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ON THE CONVERGENCE OF LAGRANGE'S INTERPOLATING POLYNOMIAL OF SMOOTH FUNCTIONS

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ABSTRACT : *Considering the Lagrange interpolating polynomial Ln-1(f, x) of a given function defined on a uniform partition with a given mesh, we shall show in the present paper that Ln-1(f, x) does not converge to the function which has nice mathematical properties of approximation.*

KEYWORD: Lagrange's Interpolating Polynomial, Smooth Functions

INTRODUCTION

Let Δ (n) be a partition of the interval [-1, 1] with partition points $x_k = x_{k,n} = -1+2(k-1)/(n-1)$, k = 1, 2, …., n, n = 2, 3, ……. The Lagrange's interpolating polynomial $L_{n-1}(f, x)$ of a function f (x) at the knots x_k is defined as

$$L_{n-1}(f, x) = \sum_{k=1}^{n} f(x_k) l_k(x),$$

where

$$l_k(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots\ldots(x-x_n)}{(x_k-x_1)(x_k-x_2)\ldots\ (x_k-x_{k-1})(x_k-x_{k+1})\ldots\ldots\ldots\ (x_k-x_n)}$$

Bernstein [1] (see, also {6}, p. 37) has shown that the interpolating polynomial $L_{n-1}(f, x)$ of f (x) = | x|, 0 ≤ | x | ≤ 1 diverges for 0 < | x | < 1 and converges to the function for x = 0, 1, -1. It is known ([8], p 92) that a given partition there exists a continuous function whose interpolant does not converge to that function. Thus, the points of interpolation influence the convergence of the interpolant. A similar behavior for another linear operator $S_n(f, x)$, where S_n (f, x) is the n-th partial sum of the Fourier series of f (x), is known, I.e. there exists a continuous function whose Fourier Series diverges ([10], p. 298).

Byrne et a1 [3] has obtained a precise limit of convergence of the interpolant for the function f (x) = | x | in the following.

Theorem A, If 0< | x | < 1, then for f (x) = | x |,

$$\lim_{n\to\infty} \sup n^{-1} \log | L_{n-1}(f,x) - f(x)| = (½)[(1 + x) \log (1 + x) + (1 - x) \log (1 - x)]$$

PRELIMINARIES

It is known that if a function is sufficiently smooth function, e. g., if the function is differentiable in the interval, then its Fourier series converges to the function ([9], p. 406). Moreover, if the function has higher number of derivatives, the convergence is faster, Precisely, the following is known ([10], p.225) :

Theorem B. Let f (x) be periodic and k times differentiable. If | $f^{(k)}$ (x) < M, then

$$E_n (f) \leq A_k Mn^{-k} , (n=1, 2, …..) ,$$

where $E_n(f)$ denote the error of best approximation by trigonometric polynomials of degree n and A_k is a positive constant depending on k only.

The function considered in Theorem A is not differentiable at the origin. Now, we wish to see the role of smoothness of the function in the convergence of its interpolant.…

MAIN RESULTS

We shall prove the following:

Theorem 1. If 0 < |x| < 1 and f_{2r} (x) = $\begin{cases} x^{2r}, x \geq 0, \\ -x^{2r}, x < 0, \end{cases}$

then

$$\lim_{n\to\infty} \sup n^{-1} \log| \ L_{n-1}(f_{2r}, x) - f_{2r}\,(x)|$$

$$= (½)[(1 + x) \log (1 + x) + (1 - x) \log (1 - x)]$$

Further, though the partial sum of Fourier series, S_n (f, x), of a continuous function f(x) does not necessarily converge to f (x). But, if we apply a suitable transformation to S_n(f, x), e.g. the following is known as cesaro transformation of order α,

$$T_n(f, x) = \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha}\, S_k(f,x)}{A_n^{\alpha}}, \quad \alpha > 0$$

where $A_n^{\alpha} = \begin{bmatrix} n+\alpha \\ \alpha \end{bmatrix}$, then T_n(f, x) tends to f (x) . We also investigate the effect of such a transformation to the interpolant of the function, and prove the following :

Theorem 2. The (C, k) mean of $L_{n-1}(f_{2r}, x)$ is not convergent for any k.

for a fixed number x in [-1, 0] and a given partition there is a j = j(n) and $\theta = \theta(n), 0 \leq \theta < 1$, such that $x=x_j+2\theta/(n-1)$. We write for a number a and integer k, $(a)_k$ = a (a+1) … (a + k - 1), $(a)_0$ = 1, a$\pm$0. We will denote the gamma function by $\Gamma(.)$. We shall first prove the theorem for r=1.

