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ON THE RECORD VALUES PROPERTY OF MUTH-PARETO DISTRIBUTION AND OTHER RELATED INFERENCE

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ABSTRACT: *We propose in this paper, a brief review of Muth-Pareto distribution. More precisely, we discuss the concept of record values and give a treatment of some selected properties of the distribution. The properties considered include the quartiles from quantile function, entropy, and limiting distribution of minimum order statistic. We provide some plots for the distribution of lower record values and studied the behavior by varying number of record values in the sample. It was observed that variability decreases by way of increasing number of record values in the sample. The quartiles related to the Muth-Pareto distribution were tabularized for some selected parameter values. It is our hope that the discoveries of this paper will be beneficial for practitioners and also a source of reference for users so as to enhance research interests related to Muth-Pareto distribution and its applications.*
KEYWORDS: Record Values, Order statistics, Quartiles, Lambert-*W* function, Entropy.

INTRODUCTION

Several distributions exist for modeling reliability and lifetime data. Among the prevailing parametric models are the Exponential, Lognormal and Pareto distributions. Pareto distribution, however, appears to be more popular than the exponential and lognormal in terms of modeling data that have heavy-tails, most commonly found in studies on finance, population size, as well as in extreme value theory.

So many researchers have attempted to add more flexibility to Pareto distribution through the use of generalization techniques. Prominent among these generalizations is the Generalized Pareto Distribution (GPD) by Pickands (1975). Some other important generalizations of the Pareto distribution include Beta-Pareto by Akinsete *et al*. (2008), Kumaraswamy Pareto by Bourguignon *et al*. (2012), Gamma-Pareto by Alzaatreh *et al*. (2012), Exponential-Pareto by Kareema and Boshi (2013), Exponentiated Weibull-Pareto by Afify *et al*. (2016) and Muth-Pareto by Sirajo (2020).

Although the Muth-Pareto distribution (MPD) received little attention in the literature if to compare with the great popularity of the above mentioned generalizations of Pareto distribution, the distribution has a closed form quantile function that can be used in generating random samples. According to Sirajo (2020), the MPD has proved to be significant in the modeling of failure times in reliability studies.

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RECORD VALUES OF THE MUTH-PARETO RANDOM VARIABLE

Theory on distribution of record values emanates from the work of Chandler (1952) where the frequency with which record weather conditions are being reported in newspapers were investigated. Considering independent simple random sampled time series values from a fixed continuous universe, we may define record values to be those elements or members of the series which are either less or greater than all preceding members. Therefore, an observation that is smaller in value than all the preceding observations in a series is called the lower record value, or analogously, an upper record value if it's value is larger than all the preceding observations. Many authors contributed to the development of this theory, recent contributions are due to Ahmadi *et al*. (2005), Balakrishnan *et al*. (2009), Ahsanullah *et al*. (2010), Shakil and Ahsanullah (2011), etc. Record values are found to arise naturally in various area of human endeavor including economics, traffic, sports, medicine as well as in life testing (also called reliability) studies where Muth-Pareto distribution (MPD) is found very useful.

Sirajo (2020) proposed to model the failure times of a system of Boeing 720 jet airplanes by using MPD. They found the MPD to be quite flexible and well-fitted for modeling extreme values, as well as heavy tailed distributed random variables. Chandler (1952) reported that record values are liable to be highly infrequent (which characterizes them as extreme values), thereby making MPD a good choice. There is no existing research on the analysis of record values from the MPD, and therefore the need for a special investigation. The cdf of MPD is defined by Sirajo (2020) as:

$$F(x)=\left[1-\left(\frac{\lambda}{x}\right)^{\theta}\right]^{-\alpha}\exp\left\{\frac{1}{\alpha}\left(1-\left[1-\left(\frac{\lambda}{x}\right)^{\theta}\right]^{-\alpha}\right)\right\},\qquad \lambda\le x<\infty \tag{1}$$

The pdf is:

$$f(x)=\frac{\theta\lambda^{\theta}}{x^{\theta+1}}\left\{1-\alpha\left[1-\left(\frac{\lambda}{x}\right)^{\theta}\right]^{\alpha}\right\}\left[1-\left(\frac{\lambda}{x}\right)^{\theta}\right]^{-2\alpha-1}\exp\left\{\frac{1}{\alpha}\left(1-\left[1-\left(\frac{\lambda}{x}\right)^{\theta}\right]^{-\alpha}\right)\right\} \tag{2}$$

Where $\lambda>0$ is a threshold parameter determining the location of the MPD random variable, and $\alpha,\theta>0$ are shape parameters demonstrating the different shapes of the MPD.

