambda k_2 &= 0 \\
\lambda k_3 &= 0
\end{align*}
\]  
(3.5)

If: \( k_3 \neq 0 \rightarrow \lambda = 0 \)  
(3.6)

and \( \mu(s) = \frac{-1}{k_1} \)  
(3.7)

We get the equation (3.1) after substituting equations (3.6) and (3.7) into (3.3)

And if \( \lambda \neq 0 \rightarrow k_3 = 0 \) and \( k_2 = 0 \)
The solution of the first two equations of (3.5) give:

\[
\begin{align*}
\mu &= c_2\cos k_1 s - c_1\sin k_1 s - \sin(k_1 s)^2 \\
\lambda &= c_1\cos k_1 s + c_2\sin k_1 s
\end{align*}
\]  
(3.8) 
(3.9)

We get the equation (3.2) after substituting equations (3.8) and (3.9) into (3.3)

**Theorem 3.2:** If we suppose that the curve which is lies on the subspace spanned by \( \{T, N_2\} \) are: \( R \rightarrow E_1^4 \), then the position vector of Space curve \( \propto \) can be written in the form:

\[
\propto (s) = [c_1\cos k_2 s + c_2\sin k_2 s]T(s) + \left[c_2\cos k_2 s - c_1\sin k_2 s - \frac{\sin(k_2 s)^2}{k_2}\right]N_2(s)
\]  
(3.10)

If \( k_3 = 0 \) and \( k_1 = 0 \)

And can be written in the form:

\[
\propto (s) = \frac{-1}{k_2}N_1(s)
\]  
(3.11)

If \( k_3 \neq 0 \) and \( k_1 \neq 0 \)

**Proof:** We can write the position vector in the form:

\[
\propto (s) = \lambda(s)T(s) + \mu(s)N_2(s)
\]  
(3.12)

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.12) after differentiation become:

\[
\propto' = \lambda' T + \lambda T' + \mu' N_2 + \mu N_2'
\]  
(3.13)

From the equations (2.3) and (3.13) we find:

\[
\begin{align*}
\lambda' - \mu k_2 &= 1 \\
\lambda k_2 + \mu' &= 0 \\
\lambda k_1 &= 0 \\
\lambda k_3 &= 0
\end{align*}
\]  
(3.14)
If \( \lambda \neq 0 \) → \( k_3 = 0 \) and \( k_1 = 0 \) 

The solution of the first two equations of (3.14) give:

\[
\lambda = c_1 \cos k_2 s + c_2 \sin k_2 s \\
\mu = c_2 \cos k_2 s - c_1 \sin k_2 s - \sin(k_2 s)^2
\]

(3.15) (3.16)

We get the equation (3.10) after substituting equations (3.15) and (3.16) into (3.12)

And If: \( k_3 \neq 0, \ k_1 \neq 0 \) → \( \lambda(s) = 0 \) (3.17) 

and \( \mu(s) = \frac{-1}{k_2} \) (3.18)

We get the equation (3.11) after substituting equations (3.17) and (3.18) into (3.12)

**Theorem 3.3:** If we suppose that the curve which is lies on the subspace spanned by \( \{T, B\} \) are: \( \alpha: \mathbb{R} \rightarrow \mathbb{E}^4_1 \) then there are two position vector of Space curve \( \alpha \) as:

(i) If \( k_2 \neq 0, \ k_1 \neq 0 \) Then the position vector can be written in the form:

\[\alpha(s) = \frac{-1}{k_3} N_1(s)\] (3.19)

(ii) If \( k_1 = 0 \) and \( k_2 = 0 \). Then the position vector can be written in the form:

\[\alpha(s) = \left[ C_1 \cos k_2 s + C_2 \sin k_2 s \right] T(s) + \left[ C_2 \cos(k_2 s)^2 - C_1 \sin(k_2 s)^2 \right] B(s)\] (3.20)

**Proof:** We can write the position vector in the form:

\[\alpha(s) = \lambda(s) T(s) + \mu(s) B(s)\] (3.21)

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The (3.21) after differentiation become:

\[\alpha' = \lambda' T + \lambda T' + \mu' B + \mu B'\] (3.22)

