International Journal of Mathematics and Statistics Studies
Vol.4, No.2, pp.11-19, April 2016
Published by European Centre for Research Training and Development UK (www.eajournals.org)

SENSITIVITY AND DUALITY ANALYSES OF AN OPTIMAL WATER TREATMENT COST MODEL FOR GHANA

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ABSTRACT: *In this paper, sensitivity and duality analyses have been performed on our earlier developed optimal water treatment cost model for Ghana. Linear Programming was used to formulate the model and tested with real data collected from Weija Water Headworks in Accra using Interior-Point Method to obtain solutions. The effects of variations of selected key parameters on the developed model have now been investigated. Marginal costs of water production in the selected water headworks have also been found. It is strongly recommended that all Water Headworks under Ghana Water Company Limited (GWCL) should employ at least one Operations Researcher to assist them in some of these post-optimality analyses.*

KEYWORDS: Sensitivity Analysis, Duality Analysis, Linear Programming, Optimal Water Treatment Cost Model, Interior-Point Method.

INTRODUCTION

Formally, sensitivity analysis is the study of how the uncertainty in the output of a mathematical model or system (numerical or otherwise) can be apportioned to different sources of uncertainty in its inputs (Saltelli et al., 2008). In Linear Programming (LP), Sensitivity analysis is concerned with how changes in an LP's parameters affect the LP's optimal solution. In the context of linear programming, duality implies that linear programming problem can be analysed in two different ways but would have equivalent solutions. Any LP problem (either maximization or minimization) can be stated in another equivalent form based on the same data. The new LP problem is called dual programming problem or in short dual. If the optimal solution to any one is known, the optimal solution to the other can readily be obtained. In fact, it is immaterial which problem is designated the primal since the dual of a dual is the primal. Because of these properties, the solution of a linear programming problem can be obtained by solving either the primal or the dual, whichever is easier (Rao, 2009). In our earlier paper, we developed an optimal water treatment cost model for the Ghana Water Company Limited (GWCL) to meet water quality standards while saving cost in order to perhaps reduce tariffs and also make potable water accessible to a larger section of the population. The effects of variations of selected key parameters on the developed model and the marginal costs of water production in the selected water headworks are worth investigating. This paper therefore seeks to do exactly that.

LITERATURE

In an LP model, the coefficients (also known as parameters) such as: (i) profit (cost) contribution (cj) per unit of a decision variable, xj (ii) availability of a resource (bi) and (iii) consumption of resource per unit of decision variables (aij), are assumed to be constant and are known with certainty during a planning period. However, in real-world situations, these values of the input parameters may change over a period of time due to dynamic nature of the business environment. Such changes in any of these parameters may raise doubt on the validity of the optimal solution of the given LP model. Thus, a decision-maker, in such situations, would like to know how changes in these parameters may affect the optimal solution and the range within which the optimal solution will remain unchanged with changes in the original input data values.

Formally, sensitivity analysis is the study of how the uncertainty in the output of a mathematical model or system (numerical or otherwise) can be apportioned to different sources of uncertainty in its inputs (Saltelli et al., 2008). Sensitivity analysis is concerned with how changes in an LP's parameters affect the LP's optimal solution. Changes in the problem which are usually studied can be classified into the following five categories:

(a) Changes in the profit or cost coefficient (cj)

(b) Changes in the availability of a resource (bi)

(c) Changes in the technological coefficient (aij)

(d) Addition/Deletion of a constraint

(e) Addition/Deletion of a variable

For detailed analysis and discussion of each of these five categories, the interested reader is referred to Sharma, 2010; Sinha, 2006 and Hillier & Lieberman, 2010.

The term 'dual' in general sense implies two or double. The concept of duality is very useful in Mathematics, Physics, Statistics, Engineering and Managerial Decision Making. In the context of linear programming, duality implies that linear programming problem can be analysed in two different ways but would have equivalent solutions. Any LP problem (either maximization or minimization) can be stated in another equivalent form based on the same data. The new LP problem is called dual programming problem or in short dual. If the optimal solution to any one is known, the optimal solution to the other can readily be obtained. In fact, it is immaterial which problem is designated the primal since the dual of a dual is the primal. Because of these properties, the solution of a linear programming problem can be obtained by solving either the primal or the dual, whichever is easier (Rao, 2009).

For example, consider the problem of production planning. By using primal LP problem, the production manager attempts to optimize resource allocation by determining quantities for each product to be produced that will maximize profit. But through a dual LP problem approach, he attempts to achieve a production plan that optimizes resource allocation in a way that each product is produced at that quantity so that its marginal opportunity cost equals its marginal return. Thus, the main of a dual problem is to find for each resource its best marginal value (also called shadow price). This value reflects the scarcity of the resources. That is the maximum additional price to be paid to obtain one additional unit of the resource constraints.