We need the following Lemmas :

Lemma 1, For – 1 < x < 0 and n = 2m + 1 , we have

$$\sum_{K=m+2}^{2m+1} (x_k)^2 l_k(x)$$

$$= (-1)^m F(j, m, \theta) \begin{bmatrix} 2m-2 \\ m-1 \end{bmatrix} \left[1 - (m+1-j-\theta)\left\{2(2m-1)\sum_{k=0}^{\infty} \frac{(j+\theta-1)k}{(2m+k)(m+1)k} - \frac{2m(2m-1)}{(m+1)} \sum_{k=0}^{\infty} \frac{(j+\theta)_k}{(2m+k+1)(m+2)k}\right\}\right]$$

Where F (j,m,θ)=m^{-2} $(-1)^{j+1}$ sin(πθ)Γ(j+ θ) Γ (2m+2-j- θ)/($\pi(2m)$!).

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Proof. For x=xj+2 θ/(n-1) and X_k =-1+2(k-1)/(n-1),j=1,2,…,m and k= m+2, m+3,…, 2m+1=n, a little simplification shows that

$l_k(x)=m^2 F(j,m,\theta)\frac{(-1)k}{k-j-\theta} \cdot \begin{bmatrix} 2m \\ k-1 \end{bmatrix}$.

Now

$$\sum_{K=m+2}^{2m+1} (x_k)^2 l_k(x)$$

$$=F(j,m,\theta)\ \sum_{K=m+2}^{2m+1}(-1)^k \frac{(k-m-1)^2}{k-j-\theta}\begin{bmatrix} 2m \\ k-1 \end{bmatrix}$$

$$=F(j,m,\theta)\sum_{K=m+2}^{2m+1}(-1)^k \frac{(k-m-1)}{k-j-\theta}\,[(k-1)-m)]\ \begin{bmatrix} 2m \\ k-1 \end{bmatrix}$$

$=\sum_1 - \sum_2$, say. (1.1)

Writing $\sum_1$ in the following form, we see that

$$\sum_1 \quad =2mF(j,m,\theta)\sum_{K=m+2}^{2m+1}(-1)^k \frac{k-m-1}{k-j-\theta}\begin{bmatrix} 2m-1 \\ k-2 \end{bmatrix}$$

Substituting 2m+1-k=p, we see that

$$\sum_{p=0}^{m-1}(-1)^\rho \frac{m-\rho}{2m+1-p-j-\theta}\begin{bmatrix} 2m-1 \\ p \end{bmatrix}$$

$$=(-2m)F\ (j,m,\theta)\left[\sum_1 \ = (-2m)F(j,m,\theta)\sum_{p=0}^{m-1}(-1)^p\begin{bmatrix} 2m-1 \\ p \end{bmatrix} - \sum_{p=0}^{m-1}(-1)^p \frac{m+1-j-\theta}{2m+1-p-j-\theta}\begin{bmatrix} 2m-1 \\ p \end{bmatrix}\right]$$

$$=(-2m)F\ (j,m,\theta)\left[\ (-1)^{m-1}\begin{bmatrix} 2m-2 \\ m-1 \end{bmatrix} -(m+1-j-\theta)\sum_{p=0}^{m-1}(-1)^p \frac{1}{2m+1-p-j-\theta}\begin{bmatrix} 2m-1 \\ p \end{bmatrix}\ \right] \quad (1.2)$$

Writing

$$\sum_{p=0}^{m-1}(-1)^p \frac{1}{2m+1-p-j-\theta}\begin{bmatrix}2m-1\\ p\end{bmatrix} = S,$$

We see that

$$S=\sum_{k=0}^{m-1}(-1)^{m-1-k}\frac{1}{m+2+k-j-\theta}\begin{bmatrix}2m-1\\ m-k-1\end{bmatrix}$$