From the equations (2.3) and (3.22) we find:

\[
\begin{align*}
\lambda' - \mu k_3 &= 1 \\
\lambda k_3 + \mu' &= 0 \\
\lambda k_1 &= 0 \\
\lambda k_2 &= 0
\end{align*}
\]

(3.23)

If \( k_1 \neq 0, \ k_2 \neq 0 \) → \( \lambda = 0 \) (3.24)

and \( \mu(s) = \frac{-1}{k_3} \) (3.25)
We get the equation (3.19) after substituting equations (3.24) and (3.25) into equation (3.21)

And if \( \lambda \neq 0 \rightarrow k_2 = 0 \) and \( k_1 = 0 \)

The solution of the first two equations of (3.23) give:

\[
\lambda(s) = C_1 \cos(k_2 s) + C_2 \sin(k_2 s) \quad (3.26)
\]

\[
\mu(s) = C_2 \cos(k_2 s) - \frac{\cos(k_2 s)^2}{k_2} - C_1 \sin(k_2 s) - \frac{\sin(k_2 s)^2}{k_2} \quad (3.27)
\]

We get the equation (3.20) after substituting equations (3.26) and (3.27) into (3.21)

**Theorem 3.4:** If we suppose that the curve which is lies on the subspace spanned by \( \{N_1, N_2\} \) are: \( \alpha: \mathbb{R} \rightarrow \mathbb{E}^4 \), then the position vector of Space curve \( \alpha \) can be written in the form:

\[
\alpha(s) = \left[ -\frac{(1 + ck_2)}{k_1} \right] N_1(s) + cN_2(s) \quad (3.28)
\]

**Proof:** We can write the position vector in the form:

\[
\alpha(s) = \lambda(s)N_1(s) + \mu(s)N_2(s) \quad (3.29)
\]

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.29) after differentiation become:

\[
\alpha' = \lambda'N_1 + \lambda N_1' + \mu'N_2 + \mu N_2' \quad (3.30)
\]

From the equations (2.3) and (3.30) we find:

\[
\begin{cases}
\lambda k_1 + \mu k_2 = -1 \\
\mu' = 0 \\
\lambda' = 0
\end{cases} \quad (3.31)
\]

The solution of the equations of (3.31) give:

\[
\mu = c \quad (3.32)
\]

\[
\lambda = \frac{-(1 + ck_2)}{k_1} \quad (3.33)
\]

So

\[
\alpha(s) = \left[ -\frac{(1 + ck_2)}{k_1} \right] N_1(s) + cN_2(s) \quad (3.34)
\]
Theorem 3.5: If we suppose that the curve which is lies on the subspace spanned by \( \{ N_1, B \} \) are: \( \propto: \mathbb{R} \rightarrow \mathbb{E}_1^4 \), then the position vector of Space curve \( \propto \) can be written in the form:

\[
\propto(s) = \left[-\frac{(1 + c k_3)}{k_1}\right] N_1(s) + c N_2(s) \tag{3.35}
\]

**Proof:** We can write the position vector in the form:

\[
\propto(s) = \lambda(s) N_1(s) + \mu(s) B \tag{3.36}
\]

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.36) after differentiation become:

\[
\propto' = \lambda' N_1 + \lambda N_1' + \mu' B + \mu B' \tag{3.37}
\]

From the equations (2.3) and (3.37) we find:

\[
\begin{cases}
\lambda k_1 + \mu k_3 = -1 \\
\mu' = 0 \\
\lambda' = 0
\end{cases} \tag{3.38}
\]

The solution of the equations of (3.38) give:

\[
\mu = c \tag{3.39}
\]

\[
\lambda = \frac{-(1 + c k_3)}{k_1} \tag{3.40}
\]

So

\[
\propto(s) = \left[-\frac{(1 + c k_3)}{k_1}\right] N_1(s) + c N_2(s) \tag{3.41}
\]

Theorem 3.6: If we suppose that the curve which is lies on the subspace spanned by \( \{ N_2, B \} \) are: \( \propto: \mathbb{R} \rightarrow \mathbb{E}_1^4 \), then the position vector of Space curve \( \propto \) can be written in the form:

\[
\propto(s) = \left[-\frac{(1 + c k_3)}{k_2}\right] N_1(s) + c N_2(s) \tag{3.42}
\]

