International Journal of Mathematics and Statistical Studies

Vol.4, No.4, pp.32-35, August 2016

Published by European Centre for Research Training and Development UK (www.eajournals.org)

ON THE STABILITY OF SOLUTIONS OF GRAND GENERAL THIRD ORDER NON LINEAR ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT: *This work deals with the stability of solutions of the nonlinear third order autonomous ordinary differential equation* $\dddot{x} + h(x,\dot{x},\ddot{x}) = 0$. *By constructing suitable Liapounov functional, the sufficient conditions for the ordinary stability and asymptotic stability of the trivial solution* $\bar{x} = \bar{0}$ *to the differential equation are established.*
KEYWORDS: Stability, trivial solution, Liapounov functional; Positive definite; negative definite

INTRODUCTION

The differential equation under consideration is $\dddot{x} + h(x,\dot{x},\ddot{x}) = 0$…(1.1)
A simple case of (1.1) is when $h(x,\dot{x},\ddot{x})$ is linear of the form
$h(x,\dot{x},\ddot{x}) = a\ddot{x} + b\dot{x} + cx$ so that (1.1) becomes
$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$ ……………………………………..………(1.2)
where a, b, c are constants. (1.2) is a well known case by Routh [4] in which the trivial solution $\bar{x} = \bar{0}$ is asymptotically stable provided the Routh – Hurwitz criteria a > 0, c > 0, ab – c > 0 are satisfied.
Now, Barbasin [1] replaced $b\dot{x}$ by a nonlinear function $g(\dot{x})$ and cx by a nonlinear function h(x) in (1.2) to obtain
$\dddot{x} + a\ddot{x} + g(\dot{x}) + h(x) = 0$ …………………………………………(1.3)
and established the asymptotic stability of the trivial solution to (1.3) by a suitability Liaponouv functional. Simanov [5] treated the case of replacing a in (1.2) by a variable function $f(x,\dot{x})$, but leaving b, c to remain constant in (1.2) to obtain $\dddot{x} + f(x,\dot{x})\ddot{x} + b\dot{x} + cx = 0$ ….(1.4)
Again by using a suitable Liapounov functional he established the asymptotic stability of the trivial solution to (1.4). Then Ezeilo [2] combined (1.3) and (1.4) to obtain
$\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = 0$ …………………………………… (1.5)
with the aim of investigating the asymptotic stability of the trivial solution
$\bar{x} = \bar{0}$ to (1.5) and to establish further generalizations of the results of (1.2), (1.3) and (1.4), and which he did by using a suitable Liapounov functional. In this paper I decided to work on equation (1.1), that is,
$\dddot{x} + h(x,\dot{x},\ddot{x}) = 0$, which is equivalent to the replacement of $a\ddot{x} + b\dot{x} + cx$ in (1.2) by a non-linear (variable) function $h(x,\dot{x},\ddot{x})$, and which is a more general form of all of (1.2), (1.3), (1.4) and (1.5) and with the intention to investigate the sufficient conditions for the ordinary stability and asymptotic stability of the trivial solution $\bar{x} = \bar{0}$ to (1.1).
To achieve our objective, we require the following definitions and theorems.

Definitions
Let $V: \mathfrak{R}^n \longrightarrow \mathfrak{R}$ be a scalar function.

ISSN 2053-2229 (Print), ISSN 2053-2210 (Online)

Definition 1: V is positive definite if $V(\bar{0}) = 0$ and $V(\bar{x}) > 0$ in some deleted neighbourhood, D, of the origin $\bar{0}$. $(ie.\ V(\bar{x}) > 0\ \forall x \in D\backslash\{\bar{0}\}$.
Definition 2: V is positive semi-definite if $V(\bar{0}) = 0$ and $V(\bar{x}) \geq 0$ in some deleted neighbourhood, D, of the origin $\bar{0}$.
Definition 3: V is negative definite if $V(\bar{0}) = 0$ and $V(\bar{x}) < 0$ in some deleted neighbourhood, D, of the origin $\bar{0}$.
Definition 4: V is negative semi-definite if $V(\bar{0}) = 0$ and $V(\bar{x}) \leq 0$ in some deleted neighbourhood, D, of the origin $\bar{0}$.
Definition 5: V is indefinite if it is neither definite nor semi-definite.

3. Liapounov's Stability Theorems for the Direct Method.

In the sequel, we shall need the following stability theorems due to A.M. Liapounov [3].

Theorem 3.1: Liapounov's stability Theorem [3]
Consider the autonomous differential system $\dot{\boldsymbol{x}} = \bar{f}(\bar{x}),\ \bar{f}(\bar{0}) = \bar{0}$.
Suppose there exists a positive definite functional
$V : \Re^n \longrightarrow \Re$ such that its time derivative $\dot{V}(\bar{x}) =$ grad $V . \bar{f}(\bar{x})$ along solution paths is negative semi-definite, then the trivial solution $\bar{x} = \bar{0}$ is stable.