Using the identities

$$\frac{1}{m+2-k-j-\theta}=\frac{1}{m+2-j-\theta}\frac{(m+2-j-\theta)_k}{(m+3-j-\theta)_k},$$

$$(m-k-1)!= (-1)^k \frac{(m-1)!}{(1-m)k} \ ,$$

$$(m+1)! = (m)!(1+m)_k$$

And $(1-m)_k=0, k=m, m+1,\ldots,$ we can write

$$S=\frac{(-1)^{m+1}}{m+2-j-\theta}\begin{bmatrix}2m-1\\ m-1\end{bmatrix}\sum_{k=0}^{\infty}\frac{(m+2-j-\theta)_k(1-m)_k(1)_k}{(m+3-j-\theta)_k(1+m)_k(k)!}$$

$$=(-1)^{m+1}\begin{bmatrix}2m-1\\ m-1\end{bmatrix}\sum_{k=0}^{\infty}\frac{(j+\theta-1)k}{(m+1)_k(2m+k)} \ , \qquad (1.3)$$

By the application of the result of Luke ([5],p.104).Also, using the result of Byrne et al [3]

(Lemma 1,p.85) we see that

$$\sum\nolimits_2 = mF((j,m,\theta)(-1)^m\left[\begin{bmatrix}2m-1\\ m-1\end{bmatrix}-(m+1-j-\theta)\begin{bmatrix}2m\\ m-1\end{bmatrix}\sum_{k=0}^{\infty}\frac{(j+\theta)k}{(2m+1+k)(m+2)k}\right]$$

(1.4)

Using (1.3) in (1.2) and combining (1.4) and (1.2) we complete the proof of Lemma 1.

Lemma 2. If the series $\sum_{k=0}^{\infty} ak$ with partial sums sn is summable (C,k), then

$a_n=o(n^k)$ and $s_n=o(n^k)$.

The result is contained in knoop [4](Th.,p.484).

Proof of Theroem 1. We first prove the theorem for r=1. Suppose n is odd i.e. n=2m+1 for some integer m. We assume that $x\in(-1,0)$. Since x^2 is a polynomial of degree less than n, we have (cf.[3], p.86)

$$x^2=\sum_{K=1}^{2m+1}(x_k)^2 l_k(x) \text{ and } \tag{1.5}$$

$$L_{n-1}(f_2, x)=\sum_{K=1}^{2m+1} f(x_k)^2 l_k(x)$$

$$=-\sum_{K+1}^{m}(x_k)^2 l_k(x)+\sum_{K=m+2}^{2m+1}(x_k)^2 l_k(x). \tag{1.6}$$

For -1<x<0, in view of (1.5) and (1.6), we get

$$L_{n-1}(f_{2,x})-f_{2(x)}=2\sum_{K=m+2}^{2m+1}(x_k)^2 l_k(x), \tag{1.7}$$

and therefore we need to estimate the R.H.S. of (1.7).

We shall start by obtaining the bounds for the infinite series evolved on the R.H.S. of the expression of Lemma 1. Using the summing formula for hyper geometric series, we have ([7], p.49)

$$1-((m+1-j-\theta)\{2(2m-1)\sum_{k=0}^{\infty}\frac{(j+\theta-1)k}{(m+1)k(2m+k)}+\frac{2m(2m-1)}{m+1}\sum_{k=0}^{\infty}\frac{(j+\theta)k}{(m+2)k(2m+k+1)}$$

$$\leq 1+(m+1-j-\theta)\{\frac{2(2m-1)}{2m}\sum_{k=0}^{\infty}\frac{(j+\theta-1)k}{(m+1)k}+\frac{2m(2m-1)}{m+1\,(2m+1)}\sum_{k=0}^{\infty}\frac{(j+\theta)k}{(m+2)k}\}<4m. \tag{1.8}$$