**Proof:** We can write the position vector in the form:

\[
\propto(s) = \lambda(s) N_2(s) + \mu(s) B \tag{3.43}
\]
Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.43) after differentiation become:

\[
\alpha' = \lambda'N_2 + \lambda N_2' + \mu'B + \mu B'
\]  \hspace{1cm} (3.44)

From the equations (2.3) and (3.44) we find:

\[
\begin{cases}
\lambda k_2 + \mu k_3 = -1 \\
\mu' = 0 \\
\lambda' = 0
\end{cases}
\]  \hspace{1cm} (3.45)

The solution of the equations of (3.45) give:

\[
\lambda = -\frac{(1 + ck_3)}{k_2} \\
\mu = c
\]  \hspace{1cm} (3.46) (3.47)

So \( \alpha (s) = \left[ -\frac{(1 + ck_3)}{k_2} \right] N_1(s) + cN_2(s) \hspace{1cm} (3.48) \)

**Theorem 3.7**: If we suppose that the curve which is lies on the subspace spanned by \( \{T, N_1, N_2\} \) are: \( \alpha: \mathbb{R} \rightarrow \mathbb{E}_1^4 \), then the position vector of Space curve \( \alpha \) can be written in the form:

\[
\alpha (s) = \left[ \frac{ck_2 - 1}{k_1} \right] N_1(s) + cN_2(s) \hspace{1cm} (3.49)
\]

If \( k_3 \neq 0 \) and if \( k_3 = 0 \) can be written as:

\[
\alpha (s) = \lambda(s)T(s) + \mu(s)N_1(s) + \nu(s)N_2(s) \hspace{1cm} (3.50)
\]

Where \( M = \sqrt{k_1^2 + k_2^2} \) and

\[
\lambda(s) = \left[ -k_1 \frac{2}{4M^3} e^{-2Ms} (-1 + e^{2Ms}) (1 + e^{2Ms}) - k_1 \frac{2}{4M^3} e^{-2Ms} (-1 + e^{2Ms}) (1 + e^{2Ms}) \\
+ \frac{e^{-2Ms} (-1 + e^{2Ms}) (1 + e^{2Ms})}{4M} + \frac{C_1}{2} e^{-Ms} (1 + e^{2Ms}) + \frac{k_1 C_2 e^{-Ms} (-1 + e^{2Ms})}{2M} \\
- \frac{k_1 C_3 e^{-Ms} (-1 + e^{2Ms})}{2M} \right] \hspace{1cm} (3.51)
\]

\[
\mu(s) = \left[ -k_1 \frac{2}{4M^4} e^{-2Ms} (-1 + e^{Ms}) (1 + e^{2Ms}) + \frac{k_1 e^{-2Ms} (-1 + e^{2Ms})^2}{4M^2} \\
- k_1 e^{-2Ms} (1 + e^{2Ms}) (k_1^2 + e^{2Ms} k_1^2 + 2 e^{Ms} k_1) + \frac{C_1}{2} e^{-Ms} (-1 + e^{2Ms}) \\
+ \frac{C_2 e^{-Ms} (k_1^2 + 2 k_1^2 e^{Ms})}{2M^2} - \frac{k_1 C_3 e^{-Ms} (-1 + e^{Ms})^2}{2M^2} \right] \hspace{1cm} (3.52)
\]
\[ \nu(s) = \left[ \frac{k_1^2 e^{-2Ms}(-1 + e^{2Ms})(1 + e^{2Ms})}{4M^3} - \frac{k_1 e^{-2Ms}(-1 + e^{2Ms})}{4M^2} \right. \\
\left. + \frac{k_1 e^{-2Ms}(-1 + e^{2Ms})(2e^{Ms}k_1^2 + k_1^2 + e^{2Ms}k_1^2)}{4M^2} - \frac{k_2 C e^{-Ms}(-1 + e^{2Ms})}{2M} \right. \\
\left. - \frac{k_1 k_2 C e^{-Ms}(-1 + e^{2Ms})}{2M^2} + \frac{C_3 e^{-Ms}(2k_1^2 e^{Ms} + k_2^2 + e^{2Ms})}{2M^2} \right] \] (3.53)

**Proof:** We can write the position vector in the form:

\[ \alpha'(s) = \lambda(s) T(s) + \mu(s) N_1(s) + \nu(s) N_2(s) \] (3.54)

