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SINC-JACOBI COLLOCATION ALGORITHM FOR SOLVING THE TIME-FRACTIONAL DIFFUSION-WAVE EQUATIONS

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ABSTRACT*: In this paper, we present a numerical method for fractional diffusion equations with variable coefficients. This method is based on Shifted Jacobi collocation scheme and Sinc functions approximation for temporal and spatial discretizations, respectively. The method consists of reducing the problem to the solution of linear algebraic equations by expanding the required approximate solution as the elements of shifted Jacobi polynomials in time and the Sinc functions in space with unknown coefficients. Some examples are provided to illustrate the applicability and the simplicity of the proposed numerical scheme.*

KEYWORDS*:* shifted Jacobi polynomials, Sinc functions, Collocation method, Fractional diffusion equation.

INTRODUCTION

Fractional calculus is a generalization of classical calculus, which provides an excellent tool to describe memory and hereditary properties of various materials and process. The field of the fractional differential equations draws special interest of researchers in several areas including chemistry, physics, engineering, finance and social sciences [1]. The fractional diffusion equations is a generalization of the usual diffusion equations. In particular, the fractional diffusion equation has been used to model many important physical phenomena ranging from amorphous, colloid, glassy, and porous materials through fractals, percolation clusters, random and disordered media to comb structures, dielectrics and semiconductors, polymers and biological systems [2].

Some researchers have proposed the numerical approximation for the time-fractional diffusion equations [3, 4, 5]. Several techniques have been suggested to solve fractional diffusion equations such that Uddin et al. [6] proposed a technique based on the radial basis functions method for nonlinear diffusion equations. Lin et al. [7] used finite difference and spectral approximations for the numerical solutions of the time-fractional diffusion equation. Zhuang et al. [8] proposed explicit and implicit Euler approximations. Saadatmandi et al. [9] and Mao et al.[10] described the numerical solution for the time-fractional diffusion equations using Sinc–Legendre and Sinc–Chebyshev respectively.
In this paper we shall consider the fractional order diffusion equation of the form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + p(x)\frac{\partial u(x,t)}{\partial x} + q(x)\frac{\partial^{2} u(x,t)}{\partial x^{2}} = f(x,t); \quad a < x < b, \ \ 0 < t < \tau \qquad \ldots(1)$$

where $p(x)$ and $q(x)$ are non-zero continuous functions and $0 < \alpha \leq 1$.

subject to the three kinds of initial and boundary conditions:

Case 1 : $u(x,0) = 0, \quad 0 \le x \le 1$ …(2)
$u(0,t) = u(1,t) = 0, \quad 0 \le t \le \tau,$ …(3)

Case 2 : $u(x,0) = 0,\ u_t(x,0) = 0, \quad 0 \le x \le 1$ …(4)
$u_x(0,t) = u_x(1,t) = 0, \quad 0 \le t \le \tau,$ …(5)

Case 3 : $u(x,0) = 0,\ u_t(x,0) = 0, \quad 0 \le x \le 1$ …(6)
$u(0,t) = u(1,t) = 0, \quad 0 \le t \le \tau,$ …(7)

We construct the solution of the fractional diffusion Eq. (1) with respect to the three cases above based on the collocation techniques. Since a fractional derivative is a global operator, it is very natural to consider a global method like the collocation method for its numerical solution [11]. Our method consists of reducing the problem to the solution of algebraic equations by expanding the required approximate solution as the elements of the shifted Jacobi polynomials in time and the Sinc functions in space with unknown coefficients. The properties of Sinc functions and shifted Jacobi polynomials are then utilized to evaluate the unknown coefficients. The proposed approach can be considered as a generalization for the works of [9] and [10] respectively since shifted Legendre polynomials and shifted Chebyshev polynomials can be considered as a special cases of shifted Jacobi polynomials[11].

Fractional derivative and Integration

In this section, we shall review the basic definitions and properties of fractional integral and derivatives, which are used further in this paper[1].