Suppose that $(X_m)_{m\ge1}$ represents a sequence of iid random variables with the cdf defined in Eq. (1). Suppose also that $Y_m=\max(\min)\{X_i\,|1\le i\le m\}$, $m\ge1$. We call X_i a lower (upper) record value of $\{X_m\,|m\ge1\}$ if $Y_i>(<)Y_{i-1},\ i>1$. Using this definition, X_1 is always a lower as

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well as an upper record value. It is quite easy to transform upper records into lower records by simply replacing $\{X_i\}$ by $\{-X_i, i \geq 1\}$. In this regard, the lower records of the new sequence automatically correspond to the upper records of the untransformed (i.e. the original) sequence. In this paper, we will restrict ourselves to simply the lower records.

Lower Record Values

We shall define the record time $L(m)$ to be the indices where the lower records arise, which are given by the sequence $\{L(m), m \geq 1\}$, where $L(m) = \min\{i \mid i > L(m-1), X_i < X_{L(m-1)}, m \geq 1\}$ and $L(1) = 1$. Consequently, the mth lower record value will be denoted by $X_{L(m)}$ or $X(m)$ for the sake of brevity. We shall define $f_m(x)$ to be the pdf of $X(m)$, $m \geq 1$. Then following Arnold *et al*. (1998) we have

$$f_m(x) = \frac{1}{\Gamma(m)}\left[-\ln F(x)\right]^{m-1} f(x), \quad x > 0 \tag{3}$$

With a corresponding cdf

$$F_m(x) = \frac{1}{\Gamma(m)}\Gamma\left(m, -\ln(F(x))\right) \tag{4}$$

where $\Gamma(a, y) = \int_y^\infty u^{a-1} e^{-u} du$ is the incomplete gamma function.

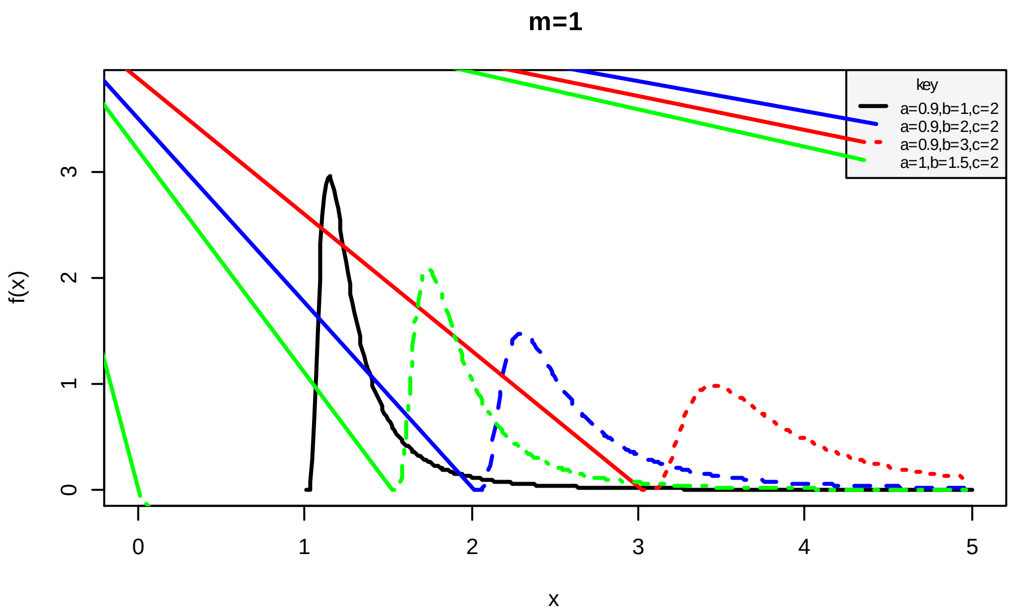

Figure 1: Density plot of MPD assuming one record value.

