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ON THE STABILITY OF SOLUTIONS OF A CERTAIN GENERAL THIRD ORDER NON LINEAR ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT: *This Work Deals With The Stability Of Solutions Of The Nonlinear Third Order Autonomous Ordinary Differential Equation* $\dddot{x} + f(x,\dot{x})\ddot{x} + g(x)\dot{x} + h(x) = 0$. *The Main Object Of This Paper Is To Establish Sufficient Conditions For The Asymptotic Stability Of The Trivial Solution* $\bar{x} = \bar{0}$ *By Constructing A Suitable Liapounov Functional.*

KEYWORDS: Stability; Trivial Solution; Liapounov Functional; Positive Definite; Negative Definite.

INTRODUCTION

Most Of The Interesting Differential Equations Are Nonlinear And, With A Few Exceptions, Are Not Solvable Analytically; But The Knowledge Of The Stability Behavior Of Their Solutions Is Very Useful. Fortunately The Stability Status Of These Solutions Can Be Determined By Constructing Suitable Liagpounov Functionals. This Paper Deals With The System

$\dddot{x} + f(x,\dot{x})\ddot{x} + g(x)\dot{x} + h(x) = 0$ ……………………………………………(1.1) Which Is An Extention Of The Work Of Ezeilo [2] On The System

$\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = 0$ …………………………………………….(1.2)

By A Suitable Liapounov Functional He Established The Asymptotic Stability Of The Trivial Solution $\bar{x} = \bar{0}$ To (1.2). Ezeilo's Work Is A Combination Of The Works Of BarbaŝIn [1] And Simanov [5]. BarbaŝIn's Paper Deals With The Case When $b\dot{x}$ And Cx Are Replaced By Nonlinear Functions $g(\dot{x})$ And $h(x)$ Respectively In The Third Order Linear System.

$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$ ……………………………………………………(1.3)

Where A, B, C Are Constants, To Obtain The System

$\dddot{x} + a\ddot{x} + g(\dot{x}) + h(x) = 0$…………………………………………….(1.4)

Simanov Considered The Case When B And C Remain Constant In (1.3), But A Is Replaced By A Variable Function $f(x,\dot{x})$ To Obtain The System

$\dddot{x} + f(x,\dot{x})\ddot{x} + b\dot{x} + cx = 0$ …………………………………………………(1.5)

In Both (1.4) And (1.5), The Asymptotic Stability Of The Trivial Solution $\bar{x} = \bar{0}$ Was Established By Suitable Liapounov Fucntionals. Observe That (1.3) Is The Well Known Case Of Routh [4] Where The Trivial Solution $\bar{x} = \bar{0}$ Is Asymptotically Stable Provided That The Routh – Hurwitz Criteria

A > 0, C > 0, Ab – C > 0 Are Satisfied.

To Achieve Our Aim The Following Definitions And Theorems Are Required.

1. Definitions

Let $V : \Re^n \longrightarrow \Re$ Be A Scalar Function.

Let $D \setminus \{\bar{0}\}$ Be A Deleted Neighbourhood, D , Of The Origin $\bar{0}$

Definition 2.1: V Is Positive Definite If $V(\bar{0}) = 0$ And

$V(\bar{x}) > 0 \ \forall x \in D\setminus\{0\}$.

Definition 2.2: V Is Positive Semi-Definite If $V(\bar{0}) = 0$ And

$V(\bar{x}) \geq 0 \ \forall x \in D\setminus\{\bar{0}\}$.

Definition 2.3: V Is Negative Definite If $V(\bar{0}) = 0$ And

$V(\bar{x}) < 0 \ \forall x \in D\setminus\{\bar{0}\}$.

Definition 2.4: V Is Negative Semi-Definite If $V(\bar{0}) = 0$ And

$V(\bar{x}) \leq 0 \ \forall x \in D\setminus\{\bar{0}\}$.

Definition 2.5: V Is Indefinite If It Is Neither Definite Nor Semi-Definite.