Definition(1):- The Riemann-Liouville fractional integral operator of order $v > 0$, is defined as

$$I^v f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt, \quad v > 0, x > 0. \qquad \ldots(8)$$
$$I^0 f(x) = f(x)$$

Definition(2):- The Riemann-Liouville fractional derivative operator of order $v > 0$,is defined as

$${}_0D_x^v f(x) = \frac{1}{\Gamma(n-v)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-v-1} f(t)dt, \quad v > 0, x > 0. \qquad \ldots(9)$$

Where n is an integer and $n-1 < v < n$.

Definition(3):- The Caputo fractional derivative operator of order $v > 0$, is defined as

$${}^cD_x^v f(x) = \frac{1}{\Gamma(n-v)} \int_0^x (x-t)^{n-v-1} \frac{d^n}{dx^n} f(t)dt, \quad v > 0, x > 0 \qquad \ldots(10)$$

Where n is an integer and $n-1 < v \le n$.
Caputo fractional derivative has an useful property:

$$I^v\ {}^cD_x^v f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!} \qquad \ldots(11)$$

Where n is an integer and $n-1 < v \le n$.
Also, for the Caputo fractional derivative we have

$${}^cD_x^v x^\beta = \begin{cases} 0 & for\ \beta \in N_0\ and\ \beta < \lceil v \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} x^{\beta-v}, & for\ \beta \in N_0\ and\ \beta \ge \lceil v \rceil\ or\ \beta \notin N\ and\ \beta > \lfloor v \rfloor. \end{cases} \qquad \ldots(12)$$

We use the ceiling function $\lceil v \rceil$ to denote the smallest integer greater than or equal to v, and the floor function $\lfloor v \rfloor$ to denote the largest integer less than or equal to v. Also $N = \{1,2,\dots\}$ and $N_0 = \{0,1,2,\dots\}$.
Recall that for $v = 0$, the Caputo differential operator concides with the usual differential operator of an integer order. Similar to the integer-order differentiation, the Caputo fractional differentiation is a linear operator; i.e.

$$ {}^{c}D_x^{v}\big(\lambda f(x) + \mu g(x)\big) = \lambda\, {}^{c}D_x^{v} f(x) + \mu\, {}^{c}D_x^{v} g(x) \qquad \dots(13) $$

Where λ and μ are constants.

Sinc functions

For the last three decades, Sinc numerical methods have been extensively used for solving differential equations, not only because of their exponential convergence rate, but also due to their desirable behavior toward problems with singularities[12].
The sinc function is defined on the whole real line $\mathrm{I} = (-\infty, \infty)$ by

$$ Sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \qquad \dots(14) $$

For each integer k and the mesh size h, the sinc basis functions are defined on R by [13]

$$ S_k(h,x) \equiv Sinc\left(\frac{x-kh}{h}\right) = \begin{cases} \frac{\sin\left(\frac{\pi}{h}(x-kh)\right)}{\left(\frac{\pi}{h}(x-kh)\right)}, & x \neq kh, \\ 1, & x = kh. \end{cases} \qquad \dots(15) $$

If a function f(x) is defined on $\mathcal{R}$, then for h > 0 the series

$$ C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) Sinc\left(\frac{x-kh}{h}\right), \qquad \dots(16) $$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of the Whittaker cardinal expansion have been extensively studied in [14]. These properties are derived in the infinite strip DS of the complex ω-plane, where for d > 0,

$$ D_S = \left\{\omega = t + is:\ |S| < d \leq \frac{\pi}{2}\right\}, \qquad \dots(17) $$

To construct the approximation over the finite interval $[a,b]$, which is used in this paper, we consider the one-to-one conformal map

$$ \omega = \phi(z) = In\left(\frac{Z-a}{b-Z}\right), \qquad \dots(18) $$

which maps the eye-shaped region

$$ D_E = \left\{z = x + iy:\ \left|\arg\left(\frac{Z-a}{b-Z}\right)\right| < d \leq \frac{\pi}{2}\right\}, \qquad \dots(19) $$

onto the infinite strip D_S. We also define the range of $\Psi = \phi^{-1}$ on the real line as

$$ \Gamma = \{\Psi(t) \in D_E : -\infty < t < +\infty\} = (0, +\infty), $$

Thus we may define the inverse images of the real line and of the evenly spaced nodes $\{kh\}_{k=-\infty}^{k=+\infty}$ as

$$ x_K = \Phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \qquad k = 0, \mp 1, \mp 2, \dots \qquad \dots(20) $$