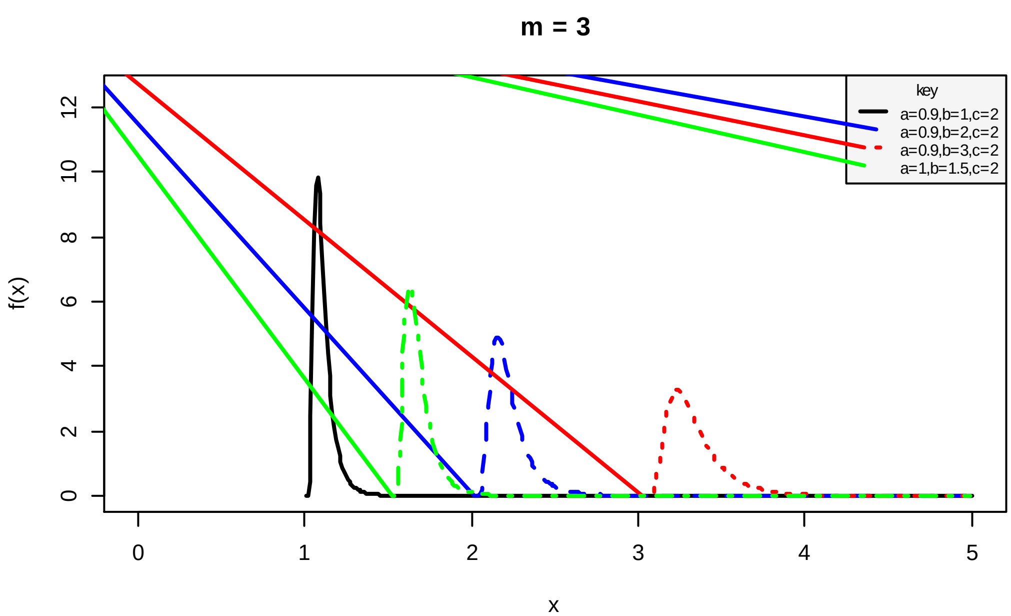

Figure 2: Density plot of MPD assuming three record values.

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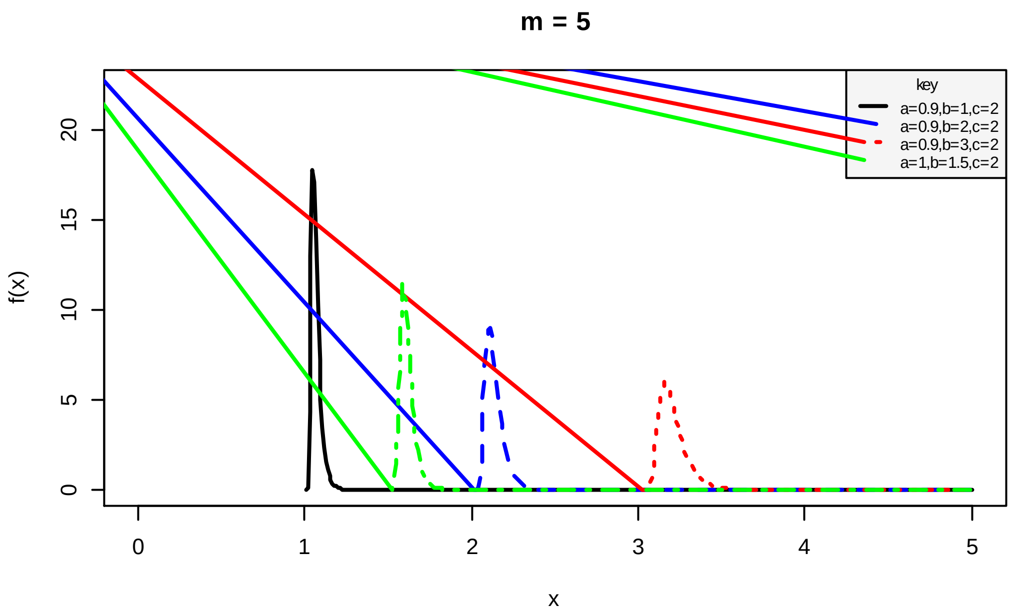

Figure 3: Density plot of MPD assuming five record values.

Figures 1-3 above shows that pdf of the *mth* lower record value of MPD is unimodal with longer right tails. The curves also shrink (i.e. variability reduces) by way of increasing *m*, the number of record values. This by implication means that, as the number gets large, the lower record values of the MPD won't be quite dispersed and would assume values very close to one another. This also means that the first few record values would be enough to make an appropriate guess of what values higher records will assume.

