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POSITION VECTORS OF SPACE CURVES WITH CONSTANT CURVATURES IN EUCLIDEAN 4- SPACE E^4

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ABSTRACT: *Position Vectors of Space Curves has important applications in numerous mathematical field, thus we study it in an arbitrary space curves according to curvatures as constant with respect to Frenet frame in Euclidean 4-space* E^4

KEYWORDS: Euclidean 4-space; Frenet Equations; Position Vectors.

INTRODUCTION

In the practical world, curves arise in many different disciplines, for instance as the megalithic art , the path of particles in physics, Mandelbrot set, the profile of technical object in medicine, conic sections, engineering etc. Recall that, a curve is a topological space locally homeomorphic to a line; which means that a curve is a set of points. Intuitively, we are thinking of a curve as the path of a particle moving in E^3 . The problem of the determination of parametric representation of the position vector of an arbitrary space curve is still open in the E^4- space. This problem is hard to solve in general case. However, we solved it with two special cases only (with constant curvatures and exactly at $\tau = \sigma$). Recently, Ali [2] was accommodate position vector of an arbitrary curve in Galilean 3-space G3. Cetin [3] was study the geometry of position vector of a unit speed curve in a regular surface in E^3. However, this problem is solved in other cases of the space curve. So, the intention beyond investigating of position vectors of the curves in a classical aims to determine behavior of the moved particle as in [1], [4], [6]. In the light of our main problem, first we establish a system of differential equations whose solution gives the components of the position vector of a curve on the Frenet axis.

Preliminaries

In this study the generalization of the concept of space curves in Euclidean 4- space E^4 considered. For that recall that if $\alpha : I \subset R \to E^4$ be an arbitrary curve in E^4 then α it said to be unit speed (or parameterized by arc length function s), if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle .,. \rangle$ is the standard scalar product in the Euclidean 4- space E^4 given by:

$$\langle a,b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 \qquad (2.1)$$

For each

$$a=(a_1,a_2,a_3,a_4)\in E^4 \quad ,b=(b_1,b_2,b_3,b_4)\in E^4$$

In particular, the norm of a vector $a\in E^4$ is given by $\|a\|=\sqrt{\langle a,a\rangle}$.

Denote by $\{T,N,B,E\}$ the moving Frenet frame along the unit Speed curve α. Then the Frenet formulas are given by [5]. And satisfying $\langle T,T\rangle=\langle N,N\rangle=\langle B,B\rangle=\langle E,E\rangle=1$ are:

$$\left.\begin{aligned} T'(\mathrm{s})&=\kappa(\mathrm{s})\cdot \mathrm{N}(\mathrm{s})\\ N'(\mathrm{s})&=-\kappa(\mathrm{s})\cdot T(\mathrm{s})+\tau(\mathrm{s})\mathrm{B}(\mathrm{s})\\ \mathrm{B}'(\mathrm{s})&=-\tau(\mathrm{s})\,\mathrm{N}(\mathrm{s})+\sigma(\mathrm{s})\cdot E(\mathrm{s})\\ \mathrm{E}'(\mathrm{s})&=-\sigma(\mathrm{s})\cdot B(\mathrm{s})\end{aligned}\right\} \tag{2.2}$$

Here T,N,B and E are called, respectively, the tangent, the normal, the binormal and the trinormal vector fields of the curves. And the functions $\kappa(\mathrm{s})$, $\tau(\mathrm{s})$ and $\sigma(\mathrm{s})$ are called, respectively, the first, the second and the third curvature of the curve α.

Definition 2.1 Let $a=(a_1,a_2,a_3,a_4)$ $,b=(b_1,b_2,b_3,b_4)$ *and* $c=(c_1,c_2,c_3,c_4)$ *be vectors in* E^4. *The vector product of a, b* and *c is defined by the determinant*

$$a\wedge b\wedge c=\begin{vmatrix} e_1 & e_2 & e_3 & e_4\\ a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\\ c_1 & c_2 & c_3 & c_4\end{vmatrix}$$

Where

$$e_1\wedge e_2\wedge e_3=e_4 \;,\; e_2\wedge e_3\wedge e_4=e_1 \;,\; e_4\wedge e_3\wedge e_1=e_2 \;,\; e_3\wedge e_2\wedge e_1=e_4 \;,\; e_4\wedge e_1\wedge e_2=e_3$$

Theorem 2.1 Let $\alpha=\alpha(\mathrm{t})$ be an arbitrary curve in E^4. Frenet tackle of the α can be calculated by the following equations.

$$T=\frac{\alpha'}{\|\alpha'\|} \tag{2.3}$$

$$N=\frac{\|\alpha'\|\alpha''-\langle\alpha',\alpha''\rangle\alpha'}{\big\|\|\alpha'\|\alpha''-\langle\alpha',\alpha''\rangle\alpha'\big\|} \tag{2.4}$$

$$B=\eta E\wedge T\wedge N \tag{2.5}$$

$$E = \eta \frac{T \wedge N \wedge \alpha'''}{\left\| T \wedge N \wedge \alpha''' \right\|} \tag{2.6}$$