Hence the numerical process developed in the domain containing the whole real line can be carried over to infinite interval by the inverse map. The basis functions on (a,b) are taken to be the composite translated sinc functions,

$$ S_k(x) \equiv S(k,h) \circ \phi(x) = Sinc\left(\frac{\phi(x) - kh}{h}\right), \qquad \dots(21) $$

Where $S(k,h) \circ \phi(x)$ is defined by $S(k,h)\big(\phi(x)\big)$.

Definition(4):- Let $B(D_E)$ be the class of functions f which are analytic in D_E, satisfy

$$\int_{\Phi^{-1}(t+L)} |f(z)dZ| \to 0, \qquad t \to \pm\infty,$$

Where $L = \left\{iv: |v| < d \le \frac{\pi}{2}\right\}$, and on the boundary of D_E, (denoted ∂D_E), satisfy

$$N(F) = \int_{\partial D_E} |f(z)dZ| < \infty.$$

Interpolation for function in $B(D_E)$ is defined in the following theorem which is proved in [13].

Theorem 1 (Interpolation, see [14]). If $f\phi' \in B(D_E)$ then for all $x \in \Gamma$

$$|f(x) - \sum_{k=-\infty}^{\infty} f(x_K)S_K(x)| \le \frac{N(f\phi')}{2\pi d \sin h(\pi d/h)} \le 2\frac{N(f\phi')}{\pi d} e^{-\pi d/h}.$$

Moreover, if $|f(x)| \le Ce^{-\gamma|\Phi(x)|}$, $x \in \Gamma$, for some positive constants C and γ, and if the selection $h = \sqrt{\frac{\pi d}{\gamma N}} \le \frac{2\pi d}{ln\,2}$, then

$$|f(x) - \sum_{k=-N}^{N} f(x_K)S_K(x)| \le C_2\sqrt{N}\exp\left(-\sqrt{\pi d\gamma N}\right), \qquad x \in \Gamma,$$

Where C_2 *depends only on* f, d *and* γ.

The above expressions show Sinc interpolation on $B(D_E)$ converges exponentially. We also require derivatives of composite Sinc functions evaluated at the nodes. The expressions required for the present discussion are [14].

$$\delta_{k,j}^{(0)} = [S_K(x)]|_{x=x_j} = \begin{cases} 1, & k = j, \\ 0, & k \ne j, \end{cases} \qquad \ldots(22)$$

$$\delta_{k,j}^{(1)} = \frac{d}{d\Phi}[S_K(x)]|_{x=x_j} = \begin{cases} 0 & , \quad k = j, \\ \frac{(-1)^{j-k}}{j-k}, & k \ne j, \end{cases} \qquad \ldots(23)$$

$$\delta_{k,j}^{(2)} = \frac{d^2}{d\Phi^2}[S_K(x)]|_{x=x_j} = \begin{cases} \frac{-\pi^2}{3} & , \quad k = j, \\ \frac{-2(-1)^{j-k}}{(j-k)^2}, & k \ne j, \end{cases} \qquad \ldots(24)$$

The Shifted Jacobi Polynomials.

The well-known Jacobi polynomials [15] are defined on the interval [-1,1] and can be generated with the aid of the following recurrence formula:

$$P_i^{(\alpha,\beta)}(t) = \frac{(\alpha+\beta+2i-1)\left\{\left(\alpha^2-\beta^2+t(\alpha+\beta+2i)\right)(\alpha+\beta+2i-2)\right\}}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)} P_{i-1}^{(\alpha,\beta)}(t)$$
$$-\frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)} P_{i-2}^{(\alpha,\beta)}(t) \qquad i = 1,2,\ldots, \qquad \ldots(25)$$