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.54) after differentiation become:

\[ \alpha' = \lambda' T + \lambda T' + \mu' N_1 + \mu N_1' + \nu' N_2 + \nu N_2' \] (3.55)

From the equations (2.3) and (3.55) we find:

\[
\begin{cases}
\lambda' - \mu k_1 + \nu k_2 = 1 \\
\lambda k_1 - \mu' = 0 \\
\lambda k_2 + \nu' = 0 \\
\lambda k_3 = 0
\end{cases}
\] (3.56)

If: \( k_3 \neq 0 \) \rightarrow \( \lambda = 0 \) \hspace{1cm} (3.57)

\[ \nu = c \] (3.58)

And \( \mu = \frac{ck_2 - 1}{k_1} \) \hspace{1cm} (3.59)

We get the equation (3.49) after substituting equations (3.57), (3.58) and (3.59) into (3.54)

And if: \( k_3 = 0 \) \rightarrow \( \lambda \neq 0 \)

\[ \lambda = - \frac{k_1^2 e^{-2Ms}(-1 + e^{2Ms})(1 + e^{2Ms})}{4M^3} - \frac{k_1 e^{-2Ms}(-1 + e^{2Ms})(1 + e^{2Ms})}{4M^2} \\
+ \frac{e^{-2Ms}(-1 + e^{2Ms})(1 + e^{2Ms})}{4M} + \frac{C_3 e^{-Ms}(1 + e^{2Ms})}{2} + \frac{k_1 C_2 e^{-Ms}(-1 + e^{2Ms})}{2M} \] (3.60)
\[
\mu = -\frac{k_1 k_2^2 e^{-2M s} (1 + e^{2M s})^2}{4M^4} + \frac{k_1 e^{-2M s}(1 + e^{2M s})(k_1^2 + e^{2M s}k_1^2 + 2e^{2M s}k_1^2)}{4M^2} + \frac{C_1 k_1 e^{-M s}(1 + e^{2M s})}{2M} + \frac{C_2 e^{-M s}(k_1^2 + k_1^2 e^{2M s} + 2k_1^2 e^{M s})}{2M^2} - \frac{k_1 k_1 C_3 e^{-M s}(1 + e^{2M s})^2}{2M^2} (3.61)
\]

\[
\nu = \frac{k_1^2 k_2 e^{-2M s} (1 + e^{2M s})^2}{4M^4} - \frac{k_1 e^{-2M s}(1 + e^{2M s})(2e^{2M s}k_1^2 + k_1^2 + e^{2M s}k_1^2)}{4M^2} - \frac{k_2 C_1 e^{-M s}(1 + e^{2M s})}{2M} + \frac{C_2 e^{-M s}(2k_1^2 e^{M s} + k_2^2 + k_2^2 e^{2M s})}{2M^2} (3.62)
\]

Where: \( M = \sqrt{k_1^2 + k_2^2} \)

We get the equation (3.50) after substituting equations (3.60), (3.61) and (3.62) into (3.51)

**Theorem 3.8:** If we suppose that the curve which is lies on the subspace spanned by \( \{ T, N_1, B \} \) are: \( \alpha: R \rightarrow E^4_1 \), then the position vector of Space curve \( \alpha \) can be written in the form:

\[
\alpha (s) = \frac{1 - c k_3}{k_1} N_1 (s) + c B (s) \quad (3.63)
\]

If: \( k_2 \neq 0 \)

And if \( k_2 = 0 \) we find that:
\[ \alpha (s) = \left[-\frac{k_1^2 e^{-2Hs}(-1 + e^{2Hs})(1 + e^{2Hs})}{4H^3} - \frac{k_2 k_3 e^{-2Hs}(-1 + e^{2Hs})(1 + e^{2Hs})}{4H^3} + \frac{e^{-2Hs}(-1 + e^{2Hs})(1 + e^{2Hs})}{4H^3} + \frac{C_1 e^{-Hs}(1 + e^{2Hs})}{2H} + \frac{C_2 k_1 e^{-Hs}(-1 + e^{2Hs})}{2H} - \frac{C_3 k_3 e^{-Hs}(-1 + e^{2Hs})}{2H} \right] T(s) + \left[-\frac{k_1 k_2 k_3 e^{-2Hs}(-1 + e^{2Hs})^2(1 + e^{2Hs})}{4H^4} + \frac{k_1 e^{-2Hs}(-1 + e^{2Hs})^2}{4H^2} - \frac{k_1 e^{-2Hs}(1 + e^{2Hs})(k_1^2 + k_2 e^{2Hs} + 2k_2 k_3 e^{Hs})}{4H^4} + \frac{C_2 e^{-Hs}(k_1^2 + 2e^{Hs} k_1 k_3) + k_1 k_3 C_3 e^{-Hs}(-1 + e^{Hs})^2}{2H^2} \right] N_1(s) + \left[\frac{e^{-2Hs}(-1 + e^{Hs})^2(1 + e^{2Hs}) k_1^2}{4H^4} - \frac{k_2 e^{-2Hs}(-1 + e^{2Hs})^2}{4H^2} + \frac{k_2 e^{-2Hs}(1 + e^{2Hs})(2e^{Hs} k_1 + k_2 k_3)}{4H^4} \right] B(s) \] (3.64)