2. Liapounov's Stability Theorems For The Direct Method.

In The Sequel, We Shall Require The Following Theorems Due To A.M. Liapounov [3].

Theorem 3.1: Liapounov's Stability Theorem [3]

Consider The Autonomous Differential System.

$$\dot{\bar{x}} = \bar{f}(\bar{x}); \ \bar{f}(\bar{0}) = \bar{0}$$

Suppose There Exists A Positive Definite Functional

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$V : \mathfrak{R}^n \longrightarrow \mathfrak{R}$ Such That Its Time Derivative $\dot{V}(\bar{x}) = grad\, V.\bar{f}(\bar{x})$ Along Solution Paths Is Negative Semi – Definite, Then The Trivial Solution $\bar{x} = \bar{0}$ Is Stable.

Theorem 3.2: Liapounov's Asymptotic Stability Theorem [3]

Consider The Autonomous Differential System.

$$\dot{\bar{x}} = \bar{f}\,(\bar{x});\ \bar{f}(\bar{0}) = \bar{0}$$

Suppose There Exists A Positive Definite Functional $V : \mathfrak{R}^n \longrightarrow \mathfrak{R}$ Such That Its Time Derivative $\dot{V}(\bar{x}) = grad\, V.\bar{f}(\bar{x})$ Along Solution Paths Is Negative Definite, Then The Trivial Solution $\bar{x} = \bar{0}$ Is Asymptotically Stable.

Theorem 3.3: Liapounov's Instability Theorem [3]

Consider The Autonomous Differential System

$$\dot{\bar{x}} = \bar{f}\,(\bar{x});\ \bar{f}(\bar{0}) = \bar{0}$$

Suppose There Exists A Positive Definite Functional $V : \mathfrak{R}^n \longrightarrow \mathfrak{R}$ Such That Its Time Derivative $\dot{V}(\bar{x}) = grad\, V.\bar{f}(\bar{x})$ Along Solution Paths Is Positive Definite, Then The Trivial Solution $\bar{x} = \bar{0}$ Is Unstable.

3. The Main Result.

Let F, G, H Be Real Valued Functions.

Then The Trivial Solution $\bar{x} = \bar{0}$ To The Third Order Nonlinear Ordinary Differential Equation Given By.

$$\dddot{x} + f(x,\dot{x})\ddot{x} + g(x)\dot{x} + h(x) = 0 \quad \text{……………………………………….(4.1)}$$

Is Asymptotically Stable If

$$\left.\begin{array}{ll} \text{(i)} & f(x,y) > 0 \\ \text{(ii)} & g(x) > 1 \\ \text{(iii)} & zh(x) > xy,\ \ x \neq 0 \\ \text{(iv)} & \frac{z}{y} >,\ 0,\ y \neq 0 \end{array}\right\} \quad \text{……………………………………..(4.2)}$$

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Proof:

An Equivalent 3 – System To (4.1) Is

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= Z\\ \dot{z} &= -f(x,\dot{x})\ddot{x} - g(x)\dot{x} - h(x)\\ &= -f(x,y)z - g(x)y - h(x)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

Let The Quadratic Form $V:\mathfrak{R}^3 \longrightarrow \mathfrak{R}$ Be

$$\left.\begin{aligned}V(X,Y,Z) &= \tfrac{1}{2}Ax^2 + \tfrac{1}{2}By^2 + \tfrac{1}{2}Cz^2 + Dxy + Exz + Fyz\\ &\quad + Gz^2\int_0^x\int_0^y f(r,s,)drds + Hy^2\int_0^x g(r)dr + J\int_0^x h(r)dr\\ \text{Where, } &\int_0^x\int_0^y f(r,s,)drds > 0;\ \int_0^x g(r)dr > 0;\ \int_0^x h(r)dr > 0.\end{aligned}\right\} .(4.4)$$