where $P_0^{(\alpha,\beta)}(t) = 1$ and $P_1^{(\alpha,\beta)}(t) = \frac{\alpha+\beta+2}{2}t + \frac{\alpha-\beta}{2}$. We also define the so-called shifted Jacobi polynomials on the interval $[0, L]$ by using the change of variable $t = \frac{2x}{L} - 1$. So Shifted Jacobi polynomials $P_i^{(\alpha,\beta)}(\frac{2x}{L} - 1)$ are denoted by $P_{L,i}^{(\alpha,\beta)}(x)$. Shifted Jacobi polynomials of x can be determined with the aid of the

$$P_{L,i}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+2i-1)\left\{\left(\alpha^2-\beta^2+(\frac{2x}{L}-1)(\alpha+\beta+2i)\right)(\alpha+\beta+2i-2)\right\}}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)} P_{L,i-1}^{(\alpha,\beta)}(x)$$
$$-\frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)} P_{L,i-2}^{(\alpha,\beta)}(x) \qquad i = 1,2,\ldots, \qquad \ldots(26)$$

where $P_{L,0}^{(\alpha,\beta)}(x) = 1$ and $P_{L,1}^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+2}{2}\left(\frac{2x}{L} - 1\right) + \frac{\alpha-\beta}{2}$. $\ldots(27)$

The analytic form of the n-degree shifted Jacobi polynomials is given by

$$P_{L,i}^{(\alpha,\beta)}(x) = \sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(\alpha+\beta)\Gamma(i+K+\alpha+\beta+1)}{\Gamma(K+\beta+1)\Gamma(i+\alpha+\beta+1)(i-K)!K!L^k}x^k,\ i = 1,2,\ldots, \qquad \ldots(28)$$

Where

$$P_{L,i}^{(\alpha,\beta)}(0) = (-1)^i\frac{\Gamma(i+\beta+1)}{i!\Gamma(\beta+1)} \quad \text{and} \quad P_{L,i}^{(\alpha,\beta)}(L) = \frac{\Gamma(i+\beta+1)}{i!\Gamma(\alpha+1)} \qquad \ldots(29)$$

Of these polynomials, the most commonly used are the shifted Gegenbauer (ultraspherical) polynomials (symmetric shifted Jacobi polynomials) $C_{L,i}^{\alpha}(x)$; the shifted Chebyshev polynomials of the first kind $T_{L,i}(x)$; the shifted Legendre polynomials $L_{L,i}(x)$; the shifted Chebyshev polynomials of the second kind $U_{L,i}(x)$; and for the nonsymmetrical shifted Jacobi polynomials, the two important special cases of shifted Chebyshev polynomials of third and fourth kinds $V_{L,i}(x)$ and $W_{L,i}(x)$ are also considered. These orthogonal polynomials are interrelated to the shifted Jacobi polynomials by the following relations

$$\left.\begin{aligned}
&C_{L,i}^{\alpha}(x) = \frac{i!\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(i+\alpha+\frac{1}{2}\right)}P_{L,i}^{(\alpha-\frac{1}{2},\beta-\frac{1}{2})}(x),\ T_{L,i}(x) = \frac{i!\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(i+\frac{1}{2}\right)}P_{L,i}^{(-\frac{1}{2},-\frac{1}{2})}(x)\\
&L_{L,i}(x) = P_{L,i}^{(0,0)}(x), \qquad U_{L,i}(x) = \frac{(i+1)i!\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(i+\frac{3}{2}\right)}P_{L,i}^{(\frac{1}{2},\frac{1}{2})}(x)\\
&V_{L,i}(x) = \frac{(2i)!!}{(2i-1)!!}P_{L,i}^{(\frac{1}{2},-\frac{1}{2})}(x), \quad W_{L,i}(x) = \frac{(2i)!!}{(2i-1)!!}P_{L,i}^{(-\frac{1}{2},\frac{1}{2})}(x)
\end{aligned}\right\} \qquad \ldots(30)$$

The orthogonality condition of shifted Jacobi polynomials is

$$\int_0^L P_{L,j}^{(\alpha,\beta)}(x)P_{L,k}^{(\alpha,\beta)}(x)W_L^{(\alpha,\beta)}(x)dx = h_k, \qquad \ldots(31)$$