Theorem 3.2: Lyapounov's Asymptotic stability Theorem [3]
Consider the autonomous differential system $\dot{\boldsymbol{x}} = \bar{f}(\bar{x}),\ \bar{f}(\bar{0}) = \bar{0}$.
Suppose there exists a positive definite functional $V : \Re^n \longrightarrow \Re$ such that its time derivative $\dot{V}(\bar{x}) =$ grad $V . \bar{f}(\bar{x})$ along solution paths is negative definite, then the trivial solution $\bar{x} = \bar{0}$ is asymptotically stable.

Theorem 3.3: Lapounov's Instability Theorem [3]
consider the autonomous differential system $\bar{x} = \bar{f}(\bar{x}),\ \bar{f}(\bar{0}) = \bar{0}$
Suppose there exists a positive definite functional $V : \Re^n \longrightarrow \Re$ such that its derivative, $\dot{V}(\bar{x}) =$ grad $V . \bar{f}(\bar{x})$ along solution paths is positive definite, then the trivial solution $\bar{x} = \bar{0}$ is unstable

4. The Main Result

Theorem 4.1: The trivial solution $\bar{x} = \bar{0}$ to the third order nonlinear ordinary differential equation given by $\dddot{x} + h(x, \dot{x}, \ddot{x}) = 0$ …………(4.1)

(1) is ordinarily stable in the sense of Liapounov if

$$\left.\begin{array}{ll}\text{(i)} & \frac{h(x,y,z)}{y} > \frac{x}{y},\ y \neq 0 \\ \text{(ii)} & \frac{h(x,y,z)}{z} > \frac{2y+z}{z},\ z \neq 0\end{array}\right\} \quad \ldots\ldots(4.2)$$

2. is asymptotically stable in the sense of Liapounov if

$$\left.\begin{array}{ll}\text{(i)} & \frac{h(x,y,z)}{x} > 0,\ x \neq 0 \\ \text{(ii)} & \frac{h(x,y,z)}{y} > \frac{2x}{y},\ y \neq 0 \\ \text{(iii)} & \frac{h(x,y,z)}{z} > \frac{2y+z}{2z},\ z \neq 0\end{array}\right\} \quad \ldots\ldots(4.3)$$

Proof:

An equivalent 3 – system to (4.1) is

$$\left.\begin{array}{rl}\dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -h(x,\dot{x},\ddot{x}) \\ &= -h(x,y,z)\end{array}\right\} \quad \ldots\ldots(4.4)$$

Take the quadratic form $V: \Re^3 \longrightarrow \Re$ as
$V(x,y,z) = \frac{1}{2}Ax^2 + \frac{1}{2}By^2 + \frac{1}{2}Cz^2 + Dxy + Exz + Fyz$

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$$+G\int_0^x\int_0^y\int_0^z h(r,s,\lambda)drdsd\lambda$$

Where, without loss of generality ……………(4.5)

$$\int_0^x\int_0^y\int_0^z h(r,s,\lambda)drdsd\lambda > 0$$

Take the time derivative of V along solution paths to obtain

$$\dot V(x,y,z) = Ax\dot x + By\dot y + Cz\dot z + D\dot x y + Dx\dot y$$
$$+E\dot x z + Ex\dot z + F\dot y z + Fy\dot z + G\dot x\int_0^y\int_0^z h(x,s,z)dsd\lambda$$
$$+G\dot y\int_0^x\int_0^z h(r,y,\lambda)drd\lambda + G\dot z\int_0^x\int_0^y h(r,s,z)drds$$
$$= Axy + Byz + Cz[-h(x),y,z] + Dy^2 + Dxz + Eyz$$
$$+Ex[-h(x,y,z)] + Fz^2 + Fy[-h(x,y,z)] + Gy\int_0^y\int_0^z h(x,s,\lambda)dsd\lambda$$
$$+Gz\int_0^x\int_0^z h(r,y,\lambda)drd\lambda + G[-h(x,y,z)]\int_0^x\int_0^y h(r,s,z)drds.$$
$$= Axy + Byz - Czh(x,yz) + Dy^2 + Dxz + Eyz - Exh(x,y,z)$$
$$+Fz^2 - Fyh(x,y,z) + Gy\int_0^y\int_0^z h(x,s,\lambda)dsd\lambda$$
$$+Gz\int_0^x\int_0^z h(r,y,\lambda)drd\lambda - Gh(x,y,z)\int_0^x\int_0^y h(r,s,z)drds \quad …….(4.6)$$

Case 1: For ordinary Stability.