2.2 Non-Central Moment of Lower Record Values of MPD

The *rth* moment of $X(m)$ may be obtained by means of equations (1), (2) and (3) as shown below:

$$\mu^{r}(m) = E\left[X^{r}(m)\right] = \frac{1}{\Gamma(m)}\int_{0}^{\infty} x^{r}\left[-\ln F(x)\right]^{m-1} f(x)dx$$

Applying change of variables by setting $y=\left[1-\left(\frac{\lambda}{x}\right)^{\theta}\right]^{-\alpha}$, we find that

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$$\mu^r(m) = \frac{-\lambda^r}{\alpha\Gamma(m)}\int_0^1 \left(1-y^{-1/\alpha}\right)^{-r/\theta}\left[-\ln\{ye^{-(y-1)/\alpha}\}\right]^{m-1}(y-\alpha)e^{-(y-1)/\alpha}dy$$

Using the Binomial series expansion

$$(1-x)^{-t} = \sum_{i=0}^{\infty}\binom{t}{i}x^i = \sum_{i=0}^{\infty}\frac{t_i}{i!}x^i \tag{5}$$

where

$$t_i = \begin{cases} t(t+1)...(t+i-1); & i>0 \\ 1 \qquad ; & i=0 \end{cases}$$

we get

$$\mu^r(m) = \frac{-\lambda^r}{\alpha\Gamma(m)}\sum_{j=0}^{\infty}\frac{(r/\theta)_j}{j!}\int_0^1 y^{-i/\alpha}\left[-\ln\{ye^{-(y-1)/\alpha}\}\right]^{m-1}(y-\alpha)e^{-(y-1)/\alpha}dy$$

By putting $u = -\ln\{ye^{-(y-1)/\alpha}\}$, we see that

$$\mu^r(m) = \frac{\lambda^r}{\Gamma(m)}\sum_{j=0}^{\infty}\frac{(r/\theta)_j}{j!}\int_0^{\infty}\left[-\alpha W_{-1}\left(\frac{-\exp(-u)}{\alpha\exp(1/\alpha)}\right)\right]^{-i/\alpha}u^{m-1}e^{-u}du \tag{6}$$

Where W_{-1} symbolizes the negative branch of Lambert-*W* function.

We briefly remind that the Lambert-*W* function is defined as the solution of the equation $W(x)\exp(W(x)) = x$.

Here, $W(x)$ is a real function such that x is a real number and $x \geq -1/e$. In this regard, the *W*-function has two branches: the real branch taking on values in $(-\infty, -1]$ which is called the negative branch and denoted by W_{-1}, as well as the real branch taking on values in $[-1, \infty)$ which is called the principal branch and denoted by W_0. The negative branch has the main property that $W \leq -1$. It decreases from $W_{-1}(-1/e) = -1$ to $W_{-1}(0^-) = -\infty$.

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The integral in Eq. (6) above shows that the analytical derivation of the moments of record values from MPD seems to be complicated. Evaluating this integral is therefore left for further investigations. However, using mixture representations of cdf and pdf of the MPD (see Sirajo, 2020), we alternatively derive these moments as follows:

$$\mu^r(m) = \frac{1}{\Gamma(m)} \int_0^\infty x^r \left\{ -\ln\left(\sum_{l=0}^\infty \eta_l \left[1 - \left(\frac{\lambda}{x} \right)^\theta \right]^l \right) \right\}^{m-1} \sum_{k=0}^\infty \eta_k \frac{\theta \lambda^\theta}{x^{\theta+1}} \left[1 - \left(\frac{\lambda}{x} \right)^\theta \right]^k dx$$

$$\mu^r(m) = \frac{\theta}{\Gamma(m)} \int_0^\infty x^{r-1} \left(\frac{x}{\lambda} \right)^{-\theta} \left\{ -\ln\left(\sum_{l=0}^\infty \eta_l \left[1 - \left(\frac{x}{\lambda} \right)^{-\theta} \right]^l \right) \right\}^{m-1} \sum_{k=0}^\infty \eta_k \left[1 - \left(\frac{x}{\lambda} \right)^{-\theta} \right]^k dx$$