Where $W_L^{(\alpha,\beta)}(x) = x^{\beta}(L-x)^{\alpha}$

And

$$h_k = \begin{cases}\dfrac{\ell^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)}, & i = j\\ 0 & i \neq j.\end{cases} \qquad \ldots(32)$$

Lemma (1)[15]: Caputo's fractional derivative of order $v > 0$ for the shifted Jacobi polynomials $P_{L,j}^{(\alpha,\beta)}(x)$ is given by :

$${}^cD_0^v P_{\tau,i}^{(\alpha,\beta)}(t) = \sum_{k=\lceil v\rceil}^{i} b_{i,k}^{(\alpha,\beta)}t^{k-v}, \quad i = \lceil v\rceil, \lceil v\rceil+1, \ldots, \qquad \ldots(33)$$

$${}^cD_0^v P_{\tau,i}^{(\alpha,\beta)}(t) = 0, \quad i = 0,1,\ldots,\lceil v\rceil-1, \quad v > 0 \qquad \ldots(34)$$

and

$$b_{i,k}^{(\alpha,\beta)} = (-1)^{i-k}\frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!\Gamma(k-\alpha+1)\tau^k} \qquad \ldots(35)$$

Lemma (2)[9] Let $1 < v < 2$ and x_k be spatial collocation points given in (20). Then the following relations hold:

$$\frac{\partial^v u_{m,n}(x_k,t)}{\partial x^\alpha} = \sum_{i=1}^{n}\sum_{r=1}^{j} c_{kj}\, b_{jr}\, t^{r-v}\,, \qquad \ldots(36)$$

$$\frac{\partial u_{m,n}(x_k,t)}{\partial x} = \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\, q_{ik}^{(1)} P_{L,j}^{(\alpha,\beta)}(t)\,, \qquad \ldots(37)$$

$$\frac{\partial^2 u_{m,n}(x_k,t)}{\partial x^2} = \sum_{i=-m}^{m}\sum_{r=0}^{n} c_{kj}\, q_{ik}^{(2)} P_{L,j}^{(\alpha,\beta)}(t)\,, \qquad \ldots(38)$$

where

$$q_{ik}^{(1)} = \phi'(x_k)\delta_{i,k}^{(1)}, \qquad \text{and} \qquad q_{ik}^{(2)} = \phi''(x_k)\delta_{i,k}^{(1)} + (\phi'(x_k))^2\delta_{i,k}^{(2)},$$

(See [9] for the proof.)

Description of the method

To solve equation (1) with respect to the initial and boundary conditions given by equations (2)-(7) individually, we first approximate $u(x,t)$ by the (n + 1) shifted Jacobi polynomials and (2 m + 1) Sinc functions as

$$u_{m,n}(x,t) = \sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\, S_i(x) P_{L,j}^{(\alpha,\beta)}(t). \qquad \ldots(39)$$

Where $\lim_{x\to a} S_i(x) = \lim_{x\to b} S_i(x) = 0$ and this guarantees that $u_{m,n}$ satisfies the equations (3),(5) and (7) respectively.

Substituting Eq. (39) into Eq. (1) we obtain

$$\frac{\partial^v u_{m,n}(x,t)}{\partial t^v} + p(x)\frac{\partial u_{m,n}(x,t)}{\partial x} + q(x)\frac{\partial^2 u_{m,n}(x,t)}{\partial x^2} = f(x,t); \quad a < x < b,\ 0 < t < \tau \quad \ldots(40)$$

and evaluating the result at the points x_k given in Eq. (20) and for $t = t_\ell$. For suitable collocation points we use the shifted Jacobi roots t_ℓ , $\ell = 1, \ldots$ n+1 of $P_{L+1,j}^{(\alpha,\beta)}(t)$. So according to Lemma(1), we have

$$\sum_{i=1}^{n}\sum_{r=1}^{j} c_{kj}\, b_{jr}\, t_\ell^{r-\alpha} + p(x_k)\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\, q_{ik}^{(1)} P_{L,j}^{(\alpha,\beta)}(t_\ell) +$$
$$q(x_k)\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\, q_{ik}^{(2)} P_{L,j}^{(\alpha,\beta)}(t_\ell) = f(x_k,t_\ell), \quad k = -m,\ldots,m, \quad \ell = 1,\ldots,n \quad \ldots(41)$$