Where \( H = \sqrt{k_1^2 + k_2 k_3} \)

**Proof:** We can write the position vector in the form:

\[ \alpha (s) = \lambda (s) T(s) + \mu (s) N_1(s) + \nu (s) B(s) \] (3.65)

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.65) after differentiation become:

\[ \alpha' = \lambda' T + \lambda T' + \mu' N_1 + \mu N_1' + \nu' B + \nu B' \] (3.66)

From the equations (2.3) and (3.66) we find:

\[
\begin{align*}
\lambda' - \mu k_1 + \nu k_3 &= 1 \\
\lambda k_1 + \mu' &= 0 \\
\lambda k_2 + \nu' &= 0 \\
\lambda k_2 &= 0
\end{align*}
\]

If: \( k_2 \neq 0 \) → \( \lambda = 0 \) (3.68)

\[ \nu = c \] (3.69)

And

\[ \mu = \frac{1-c k_3}{k_1} \] (3.70)
So we get
\[ \alpha (s) = \frac{1-ck_3}{k_1} N_1(s) + cN_2(s) \] (3.71)

And if \( k_2 = 0 \rightarrow \lambda \neq 0 \)
\[
\lambda = -\frac{k_1^2 e^{-2Hs}(-1 + e^{2Hs})(1 + e^{2Hs}) - k_2 k_3 e^{-2Hs}(-1 + e^{2Hs})(1 + e^{2Hs})}{4H^3} + \frac{e^{-2Hs}(-1 + e^{2Hs})(1 + e^{2Hs})}{4H^3} + \frac{2}{C_1} e^{-Hs}(1 + e^{2Hs}) + \frac{C_2 k_1 e^{-Hs}(-1 + e^{2Hs})}{2H} \]
(3.72)

\[
\mu = -\frac{k_1 k_2 k_3 e^{-2Hs}(-1 + e^{Hs})^2(1 + e^{2Hs})}{4H^4} + \frac{k_1 e^{-2Hs}(-1 + e^{2Hs})^2}{4H^2} - \frac{k_1 e^{-2Hs} (1 + e^{2Hs}) (k_1^2 + k_2 e^{2Hs} + 2k_2 k_3 e^{Hs})}{2H^2} + \frac{C_1 k_1 e^{-Hs}(-1 + e^{2Hs})}{2H^2} \]
(3.73)

\[
v = \frac{e^{-2Hs}(-1 + e^{Hs})^2(1 + e^{2Hs})k_1^2 k_2}{4H^4} - \frac{k_2 e^{-2Hs}(-1 + e^{2Hs})^2}{4H^2} \]

Where \( H = \sqrt{k_1^2 + k_2 k_3} \)

We get the equation (3.64) after substituting equations (3.72), (3.73) and (3.74) into (3.65)

**Theorem 3.9:** If we suppose that the curve which is lies on the subspace spanned by \( \{T, N_2, B\} \) are: \( \alpha: R \rightarrow E_4 \), then the position vector of Space curve \( \alpha \) can be written in the form:

\[
\alpha (s) = c N_2(s) + \left[ \frac{-(1 + ck_2)}{k_3} \right] B(s) \] (3.75)