$$\dot{V}(x,y,z) = Ax\dot{x} + By\dot{y} + Cz\dot{z} + D\dot{x}y + Dx\dot{y} + E\dot{x}z + Ex\dot{z}$$

$$+F\dot{y}z + Fy\dot{z} + 2Gz\dot{z}\int_0^x\int_0^y f(r,s)drds$$

$$+Gz^2\left[\dot{x}\int_0^y f(x,s)ds + \dot{y}\int_0^x f(r,y)dr\right]$$

$$+2Hy\dot{y}\int_0^x g(r)dr + Hy^2\dot{x}g(x) + J\dot{x}h(x).$$

$$= Axy + Byz + Cz[-f(x,y)z - g(x)y - h(x)]$$

$$+Dy^2 + Dxz + Eyz + Ex[-f(x,y)z - g(x)y - h(x)]$$

$$+Fz^2 + Fy[-f(x,y)z - g(x)y - h(x)]$$

$$+ 2Gz[-f(x,y)z - g(x)y - h(x)]\int_0^x\int_0^y f(r,s)drds + Gz^2y\int_0^y f(x,s)ds$$

$$+ Gz^3\int_0^x f(r,y)dr + 2Hyz\int_0^x g(r)dr + Hy^3g(x) + Jyh(x).$$

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$$\left.\begin{aligned}
&= \; Axy + Byz - Cz^2 f(x,y) - Cyzg(x) - Czh(x) \\
&+ \; Dy^2 + Dxz + Eyz - Exzf(x,y) - Exyg(x) - Exh(x) \\
&+ Fz^2 - Fyzf(x,y) - Fy^2 g(x) - Fyh(x) \\
&+ \; [-2Gz^2 f(x,y) - 2Gzyg(x) - 2Gzh(x)] \int_0^x \int_0^y f(r,s)drds \\
&+ Gz^2 y \int_0^y f(x,s)ds + Gz^3 \int_0^x f(r,y)dr + 2Hyz \int_0^x g(r)dr \\
&+ Hy^3 g(x) + Jyh(x).
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4.5)$$

In (4.5) And (4.4) Set A = B = C = I And D = E = F = G = H = J = 0 To Obtain

$$\dot{V}(x,y,z) \; = \; xy + yz - z^2 f(x,y) - yzg(x) - zh(x)$$

$$= \quad xy - z\,h(x) + yz - yz\,g(x) - z^2 f(x,y)$$

$$= \quad x^2 \frac{y}{x} - zx^2 \frac{h(x)}{x^2} + y^2 \frac{z}{y} - y^2 \frac{z}{y} g(x) - z^2 f(x,y)$$

$$= \quad -x^2 \left[\frac{zh(x)}{x^2} - \frac{y}{x}\right] - \; y^2 \left[\frac{z}{y}(g(x) - 1)\right] - z^2 f(x,y) \;\ldots\ldots.(4.6)$$

Which Is Negative Definite, Provided

$$\frac{zh(x)}{x^2} > \frac{y}{x}\,, x \neq 0; \; \frac{z}{y} > 0, y \neq 0; g(x) > \; 1; f(x,y) > \; 0$$

The Condition $\frac{zh(x)}{x^2} > \frac{y}{x} \implies zh(x) \; > xy$, Since $x^2 > 0$ As $x \neq 0$

And $V(x,y,z) \quad = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4.7)$

Which Is Positive Definite.

By (4.7) V Is Positive Definite And By (4.6) $\dot{V}$ Is Negative Definite, Hence The Trivial Solution $\bar{x} \; = \; \bar{0}$ Is Asymptotically Stable In The Sense Of Liapounov And With $V(x,y,z) \quad = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$ As A Good Liapounov Functional, Provided The Following Conditions Are Satisfied:

i. $f(x,y) \; > \; 0$
ii. $g(x) > \; 1$
iii. $zh(x) \; > \; xy, \;\; x \; \neq \; 0$

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iv. $\frac{z}{y} >, 0, y \neq 0$

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