For case 1 by applying $u_{m,n}$ to Eq.(2), so we have

$$\sum_{i=-m}^{m}\sum_{j=0}^{n} c_{ij}\, S_i(x) L_j^h(0) = \Psi(x). \qquad \ldots(42)$$

and Collocating Eqs. (42) in $(2m+1)$ points x_k , we obtain

$$\sum_{j=0}^{n} c_{kj}\,(-1)^j \frac{\Gamma(j+\beta+1)}{\Gamma(\beta+1)j!\tau} = \Psi(x_k), \quad k = -m,\ldots,m.$$

For case 2 and 3 respectively applying $u_{m,n}$ to equations (4) and (6) and collocating at $(2m+1)$ points x_k we get

$$\sum_{j=0}^{n} c_{kj}\,(-1)^{j-1} \frac{\Gamma(j+\beta+1)\Gamma(j+\alpha+\beta+2)}{\Gamma(\beta+2)\Gamma(j+\alpha+\beta+1)(j-1)!\tau} = \varphi(x_k), \quad k = -m,\ldots,m. \qquad \ldots(43)$$

According to the all cases above we have $(n+1)(2m+1)$ system of linear equations with $(n+1)(2m+1)$ unknown coefficients say c_{ij} $i = 1,2,\ldots,2m+1$, $j = 1,2,\ldots,n+1$. And this system can be expressed in a matrix form

$$AC = B. \qquad \ldots(44)$$

Equation (44) can be solved easily for the unknown coefficients c_{ij}. Consequently $u_{m,n}(x,t)$ given in (39) can be calculate

Numerical examples

In order to verify the performance and functionality of the proposed method three examples are examined

Example 1. Consider the following time-fractional diffusion equation

$$\frac{\partial^v u_{m,n}(x,t)}{\partial x^v} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + \frac{6t^{(3-v)}}{\Gamma(4-v)}(x^3 - x^4) + t^3(-6x + 15x^2 - 4x^3),$$

$$0 < x < 1,\ 0 < t < 1,\ 0 < v < 1$$

With respect to the initial condition

$$u(x,0) = 0,\ 0 \le x \le 1$$

And the boundary conditions

$$u(0,t) = u(1,t) = 0,\ 0 \le t \le 1$$

The exact solution to this problem is given by [16] :

$$u(x,t) = t^3 x^3 (1-x).$$

We solved the problem, by applying the technique described in Section 3 ($v = 0.5$), we chose $\gamma = 1$ and $d = \frac{\pi}{2}$, and this leads to $h = \frac{\pi}{\sqrt{2m}}$ and $\alpha = \beta = 0$.

following Figure.1 represent a comparison between the exact and numerical solution given by the proposed method for $m = 15$ and $n = 8$.