Set A = C = F = 1; B = 2 and D = E = G = 0 in (4.5) and (4.6) to obtain .

$$\dot V(x,y,z) = xy + 2yz - zh(x,y,z) + z^2 - yh(x,y,z)$$
$$= \frac{x}{y}y^2 + 2\frac{y}{z}.z^2 - z^2\frac{h(x,y,z)}{z} + z^2 - y^2\frac{h(x,y,z)}{y}$$
$$= -y^2\left[\frac{h(x,y,z)}{y} - \frac{x}{y}\right] - z^2\left[\frac{h(x,y,z)}{z} - \frac{2y}{z} - 1\right]$$
$$= -y^2\left[\frac{h(x,y,z)}{y} - \frac{x}{y}\right] - z^2\left[\frac{h(x,y,z)}{z} - \left(\frac{2y+z}{z}\right)\right]$$

Which is negative semi-definite, provided …(4.7)

$$\frac{h(x,y,z)}{y} > \frac{x}{y},\ y \neq 0;\ \text{and}\ \frac{h(x,y,z)}{z} > \frac{2y+z}{z},\ z \neq 0.$$

and $V(x,y,z) = \frac{1}{2}x^2 + y^2 + \frac{1}{2}z^2 + yz$

$$= \frac{1}{2}x^2 + \frac{1}{2}y^2 + \left(\frac{1}{2}y^2 + yz + \frac{1}{2}z^2\right)$$
$$= \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}(y^2 + yz + z^2)$$
$$= \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}(y+z)^2$$

Which is positive definite. ……………………(4.8)

Now, by (4.8) V is positive definite and by (4.7) $\dot V$ is negative semi-definite; hence by Theorem 3.1 the trivial solution $\bar x = \bar 0$ to (4.1) is ordinarily stable in the sense of Liapounov, provided

(i) $\frac{h(x,y,z)}{y} > \frac{x}{y},\ y \neq 0$; and

(ii) $\frac{h(x,y,z)}{z} > \frac{2y+z}{z},\ z \neq 0$

and with $V(x,y,z) = \frac{1}{2}x^2 + y^2 + \frac{1}{2}(y+z)^2$ as a valid Liapounov functional.

Case 2: For asymptotic stability.

Set A = C = 2; B = E = F = 1; D = G = 0 in (4.5) and (4.6) to obtain

$$\dot V(x,y,z) = 2xy + yz - 2zh(x,y,z) + yz - xh(x,y,z) + z^2 - yh(x,y,z).$$
$$= 2\frac{x}{y}.y^2 + \frac{y}{z}z^2 - 2z^2\frac{h(x,y,z)}{z} + \frac{y}{z}z^2$$

ISSN 2053-2229 (Print), ISSN 2053-2210 (Online)

$$-x^2\frac{h(x,y,z)}{x} + z^2 - y^2\frac{h(x,y,z)}{y}$$
$$= -x^2\frac{h(x,y,z)}{x} - y^2\left[\frac{h(x,y,z)}{y} - \frac{2x}{y}\right] - z^2\left[\frac{2h(x,y,z)}{z} - \frac{y}{z} - \frac{y}{z} - 1\right]$$
$$= -x^2\frac{h(x,y,z)}{x} - y^2\left[\frac{h(x,yz)}{y} - \frac{2x}{y}\right] - z^2\left[\frac{2h(x,y,z)}{z} - \frac{(2y+z)}{z}\right]$$

which is negative definite, provided .(4.9)

$\frac{h(x,y,z)}{x} > 0, x \neq 0\,; \frac{h(x,y,z)}{y} > \frac{2x}{y}, y \neq 0; \frac{h(x,y,z)}{z} > \frac{2y+z}{2z}, z \neq 0$

and $V(x,yz) = x^2 + \frac{1}{2}y^2 + z^2 + xz + yz$

$$= \frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + \frac{1}{2}z^2 + xz + yz$$
$$= \frac{1}{2}x^2 + \left(\frac{1}{2}x^2 + xz + \frac{1}{2}z^2\right) + \left(\frac{1}{2}y^2 + yz + \frac{1}{2}z^2\right) 4$$
$$= \frac{1}{2}x^2 + \frac{1}{2}(x^2 + 2xz + z^2) + \frac{1}{2}(y^2 + 2yz + z^2)$$
$$= \frac{1}{2}x^2 + \frac{1}{2}(x+z)^2 + \frac{1}{2}(y+z)^2$$

which is positive definite. ……………(4.10)

Now, by (4.10) V is positive definite and by (4.9) $\dot{V}$ is negative definite; hence by theorem 3.2 the trivial solution $\bar{x} = \bar{0}$ to (4.1) is asymptotically stable in the sense of Liaponov, with

$V(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}(x+z)^2 + \frac{1}{2}(y+z)^2$ as a a valid Liapounov functional provided that:

(i) $\frac{h(x,y,z)}{x} > 0\,,\ x \neq 0$

(ii) $\frac{h(x,y,z)}{y} > \frac{2x}{y},\ y \neq 0$

(iii) $\frac{h(x,y,z)}{z} > \frac{2y+z}{2z},\ z \neq 0$

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ISSN 2053-2229 (Print), ISSN 2053-2210 (Online)