Setting $y = \left(\frac{x}{\lambda} \right)^{-\theta}$ we obtain

$$\mu^r(m) = \frac{\lambda^r}{\Gamma(m)} \sum_{k=0}^\infty \eta_k \int_0^1 y^{-1/\theta} \left\{ -\ln\left(\sum_{l=0}^\infty \eta_l [1-y]^l \right) \right\}^{m-1} (1-y)^k dy$$

For $m=1$ in the expression above, we obtain the following:

$$\mu^r(1) = \frac{\lambda^r}{\Gamma(1)} \sum_{k=0}^\infty \eta_k \int_0^1 y^{(\frac{\theta-1}{\theta})-1} (1-y)^{(k+1)-1} dy$$

$$\mu^r(1) = \lambda^r \sum_{k=0}^\infty \eta_k B\left(\frac{\theta-1}{\theta}, k+1 \right)$$

which corresponds to the *rth* moment of the MPD random variable. Thus, we say that the *rth* moment of a single lower record value $X(1)$ from the MPD is the *rth* non-central moment of the original series from which the record value was generated.

Note: $B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1} du$ is the beta function, while η_k and η_l are as defined in Sirajo (2020).

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RENYI ENTROPY OF MPD

Entropies offer some exceptional means of measuring the amount of information contained in a random sample with regards to the distribution or population the sample comes from. In various fields of science and probability, the application of entropy has proved useful in measuring the amount of uncertainty associated with random variables, where a large entropy value suggests a greater uncertainty in the data. Two most important and well-known entropy measures are the Renyi and Shanon entropies. The most general among these measures is the Renyi entropy defined by:

$$\Im(\xi) = \frac{1}{1-\xi}\log\left(\int_{\mathbb{R}_+} f^{\xi}(x)dx\right), \quad \text{for} \quad \xi > 1 \tag{7}$$

The Shannon entropy, which is defined by $E\{-\ln[f_X(x)]\}$, happens to be a special case of the Renyi entropy derived from taking $\lim_{\xi\to 1}\Im(\xi)$.

To derive the Renyi entropy of MPD, we proceed using Eq. (2) and the definition above as follows:

$$\int_{\lambda}^{\infty} f^{\xi}(x)dx = \int_{\lambda}^{\infty} e^{\xi/\alpha}\left(\frac{x}{\lambda}\right)^{-\xi\theta}\left\{1-\alpha\left[1-\left(\frac{x}{\lambda}\right)^{-\theta}\right]^{\alpha}\right\}^{\xi}\left[1-\left(\frac{x}{\lambda}\right)^{-\theta}\right]^{-2\xi\alpha-\xi}\exp\left\{-\frac{\xi}{\alpha}\left[1-\left(\frac{x}{\lambda}\right)^{-\theta}\right]^{-\alpha}\right\}dx$$

Let $y = \frac{x}{\lambda}$ so that $x = \lambda y$ then,

$$= \frac{e^{\xi/\alpha}\theta^{\xi}}{\lambda^{\xi+1}}\int_{1}^{\infty} y^{-\xi-\xi\theta}\left\{1-\alpha\left[1-y^{-\theta}\right]^{\alpha}\right\}^{\xi}\left[1-y^{-\theta}\right]^{-2\xi\alpha-\xi}\exp\left\{-\frac{\xi}{\alpha}\left[1-y^{-\theta}\right]^{-\alpha}\right\}dy$$

Using the expansion $(1-x)^t = \sum_{i=0}^{\infty}\frac{t_i}{i!}(-1)^i x^i$, the above expression reduces to

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$$=\frac{e^{\xi/\alpha}\theta^{\xi}}{\lambda^{\xi+1}}\sum_{i=0}^{\infty}\frac{\xi_i}{i!}(-1)^i\alpha^i\int_1^{\infty}y^{-\xi-\xi\theta}\left[1-y^{-\theta}\right]^{\alpha i-2\xi\alpha-\xi}\exp\left\{-\frac{\xi}{\alpha}\left[1-y^{-\theta}\right]^{-\alpha}\right\}dy$$

Applying this expansion again and using the one defined in Eq. (5) we get

$$=\frac{e^{\xi/\alpha}\theta^{\xi}}{\lambda^{\xi+1}}\sum_{i=0}^{\infty}\frac{\xi_i}{i!}(-1)^i\alpha^i\sum_{j=0}^{\infty}\binom{\alpha i-2\xi\alpha-\xi}{j}(-1)^j\int_1^{\infty}y^{-\xi-\xi\theta-\theta j}\exp\left\{-\frac{\xi}{\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}y^{-\theta k}\right\}dy$$

Using the definition of incomplete gamma function, and following Bensid and Zeghdoudi (2017) we note that $\int x^n e^{-ax}dx=\frac{1}{a^{n+1}}\Gamma(n+1,\,a)$ and $\int x^n\exp(-ax^k)dx=\frac{1}{ka^{(n+1)/k}}\Gamma\left[(n+1)/k,\,a\right]$.