If: \( k_1 \neq 0 \)
And if \( k_1 = 0 \) it can be written in the form:
\[ \alpha (s) = \frac{e^{-2F_s(-1 + e^{2F_s})}(-1 + e^{2F_s}) + k_3^2 e^{-2F_s}(-1 + e^{2F_s})}{4F} \]
\[ - \frac{k_2^2 e^{-2F_s}(-1 + e^{2F_s})(1 + e^{2F_s})}{4F(k_2^2 + k_3^2)} + \frac{C_1}{2} e^{-F_s(1 + e^{2F_s})} - \frac{C_2 k_2 e^{-F_s}(-1 + e^{2F_s})}{2(k_2^2 + k_3^2)} \]
\[ - \frac{C_3 k_3 e^{-F_s}(-1 + e^{2F_s})}{2(k_2^2 + k_3^2)} \]
\[ T(s) \]
\[ + \frac{k_2 e^{-2F_s}(-1 + e^{2F_s})^2}{4(k_2^2 + k_3^2)} + \frac{k_2 k_3^3 e^{-2F_s}(-1 + e^{2F_s})^2}{4F^2(k_2^2 + k_3^2)} \]
\[ - \frac{k_2 e^{-2F_s}(1 + e^{2F_s})(2F_s k_3^2 + k_2^2 + e^{2F_s} k_2^2)}{4F^2(k_2^2 + k_3^2)} + \frac{C_1 k_2 e^{-F_s}(-1 + e^{2F_s})}{2(k_2^2 + k_3^2)} \]
\[ C_2 e^{-F_s}(2F_s k_3^2 + k_2^2 + e^{2F_s} k_2^2) \]
\[ + \frac{2}{2(k_2^2 + k_3^2)} N_2(s) \]
\[ + \frac{e^{-2F_s}(-1 + e^{2F_s})^2 k_3}{4F^2} + \frac{e^{-2F_s}(-1 + e^{2F_s})^2(1 + e^{2F_s}) k_3 k_2^2}{4F^2(k_2^2 + k_3^2)} \]
\[ - \frac{k_3 e^{-2F_s}(1 + e^{2F_s})(k_3^2 + e^{2F_s} k_3^2 + 2 e^{2F_s} k_2^2)}{4F^2(k_2^2 + k_3^2)} + \frac{C_1 e^{-F_s}(-1 + e^{2F_s}) L F}{2(k_2^2 + k_3^2)} \]
\[ + \frac{C_2 e^{-F_s}(-1 + e^{2F_s}) k_2 k_3}{2(k_2^2 + k_3^2)} + \frac{C_3 e^{-F_s}(k_3^2 + e^{2F_s} k_3^2 + 2 e^{2F_s} k_2^2)}{2(k_2^2 + k_3^2)} B(s) \]

Where: \( F = \sqrt{-k_2^2 - k_3^2} \)

**Proof:** We can write the position vector in the form:

\[ \alpha (s) = \lambda(s) T(s) + \mu(s) N_2(s) + v(s) B(s) \]  

(3.77)

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.77) after differentiation become:

\[ \alpha' = \lambda' T + \lambda T' + \mu' N_2 + \mu N_2' + v' B + v B' \]  

(3.78)

From the equations (2.3) and (3.78) we find:

\[ \begin{align*}
\lambda' - \mu k_2 - v k_3 &= 0 \\
\lambda k_2' + \mu' &= 0 \\
\lambda k_3' + v' &= 0 \\
\lambda k_1 &= 0
\end{align*} \]

(3.79)

If \( k_1 \neq 0 \) \( \rightarrow \) \( \lambda = 0 \)  

(3.80)

\( \mu = c \)  

(3.81)
\[ \nu = \frac{-(1 + ck_2)}{k_3} \quad (3.82) \]

Then
\[ \alpha (s) = c N_2(s) + \left[ \frac{-(1 + ck_2)}{k_3} \right] B(s) \quad (3.83) \]