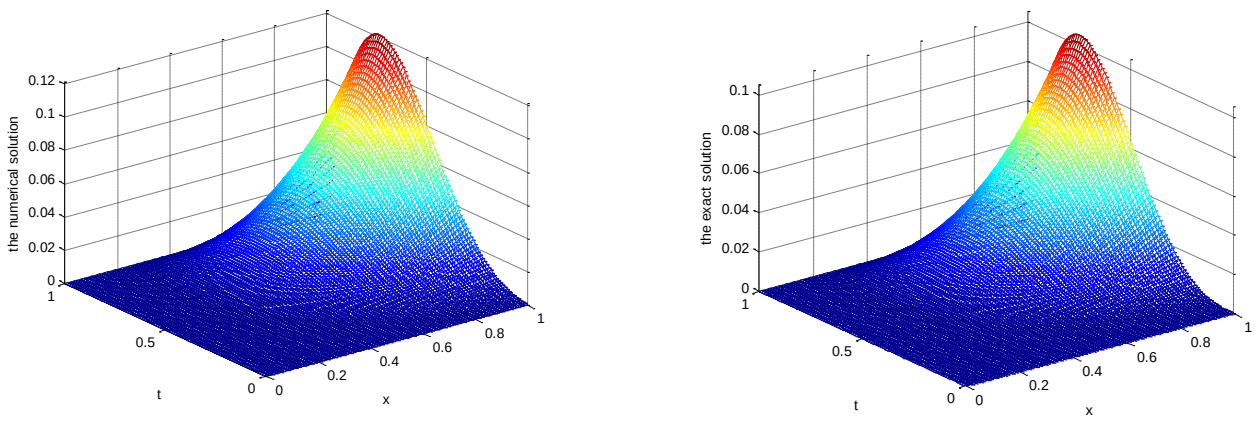

Figure 1: Comparison of the numerical and exact solution in the domain [0,1] × [0,1] for Example 1

Example 2. Consider the following fractional wave equation:

$$\frac{\partial^v u(x,t)}{\partial t^v} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 < x < 1,\ 0 < t \le \tau,\ 1 < v < 2$$

With respect to the initial conditions

$$u(x,0) = 0,\ u_t(x,0) = 0, \qquad 0 \le x \le 1,$$

With Neumann Boundary Conditions

$$u_x(0,t) = u_x(1,t) = 0, \qquad 0 < t \le \tau,$$

Where $f(x,t) = \frac{\Gamma(v+3)}{2} t^2 e^x x^2 (1-x)^2 - e^x t^{v+2}(2 - 8x + x^2 + 6x^3 + x^4)$

is the the corresponding forcing term. The exact solution to this problem is given by[17] :

$$u(x,t) = e^x x^2 (1-x)^2 t^{\alpha+2}.$$

We have been solved the problem, by applying the technique described in Section 3, and for this purpose we chose $\gamma = 1$ and $d = \frac{\pi}{2}$, and this leads to $h = \frac{\pi}{\sqrt{2m}}$, Figure.2 shows the maximum absolute error function $|u(x,t) - u_{m,n}(x,t)|$ obtained by the presented method with $m = 15$ and $n = 8$ when ($v = 1.5$, $\alpha = \beta = 0$ left and $\alpha = \beta = -0.5$ right

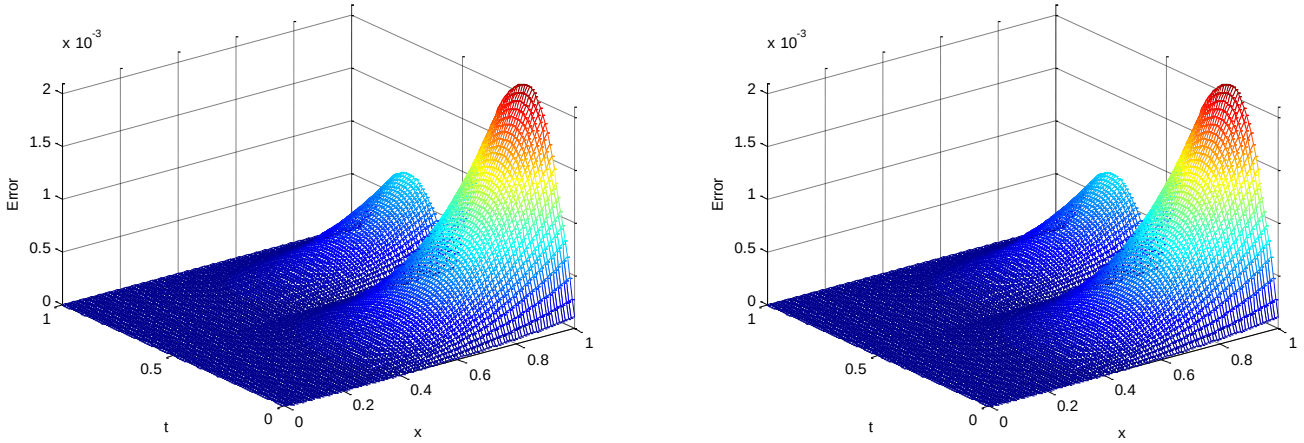

Figure 2: Plot of the absolute error for Example 2 at $v = 1.3$, $\alpha = \beta = 0$ left and $\alpha = \beta = -0.5$ right