MAIN RESULTS

Theorem 3.1. The position vector $\alpha(s)$ of space curve with constant curvatures in Euclidean 4- space with respect to Frenet frame is given by:

$$\psi(s) = \left(c_1 e^{\int \kappa(s)ds} + c_2 e^{-\int \kappa(s)ds} + c_3 e^{\frac{-1}{m}\int \kappa(s)ds} \right) T(s) + \left(c_1 e^{\int \kappa(s)ds} - c_2 e^{-\int \kappa(s)ds} - \frac{c_3}{m} e^{\frac{-1}{m}\int \kappa(s)ds} \right) N(s)$$

$$+ c_3\, m \left(1 - \frac{1}{m} \right) e^{\frac{-1}{m}\int \kappa(s)ds} B(s) + \left[\left(c_1 e^{\int \kappa(s)ds} - c_2 e^{-\int \kappa(s)ds} - \frac{c_3}{m} e^{\frac{-1}{m}\int \kappa(s)ds} \right) \left(\frac{\tau}{\sigma} \right) + c_3 \left(\frac{1}{m} - 1 \right) e^{\frac{-1}{m}\int \kappa(s)ds} \right] E(s) \tag{3.1}$$

Proof. Let $\psi(s)$ be an arbitrary curve in Euclidean space E^4, then its position vector can by expressed as follows:

$$\psi(s) = \lambda(s)T(s) + \mu(s)N(s) + \nu(s)B(s) + \delta(s)E(s) \tag{3.2}$$

Where $\lambda(s), \mu(s), \nu(s)$ and $\sigma(s)$ are differentiable functions of $s \in I \subset R$. Differentiating the above equation with respect to s and using the Frenet equations, we get the following:

$$\left. \begin{array}{l} \lambda' - \mu\kappa = 0 \\ \lambda + \mu' - \nu\tau = 0 \\ \mu\tau + \nu' + \delta\sigma = 0 \\ \nu\sigma + \delta' = 0 \end{array} \right\} \tag{3.3}$$

By means of the case $\tau(s) = \sigma(s)$, and the change of variables as in [7] $\theta = \int \kappa(s)ds$ the first equation of (3.3) leads to:

$$\dot{\lambda} = \mu \tag{3.4}$$

Where $f(\theta) = \dfrac{\kappa}{\tau}$ and dot denote the derivative with respect to θ the second equation of (3.3) becomes

$$\nu = \left(\lambda - \ddot{\lambda} \right) \left(\frac{\kappa}{\tau} \right) \tag{3.5}$$

The equation (3.3) can be written as:

$$\delta = \dot{\lambda}\left(\frac{\tau}{\sigma}\right)+\left(\frac{\kappa}{\sigma}\right)\left[\left(\lambda-\ddot{\lambda}\right)\left(\frac{\kappa}{\tau}\right)\right]^{\prime} \tag{3.6}$$

If we substitute the equations (3.4), (3.5) and (3.6) into the last equation of (3.3) we will get the next equation:

$$\left(\lambda-\ddot{\lambda}\right)\left(\frac{\kappa}{\tau}\right)+\left(\frac{\kappa}{\sigma}\right)\left[\left(\lambda-\ddot{\lambda}\right)\left(\frac{\kappa}{\tau}\right)\right]^{\prime} \tag{3.7}$$

By solving the equation (3.7), we obtain the position vector of an arbitrary curve in the Frenet frame. Here, we take a special case when $f\ (\theta)=\mathrm{m}$. The equation above becomes:

$$m\dddot{\lambda}+\ddot{\lambda}-m\dot{\lambda}-\lambda=0 \tag{3.8}$$

The general solution of equation (3.8) is:

$$\lambda(\theta)=\mathrm{c}_1 e^{\theta}+c_2 e^{-\theta}+c_3 e^{\frac{-1}{m}\theta} \tag{3.9}$$

Where c_1, c_2, c_3 are arbitrary constants. From (3.4), the function $\mu(\theta)$ is given by:

$$\mu(\theta)=\mathrm{c}_1 e^{\theta}-c_2 e^{-\theta}-\frac{c_3}{m}e^{\frac{-1}{m}\theta} \tag{3.10}$$

From (3.5), the function $\nu(\theta)$ is given by:

$$\nu(\theta)=\mathrm{c}_3\, m\left(1-\frac{1}{m}\right)e^{\frac{-1}{m}\theta} \tag{3.11}$$

From (3.6), the function $\delta(\theta)$ is given by:

$$\delta(\theta)=\left(\mathrm{c}_1 e^{\theta}-c_2 e^{-\theta}-\frac{c_3}{m}e^{\frac{-1}{m}\theta}\right)\left(\frac{\tau}{\sigma}\right)+c_3\left(\frac{1}{m}-1\right)e^{\frac{-1}{m}\theta} \tag{3.12}$$

By substituting equations (3.9), (3.10), (3.11) and (3.12) in to equation (3.2) then equation (3.1) hold, thus theorem proved.

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