And if \( k_1 = 0 \rightarrow \lambda \neq 0 \)

\[ \lambda = \frac{e^{-2Fs}(-1 + e^{2Fs})^2}{4F} - \frac{k_3^2 e^{-2Fs}(-1 + e^{2Fs})(1 + e^{2Fs})}{4F(k_2^2 + k_3^2)} - \frac{k_2^2 e^{-2Fs}(-1 + e^{2Fs})(1 + e^{2Fs})}{4F(k_2^2 + k_3^2)} \]

\[ + \frac{C_1}{2} e^{-Fs}(1 + e^{2Fs}) - \frac{C_2k_2 e^{-Fs}(-1 + e^{2Fs})}{2(k_2^2 + k_3^2)} - \frac{C_3k_3 e^{-Fs}(-1 + e^{2Fs})}{2(k_2^2 + k_3^2)} \quad (3.84) \]

\[ \mu = \frac{k_2 e^{-2Fs}(-1 + e^{2Fs})^2}{4(k_2^2 + k_3^2)} + \frac{k_3^2 e^{-2Fs}(-1 + e^{2Fs})^2(1 + e^{2Fs})}{4F^2(k_2^2 + k_3^2)} \]

\[ + \frac{4F^2(k_2^2 + k_3^2)}{k_2 e^{-2Fs}(-1 + e^{2Fs})(2e^{Fs}k_2^2 + k_2^2 + e^{2Fs}k_2^2)} + \frac{C_1 k_2 e^{-Fs}(-1 + e^{Fs})}{2(k_2^2 + k_3^2)} \]

\[ + \frac{C_2 e^{-Fs}(2e^{Fs}k_3^2 + k_2^2 + e^{2Fs}k_2^2)}{2(k_2^2 + k_3^2)} + \frac{C_3 k_2k_3 e^{-Fs}(-1 + e^{Fs})^2}{2(k_2^2 + k_3^2)} \quad (3.85) \]

\[ \nu = \frac{e^{-2Fs}(-1 + e^{2Fs})^2 k_2}{4F^2} + \frac{e^{-2Fs}(-1 + e^{2Fs})^2(1 + e^{2Fs})k_3k_2^2}{4F^2(k_2^2 + k_3^2)} \]

\[ + \frac{k_3 e^{-2Fs}(1 + e^{2Fs})(k_3^2 + e^{2Fs}k_2^2 + 2e^{Fs}k_2^2)}{4F^2(k_2^2 + k_3^2)} + \frac{C_1 e^{-Fs}(-1 + e^{2Fs})LF}{2(k_2^2 + k_3^2)} \]

\[ + \frac{C_2 e^{-Fs}(-1 + e^{Fs})^2 k_2k_3}{2(k_2^2 + k_3^2)} + \frac{C_3 e^{-Fs}(k_3^2 + e^{2Fs}k_2^2 + 2e^{Fs}k_2^2)}{2(k_2^2 + k_3^2)} \quad (3.86) \]

Where: \( F = \sqrt{-k_2^2 - k_3^2} \)

We get the equation (3.76) after substituting equations (3.84), (3.85) and (3.86) into (3.77)

**Theorem 3.10:** If we suppose that the curve which is lies on the subspace spanned by \( \{N_1, N_2, B\} \) are: \( \alpha : R \rightarrow E_4^4 \), then the position vector of Space curve \( \alpha \) can be written in the form:

ISSN 2055-009X(Print), ISSN 2055-0103(Online)
\[ \alpha (s) = cT(s) + c_1N_2(s) + \left[ \frac{(1 + c_1 k_2 + c k_3)}{k_3} \right] B(s) \]  

(3.87)

**Proof:** We can write the position vector in the form:

\[ \alpha (s) = \lambda(s)N_1(s) + \mu(s)N_2(s) + \nu(s)B(s) \]  

(3.88)

Where \( \lambda = \lambda(s) \) and \( \mu = \mu(s) \) are differentiable functions of \( s \). The equation (3.88) after differentiation become:

\[ \alpha' = \lambda' N_1 + \lambda N_1' + \mu N_2' + \mu' N_2 + \nu' B + \nu B' \]  

(3.89)

From the equations (2.3) and (3.89) we find:

\[
\begin{cases}
\mu k_2 + \lambda k_1 + \nu k_3 = -1 \\
\mu' = 0 \\
\lambda' = 0 \\
\nu' = 0
\end{cases}
\]  

(3.90)

\[ \lambda = c \]  

(3.91)

\[ \mu = c_1 \]  

(3.92)

\[ \nu = \frac{-(1 + c_1 k_2 + c k_3)}{k_3} \]  

(3.93)

Then

\[ \alpha (s) = cT(s) + c_1N_2(s) + \left[ \frac{(1 + c_1 k_2 + c k_3)}{k_3} \right] B(s) \]  

(3.94)

**REFERENCES**