The shadow price is also defined as the rate of change in the optimal objective function value with respect to the unit change in the availability of a resource. To be more precise for any constraint, we have

$$\text{Shadow price} = \frac{\text{Change in optimal objective function value}}{\text{Unit change in the availability of resource.}}$$

The interpretation of rate of change (increases or decrease) in the value of objective function depends on whether we are solving a maximization of minimization LP problem. The shadow price for a less than or equal to (≤) type constraint will always be greater than or equal to zero. This is because increasing the right-hand side resource value cannot make the value of the objective function worse. Similarly, the shadow price for a greater than or equal to (≥) type constraint will always be less than or equal to zero because increasing the right-hand side value cannot improve the value of the objective function.

In general, the primal-dual relationship between a pair of LP problems can be expressed as follows:

Primal

$$Max \;\; Z_x = \sum_{j=1}^{n} c_j x_j$$

$$Subject\ to \sum_{j=1}^{n} a_{i\,j} x_j \le b_i,\; i = 1, 2 \ldots, m$$

$$x_j \ge 0, \qquad j = 1, 2, \ldots, n$$

Dual

$$Min \;\; Z_y = \sum_{i=1}^{m} b_i y_i$$

$$Subject\ to \sum_{i=1}^{m} a_{j\,i} y_j \ge c_j,\;\; j = 1, 2, \ldots m$$

$$y_i \ge 0, \qquad i = 1, 2, \ldots, m$$

A summary of the general relationships between primal and dual LP problems is given in Table 1.

Table 1: Primal-Dual Relationships.

If Primal	**If Dual**
i) Objective is to maximize	i) Objective is to minimize
ii) Objective is to minimize	ii) Objective is to maximize
iii) jth primal variable, x_j	iii) jth dual constraint
iv) ith primal constraint	iv) ith dual variable, y_i
vi) Primal variable x_j unrestricted in sign	v) Dual constraint j is = type.
vi) Primal constraint i is = type	vi) Dual variable, y_i is unrestricted in sign
vii) Primal constraints ≤ type	vii) Dual constraints ≥ type.

Source: Sharma, 2010

The interested reader is referred to Hillier & Lieberman, 2010; Sharma, 2010 and Thie & Keough, 2008 for detailed analysis and discussion of the duality theory.

METHODOLOGY

Linear Programming (LP) was used to formulate the model for Ghana Water Company Limited. It was then tested with real data collected from Weija Water Headworks in Accra using Interior-Point Method. Interactive Operations Research Tutorial software developed by Hillier et al (2000) was used to run the model.

Linear programming (LP) also called linear optimization is a technique for the optimization of a linear objective function, subject to linear equality or inequality constraints. The objective function may either be maximized or minimized. There are four main assumptions inherent in a LP model that must be taken into account in any application. They are proportionality, additivity, divisibility, and certainty (Hillier and Lieberman, 2000).

Generally, Interior-Point Method searches for an optimal solution of a problem by traversing the interior or inside of the feasible region instead of the boundaries as in Simplex Method. The interested reader is referred to Padberg (1995), Rardin (1998), Sierksma (2002), Roos et al (2006) and Hillier & Lieberman, 2010 for a detailed discussion of Interior-Point Methods.

Sensitivity and duality analyses have been performed on our earlier developed optimal water treatment cost model for Ghana.

EARLIER RESULTS

The Developed Model

The developed optimal water treatment cost model was given as:

$$\text{Minimize } C_T = \alpha X_1 + \beta X_2 + \gamma X_3 + \lambda X_4 + \delta X_5$$

$$\text{Subject to}$$

$$\begin{aligned} \alpha X_1 + \mu X_3 &\geq \Psi \\ \beta X_2 + \rho X_3 &\geq \phi \\ \gamma X_3 &\geq \tau \\ \lambda X_4 &\geq N \\ \delta X_5 &\geq M \\ X_1, X_2, X_3, X_4, X_5 &\geq 0 \end{aligned} \qquad [1]$$

where

C_T = Total Treatment Cost
X_1 = Average Quantity of Chemicals
X_2 = Average Quantity of Electricity
X_4 = Number of Personnel
α = Unit Chemical Cost
λ = Unit Personnel Cost
γ = Unit Fuel Cost
ρ = Unit Fuel Cost in the Pumping House
ϕ = Average Cost in the Pumping House
N = Average Personnel Cost
M = Average Maintenance Cost
X_3 = Average Quantity of Fuel
X_5 = Number of Maintenances in a month
β = Unit Electricity Cost
δ = Unit Maintenance Cost
μ = Unit Fuel Cost in the Chemical House
Ψ = Average Cost in the Chemical House
τ = Average Transportation Cost.