Thus;

$$=\frac{e^{\xi/\alpha}\theta^{\xi}}{\lambda^{\xi+1}}\sum_{i=0}^{\infty}\frac{\xi_i}{i!}\alpha^i\sum_{j=0}^{\infty}\binom{\alpha i-2\xi\alpha-\xi}{j}(-1)^{i+j}\frac{1}{-\theta k\left[\frac{\xi}{\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}\right]^{(\xi+\xi\theta+\theta k-1)/\theta i}}\Gamma\left(\left(\xi+\xi\theta+\theta k-1\right)/\theta i,\ \frac{\xi}{\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}\right)$$

$$=\frac{e^{\xi/\alpha}\theta^{\xi-1}}{k\lambda^{\xi+1}\left[\frac{\xi}{\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}\right]^{(\xi+\xi\theta+\theta k-1)/\theta i}}\sum_{i=0}^{\infty}\frac{\xi_i}{i!}\alpha^i\sum_{j=0}^{\infty}\binom{\alpha i-2\xi\alpha-\xi}{j}(-1)^{i+j+1}\Gamma\left[\left(\xi+\xi\theta+\theta k-1\right)/\theta i,\ \frac{\xi}{\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}\right]$$

The Renyi entropy of MPD is finally obtained using Eq. (7) as:

$$\frac{1}{1-\xi}\ln\left\{\frac{e^{\xi/\alpha}\theta^{\xi-1}}{k\lambda^{\xi+1}\left[\frac{\xi}{\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}\right]^{(\xi+\xi\theta+\theta k-1)/\theta i}}\sum_{i=0}^{\infty}\frac{\xi_i}{i!}\alpha^i\sum_{j=0}^{\infty}\binom{\alpha i-2\xi\alpha-\xi}{j}(-1)^{i+j+1}\Gamma\left[\left(\xi+\xi\theta+\theta k-1\right)/\theta i,\ \frac{\xi}{\alpha}\sum_{k=0}^{\infty}\binom{\alpha}{k}\right]\right\}$$

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ASYMPTOTIC DISTRIBUTION OF SAMPLE MINIMUM

Let $X_1,...,X_n$ be a random sample of size n, and let $\bar{X}$ be the sample mean. Then by the well-known central limit theorem, the distribution of $\bar{X}$ approaches standard normal as $n \to \infty$. If $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ is the order statistics obtained by arranging $X_i,\ i=1,...,n$ in order of increasing magnitude. We call $X_{1:n}$ the minimum order statistic and $X_{n:n}$ the maximum order statistic. In this section, we study the distribution of the minimum order statistic of MPD as the sample size increases indefinitely. Intuitively, $X_{1:n}$ may be represented by the location parameter λ since the MPD random variable is bounded below by this parameter. Thus, we derive the asymptotic distribution of the sample minimum from MPD using the theorems in Arnold *et al*. (1998) adopted by Bensid and Zeghdoudi (2017).

The asymptotic distribution of the sample minimum $X_{1:n}$ of MPD may be defined as:

$$P\left\{c_n\left(X_{1:n}-d_n\right)\le x\right\} \xrightarrow{D} 1-\exp\left(-\lim_{t\to 0}\frac{F(tx)}{F(t)}\right)$$

Where the norming constants $c_n, d_n > 0$ are as defined in Leadbetter *et al.* (1983).