Also,

$$\left|1-2(2m-1)(m+1-j-\theta)\left\{\sum_{k=0}^{\infty}\frac{(j+\theta-1)k}{(m+1)_k(2m+k)}-\frac{m}{m+1}\sum_{k=0}^{\infty}\frac{(j+\theta)k}{(m+2)_k(2m+k+1)}\right\}\right|$$

$$\geq \left|\left|1+\frac{2(2m-1)(m+1-j-\theta)2}{m+1}\sum_{k=0}^{\infty}\frac{(j+\theta)k}{(2m+k+1)(m+2)_k}\right| - \left|\frac{(2m-1)(m+1-j-\theta)}{m}\right|\right|$$

$$=\left|1+\frac{2(2m-1)(m+1-j-\theta)^2}{m+1}\sum_{k=0}^{\infty}\frac{(j+\theta)k}{(2m+k+1)_k(m+2)_k}-\frac{(2m-1)(m+1-j-\theta)}{m}\right|$$

Now, as the function G()m defined by G(m)=1-k_m+g(m),where k_m is a constant and g(m) is a function of m such that for some constant $k'_m<k_m$

$$g(m) \leq k'_m < k_m$$

(g(m) → ∞ as m → ∞, k_m→ ∞ as→ ∞), gives minimum absolute value if we take k'_m in place of g (m)in the definition of G(m), that is $|1-km+g(m)| \geq |1-km+k'm,|$

and observing that

$$\frac{2(m+1-j-\theta)^2(2m-1)}{m+1}\sum_{k=0}^{\infty}\frac{(j+\theta)k}{(2m+k+1)(m+2)_k}<\frac{2(2m-1)(m+1-j-\theta)}{2m}\left[1-\frac{1}{2m+1}\right],$$

we get

$$\left|1+\frac{2(2m-1)(m+1-j-\theta)^2}{m+1}\sum_{k=0}^{\infty}\frac{(j+\theta)_k}{(2m+k+1)(m+2)_k}-\frac{(2m-1)(m+1-j-\theta)}{m}\right|$$

$$\left|1+\frac{2(2m-1)(m+1-j-\theta)}{2m+1}-\frac{(2m-1)(m+1-j-\theta)}{m}\right|$$

$$=\left|1-\frac{(2m-1)(m+1-j-\theta)}{m(2m+1)}\right|$$

$$>1-\frac{2m-1}{2m+1}$$

$$>\frac{1}{2m+1}. \qquad (1.9)$$

Now, using (1.8)and (1.9)in the part of the expression of Lemma 1, we get

$$|\mathrm{F}(\mathrm{j}, \mathrm{m}, \theta)| \begin{bmatrix} 2m-2 \\ m-1 \end{bmatrix} \frac{1}{2\mathrm{m}+1} \leq \left| \sum_{\mathrm{k}=\mathrm{m}+2}^{2\mathrm{m}+1} (x_k)^2 l_k(x) \right|$$

$$\leq |\mathrm{F}(\mathrm{j}, \mathrm{m}, \theta)| \begin{bmatrix} 2m-2 \\ m-1 \end{bmatrix} 4\mathrm{m}. \quad (1.10)$$

The second inequality in (1.10) gives

$$\left|\sum_{\mathrm{k}=\mathrm{m}+2}^{2\mathrm{m}+1} (x_k)^2 l_k(x)\right| \leq \frac{\Gamma(\mathrm{j}+\theta)\Gamma(2\mathrm{m}+2-\mathrm{j}-\theta)}{(\Gamma(\mathrm{m}+1))^2} \frac{2}{2\mathrm{m}-1}$$

Now, following the result fo Luke [5] (also,[7]p.30-31), we have

$$\mathrm{Log}\ \Gamma\ (\mathrm{y})=(\mathrm{y}-\tfrac{1}{2})\ \mathrm{logy}-\mathrm{y}+\frac{\log 2\pi}{2}+0(\tfrac{1}{\mathrm{y}}),$$

And (logy/y) → 0 as y → ∞. Since 1+x=(j-1+ θ)/m, we find that as m→ ∞, j tends to ∞. Also, 2m-j=m(1-x)+(θ − 1) → ∞as m→ ∞, Hence

$$\frac{1}{\mathrm{m}} \log \left| \sum_{\mathrm{k}=\mathrm{m}+2}^{2\mathrm{m}+1} (x_k)^2 l_k(x) \right|$$

$$\leq \left(\frac{\mathrm{j}+\theta-(1/2)}{\mathrm{m}}\right)\log\left(\frac{\mathrm{j}+\theta}{\mathrm{m}+1}\right)+\left(\frac{2\mathrm{m}+\left(\frac{3}{2}\right)-\mathrm{j}-\theta}{\mathrm{m}}\right)\log\left(\frac{2\mathrm{m}+2-\mathrm{j}-\theta}{\mathrm{m}+1}\right)+\mathrm{o}\left(\frac{\log \mathrm{m}}{\mathrm{m}}\right) \quad (1.11)$$