Example 3. Consider the following time-fractional diffusion equation

$$\frac{\partial^v u_{m,n}(x,t)}{\partial x^v} = \frac{\partial^2 u(x,t)}{\partial x^2} + sin(\pi x), \quad 0 < x < 1,\ 0 < t \le 1,\ 1 < v < 2$$

With respect to the initial condition

$$u(x,0) = 0,\ u_t(x,0) = 0, \quad 0 \le x \le 1$$

And the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 1$$

The exact solution to this problem is given by [10].

$$u(x,t) = \frac{1}{\pi^2}[1 - E_\alpha(-\pi^2 t^\alpha)]\sin(\pi x).$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(vk+1)}$ is the one-parameter Mittag-Leffler function. We have been solved the problem, by applying the technique described in Section 3 for ($v = 1.7$), and to this aim we chose $\gamma = 1$ and $d = \frac{\pi}{2}$, and this leads to $h = \frac{\pi}{\sqrt{2m}}$ $and\ \alpha = \beta = -0.5$. following Figure.3 represent a comparison between the exact and numerical solution given by the proposed method for $m = 15\ and\ n = 8$.

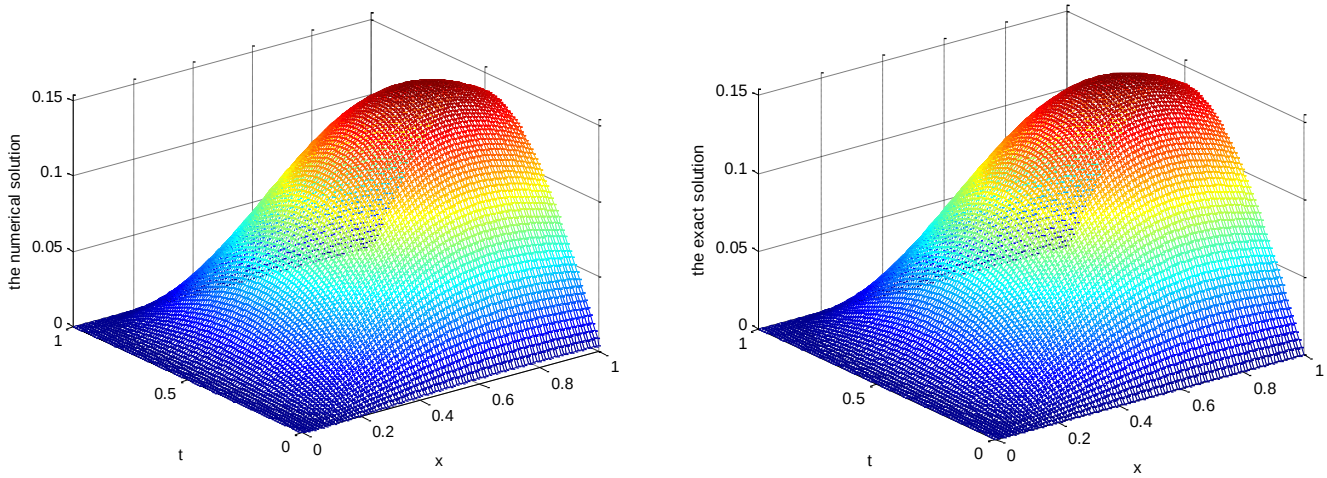

Figure 3: Comparison of the numerical and exact solution $\alpha = \beta = -0.5$ for Example 3

CONCLUSION

In this paper, we develop and analyze the efficient numerical algorithm for the fractional diffusion wave-equation Based on the collocation technique, the sinc functions and shifted Jacobi polynomials are used to reduce the solution of the fractional diffusion wave-equation with respect to the initial and boundary conditions given in cases (1), (2) and (3) respectively to the solution of a system of linear algebraic equations. Also it is important to mention that the proposed approach can be considered as a generalization to the works given in [9] and [10]. From the computational point of view, the solution obtained by this method is in excellent agreement with the exact one.

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