We proceed by evaluating $\lim_{t\to 0}\frac{F(tx)}{F(t)}$, which by applying l'Hospital's rule becomes

$$\lim_{t\to 0}\frac{F(tx)}{F(t)} = x\lim_{t\to 0}\frac{f(tx)}{f(t)}, \qquad \forall\ x>0$$

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$$= x\lim_{t\to 0}\frac{\frac{\theta\lambda^{\theta}}{(tx)^{\theta+1}}\left\{1-\alpha\left[1-\left(\frac{\lambda}{tx}\right)^{\theta}\right]^{\alpha}\right\}\left[1-\left(\frac{\lambda}{tx}\right)^{\theta}\right]^{-2\alpha-1}\exp\left\{\frac{1}{\alpha}\left(1-\left[1-\left(\frac{\lambda}{tx}\right)^{\theta}\right]^{-\alpha}\right)\right\}}{\frac{\theta\lambda^{\theta}}{t^{\theta+1}}\left\{1-\alpha\left[1-\left(\frac{\lambda}{t}\right)^{\theta}\right]^{\alpha}\right\}\left[1-\left(\frac{\lambda}{t}\right)^{\theta}\right]^{-2\alpha-1}\exp\left\{\frac{1}{\alpha}\left(1-\left[1-\left(\frac{\lambda}{t}\right)^{\theta}\right]^{-\alpha}\right)\right\}}$$

$$= x^{-\theta}$$

$\therefore\ P\left\{c_n\left(X_{1:n}-d_n\right)\le x\right\}\xrightarrow{D}1-\exp\left(-x^{-\theta}\right)$ or equivalently, $\lambda\xrightarrow{D}1-\exp\left(-x^{-\theta}\right)$.

Thus, we say that as the sample size gets large, the location parameter of the MPD converges in distribution to $1-\exp\left(-x^{-\theta}\right)$.

QUARTILES

This section computes the quartiles of MPD. The well-known quartiles of an observed variable divide the distribution into four equal parts using three quartile measures, namely, the first or lower quartile, which is the value that cuts off the first 25% of the data when it is sorted in ascending order. The second quartile or median, is the value that cuts off the first 50%. The third quartile or upper quartile, is the value that cuts off the first 75%. These quartiles for the MPD can be derived using its quantile function. The *uth* quantile of the MPD random variable is that value x such that $P(X<x)=u$. We call this value the median if $u=0.5$, the upper quartile if $u=0.75$, or the lower quartile if $u=0.25$. We remind that the quantile function of the MPD is obtained in terms of Lambert-*W* function as:

$$Q(u)=\frac{\lambda}{\left[1-\left\{-\alpha W_{-1}\left(-\frac{u}{\alpha\exp(1/\alpha)}\right)\right\}^{-1/\alpha}\right]^{1/\theta}},\qquad 0<u<1$$

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where W_{-1} denotes the negative branch of the Lambert-*W* function.

Thus, by numerically evaluating the above equation using R program, quartiles associated with the MPD random variable are computed for some selected values of the parameters. These are provided in Tables 1-3.

Table 1. Quartiles of the MPD for $\alpha = 0.9$**,** $\theta = 2$ **and** $\lambda = 1, 2, 3, 4$

λ	Q_1	$Q_2 = median$	Q_3
1	1.164593	1.264696	1.456109
2	2.329186	2.529392	2.912219
3	3.493778	3.794088	4.368328
4	4.658371	5.058785	5.824437

Table 2. Quartiles of the MPD for $\lambda = 1$**,** $\theta = 1$ **and** $\alpha = 0.1, 0.5, 0.8, 1.0$

α	Q_1	$Q_2 = median$	Q_3
0.1	1.317756	1.91411	3.709755
0.5	1.311166	1.678135	2.709166
0.8	1.342334	1.608244	2.224393
1.0	1.371383	1.595824	2.040281

Table 3. Quartiles of the MPD for $\alpha = 0.5$**,** $\lambda = 3$ **and** $\theta = 1, 2, 3, 4$

θ	Q_1	$Q_2 = median$	Q_3
1	3.933499	5.034405	8.127498
2	3.435185	3.886285	4.937863
3	3.283527	3.565033	4.182152
4	3.210227	3.414507	3.848843

CONCLUDING REMARKS

We have discussed in this paper, the distribution of record values (simply referred to as the records), for the case when MPD is the parent distribution. To describe the possible shapes of the associated pdf, the respective plots are provided. It is observed from the plotted figures of pdfs, for selected values of the parameters, that the distributions of the random variable $X\ (m)$ are unimodal and right skewed with longer and heavier right tails. Other related inferences concerning the quantile function of the MPD were considered and the quartiles tabulated. Renyi entropy has also been derived. We hope that the discoveries of this paper will be a useful reference for the specialists in various fields of studies and further enhancement of research concerning the applications of Muth-Pareto distribution in general, and record value theory in particular.

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