The interested reader is referred to Boah et al., 2016 for a detailed discussion of how the optimal water treatment cost model was developed.

Practical Application of the Model

The developed water treatment cost model was applied to Weija Water Headworks in Accra using secondary data (water treatment/production data for 2014).

Water Treatment Cost Model for Weija Water Headworks based on the collected data on water treatment/ production for 2014 was obtained as follows:

$$\begin{aligned}
&\text{Minimize } C_T = 1.06\,X_1 + 0.56\,X_2 + 3\,X_3 + 2703.5\,X_4 + 4368.23\,X_5 \\
&\text{Subject to} \\
&\qquad 1.06\,X_1 + 0.015\,X_3 \geq 729824.87 \\
&\qquad 0.56\,X_2 + 0.649\,X_3 \geq 1522188.03 \\
&\qquad 3\,X_3 \geq 5040.16 \qquad\qquad [2] \\
&\qquad 2703.5\,X_4 \geq 162210.06 \\
&\qquad 4368.23\,X_5 \geq 17472.92 \\
&\qquad X_1, X_2, X_3, X_4, X_5 \geq 0
\end{aligned}$$

Table 2 below gives a detailed and optimal (asterisked) solution of this model [2] using Interior-Point Method.

TABLE 2: OPTIMAL SOLUTION FOR WATER PRODUCTION AT WEIJA WATER HEADWORKS IN 2014 (Using Interior-Point Method).

ITERATION	X_1	X_2	X_3	X_4	X_5	C_T
0	688495.000	2716250.000	1685.000	60.000	4.000	2435642.680
1	688493.969	2716250.495	1682.527	60.000	4.000	2435634.444
2	688492.693	2716249.187	1681.290	60.000	4.000	2435628.649
3	688491.590	2716247.689	1680.672	60.000	4.000	2435624.787
4	688490.926	2716246.772	1680.362	60.000	4.000	2435622.641
5	688490.588	2716246.306	1680.208	60.000	4.000	2435621.558
6	688490.418	2716246.074	1680.131	60.000	4.000	2435621.017
7	688490.334	2716245.958	1680.092	60.000	4.000	2435620.746
8	688490.291	2716245.900	1680.073	60.000	4.000	2435620.610
9	688490.269	2716245.874	1680.063	60.000	4.000	2435620.544
10	688490.258	2716245.861	1680.058	60.000	4.000	2435620.510
11	688490.251	2716245.864	1680.056	60.000	4.000	2435620.497
12	688490.241	2716245.866	1680.055	60.000	4.000	2435620.484
13	688490.241	2716245.867	1680.054	60.000	4.000	2435620.483
14	688490.241	2716245.542	1680.054	60.000	4.000	2435620.300
15*	688490.201*	2716245.543*	1680.053*	60.000*	4.000*	2435620.258*

Table 3 gives a detailed comparison of the optimal and original (old) values of the decision variables in relation to the developed model for Weija Water Headworks.

TABLE 3: COMPARISON OF THE OPTIMAL AND ORIGINAL (OLD) VALUES OF THE DECISION VARIABLES OF THE MODEL FOR WEIJA WATER HEADWORKS.

DECISION VARIABLE	OPTIMAL VALUE	ORIGINAL VALUE	DIFFERENCE
X_1	688490.201	691695.830	3205.629 Kg
X_2	2716245.543	2719245.250	2999.707 KWh
X_3	1680.053	2156.860	476.807 Litres
X_4	60.000	60.000	0.000
X_5	4.000	4.000	0.000
C_T	**2435620.258**	**2436736.040**	**GH₵ 1115.782**

It is very conspicuous from Table 3 that, Weija Water Headworks would have saved an amount of GH₵ 1115.782 monthly and a total amount of GH₵ 13389.384 in 2014 if this optimal water treatment cost model was available.

NEW RESULTS AND DISCUSSIONS

TABLE 4: SENSITIVITY ANALYSIS OF WATER TREATMENT COST MODEL FOR WEIJA WATER HEADWORKS IN 2014.