Since j/m →1+x as m → ∞, we obtain

$$\lim_{n\to\infty} \sup \frac{1}{\mathrm{m}} \log \left| \sum_{\mathrm{k}=\mathrm{m}+2}^{2\mathrm{m}+1} (x_k)^2 l_k(x) \right| \leq [(1+\mathrm{x}) \log (1+\mathrm{x}) + (1-\mathrm{x}) \log (1-\mathrm{x})]$$
(1.12)

Finally, for the left hand inequality of (1.10), we observe that for each x such that $x=x_j+\theta/m$, $0\leq \theta <1$, there exist an increasing sequence $\{r_m\}$ of positive integers (cf.[2],) Lemma 1) such that, when we write

$$X=xj+\theta/r_m, 0<\theta<1,$$

Where j=j(n) and $\theta = \theta(n)$, the inequalities

$$K1 \leq \theta \leq k2$$

Hold for all m and $o<k_1<k_2<1$. Since $\sin \pi \theta$ has a positive lower bound, that is to say, $(\sin \pi \theta)/\pi = c$,from (1.10),we obtain

$$\frac{1}{r_m}\log \left\|\left[\sum_{k=r_{m+2}}^{2r_{m+1}} (x_k)^2 l_k(x)\right]\right\|$$

$$\geq \frac{1}{r_m}\log \Gamma(j+\theta)+\frac{1}{r_m}\log\Gamma(2r_m+2-j-\theta)-\frac{2}{r_m}\log\Gamma(r_m+1)+0(\frac{1}{r_m}) \quad (1.13)$$

As we have already shown that the right hand side of the inequality (1.10) approaches (1+x) log (1+x)+(1-x)log(1-x),as $m\rightarrow\infty$, (1.13)also approaches to the same limit. This proves the theorem for r=1. Theorem 1 follows after a similar computations and using the fact that (k-m-1) μ can be written as

$$(k-m-1)^{\mu} = \sum_{t=0}^{\mu} g_t^{\mu}(m) \prod_{i=0}^{\mu-t}(k-i)$$

where g_t μ(m) denote polynomials in m of degree t and is defined by the following recurrence relations:

$g_0^1(m)=1, g_1^1(m)=-m, g_0^2(m)=1, g_1^2(m)=1-2m, g_2^2(m)=m2$, $g_{\mu+1}^{\mu+1}(m)=-m g_{\mu}^{\mu}(m)$,

$g_{\mu}^{\mu+1}(m)=(1-m)\, g_{\mu-1}^{\mu}(m)+ g_{\mu}^{\mu}(m)$ and $g_0^{\mu}(m)= g_0^{\mu+1}(m)$, and

2m+1

$X^{2r}=\sum_{k=1}^{2m+1}(x_k)^2 l_k(x)$,

and $L_{n-1}(f_{2r},x)-f_{2r}(x)=2\sum_{k=m+1}^{2m+1}(x_k)^2 l_k(x)$.

the case n= 2m is very similar to that of n=2m+1.

Proof of theorem 2. We know that $L_{n-1}(f,x)$, n=2,3,… is convergent for $x \in \{-1,0,1\}$.Let x be a point different than {-1,0,1} and its distance from zero be a. Now, from Theorem 1, from arbitrary $\in>0$ there exists a positive integer sequence $r_1<r_2<\ldots r_m\ldots$such that

$$\left|L_{r_m-1}(f_{2r},x)-f_{2r}(x)\right| \geq [(1+a)^{1+a}(1-a)^{1-a}-\in]^{r_m a/2}$$

For all m Therefore

$$|L_{n-1}(f_{2r},x)-f_{2r}(x)| \neq o(n^k), \quad (n \to \infty) \qquad (1.14)$$

Thus ,theorem 2 follows when we combine (1.14) with lemma 2.

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