Original (old) water treatment cost in 2014 = GH₵ 2436736.040

Change in Average Cost	Average Cost in the Chemical House	Average Cost in the Pumping House	Average Transportation Cost	Average Personnel Cost	Average Maintenance Cost	Optimal Cost
0	729824.87	1522188.03	5040.16	162210.06	17472.92	2435620.485
0.01	729824.88	1522188.04	5040.17	162210.07	17472.93	2435620.532
0.50	729825.37	1522188.53	5040.66	162210.56	17473.42	2435622.874
1.00	729825.87	1522189.03	5041.16	162211.06	17473.92	2435625.263
2.00	729826.87	1522190.03	5042.16	162212.06	17474.92	2435630.042
10.00	729834.87	1522198.03	5050.16	162220.06	17482.92	2435668.270
20.00	729844.87	1522208.03	5060.16	162230.06	17492.92	2435716.058
40.00	729864.87	152228.03	5080.16	162250.06	17512.92	2435811.631
80.00	729904.87	1522268.03	5120.16	162290.06	17552.92	2436002.778
100.00	729924.87	1522288.03	5140.16	162310.06	17572.92	2436098.351
200.00	730024.87	1522388.03	5240.16	162410.06	17672.92	2436576.218
230.00	730054.87	1522418.03	5270.16	162440.06	17702.92	2436719.578
233.00	730057.87	1522421.03	5273.16	162443.06	17705.92	2436733.914
234.00*	730058.87	1522422.03	5274.16	162444.06	17706.92	2436738.693*
235.00*	730059.87	1522423.03	5275.16	162445.06	17707.92	2436743.471*
240.00*	730064.87	1522428.03	5280.16	1624450.06	17712.92	2436767.365*
250.00*	730074.87	1522438.03	5290.16	1624460.06	17722.92	2436815.151*
300.00*	730124.87	1522488.03	5340.16	162510.06	17772.92	2437054.085*
500.00*	730324.87	1522688.03	5540.16	162710.06	17972.92	2438009.818*

From Table 4, it can be observed that, the developed model will not minimize the original (old) treatment cost in 2014 if the Average Cost in the Chemical House, Average Cost in the Pumping House, Average Transportation Cost, Average

Personnel Cost and Average Maintenance Cost are increased by GH₵ 234.00 or more.

DUALITY ANALYSIS OF WATER PRODUCTION IN WEIJA WATER HEADWORKS IN 2014

The Dual of the Water Treatment Cost Model [2] for Weija Water Headworks in 2014 was formulated as shown below:

Maximize $= 729824.87\, Y_1 + 1522188.03\, Y_2 + 5040.16\, Y_3 + 162210.06\, Y_4 + 17472.92\, Y_5$

Subject to

$$\begin{aligned} &1.06\, Y_1 \leq 1.06 \\ &0.56\, Y_2 \leq 0.56 \\ &0.015\, Y_1 + 0.649\, Y_2 + 3\, Y_3 \leq 3.00 \qquad [3] \\ &2703.5\, Y_4 \leq 2703.5 \\ &4368.23\, Y_5 \leq 4368.23 \end{aligned}$$

$$Y_1, Y_2, Y_3, Y_4, Y_5, \geq 0$$

Table 5 gives the resulting solution of this dual problem or model [3].

TABLE 5: OPTIMAL SOLUTION FOR WEIJA WATER HEADWORKS (DUAL CASE)

PARAMETER	VALUE
Optimal Treatment Cost (C_T)	2435620.258
Y_1	1.000
Y_2	1.000
Y_3	0.779
Y_4	1.000
Y_5	1.000

From Table 5:

- The Optimal Treatment Cost (CT) for both the Primal Model [2] and the Dual Model [3] are the same. That is GH₵ 2435620.258.

- Shadow price or Marginal cost, Y_1 = 1.000 means that if the Average Cost in the Chemical House is increased or decreased by GH₵ 1.000, the optimal treatment cost will increase or decrease respectively by GH₵ 1.000. Similarly, Y_2 = 1.000, Y_4 = 1.000 and Y_5 = 1.000 mean that if the Average Cost in the Pumping House, Average Personnel Cost and Average Maintenance Cost respectively are each increased or decreased by GH₵ 1.000, the optimal treatment cost will increase or decrease respectively by 1.000 + 1.000 + 1.000 = GH₵ 3.00. Finally, Y_3 = 0.779 means that if the Average Transportation Cost is increased or decreased by GH₵ 1.00, the optimal treatment cost will increase or decrease respectively by GH₵ 0.779.

CONCLUSION

In this paper, sensitivity and duality analyses have been performed on our earlier developed optimal water treatment cost model for Ghana. Linear Programming was used to formulate the model and tested with real data collected from Weija Water Headworks in Accra using Interior-Point Method to obtain solutions. The effects of variations of selected key parameters on the developed model have now been investigated. The developed model will not minimize the original (old) treatment cost in 2014 if the Average Cost in the Chemical House, Average Cost in the Pumping House, Average Transportation Cost, Average Personnel Cost and Average

Maintenance Cost are increased by GH₵ 234.00 or more. Marginal costs of water production in the selected water headworks have also been found. It is strongly recommended that all Water Headworks under Ghana Water Company Limited (GWCL) should employ at least one Operations Researcher to assist them in some of these post-optimality analyses.

Acknowledgement

We are very grateful to Weija Water Headworks in Accra which is under Ghana Water Company Limited for their data on water treatment.

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