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ON SOME SIGNED SEMIGROUPS OF ORDER – PRESERVING TRANSFORMATION

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ABSTRACT: *Let $X_n = \{1, 2, 3, .., n\}$ and α be a transformation from $X_n \longrightarrow {X_n}^*$ where $X_n^* = \{-1, 1, -2, 2, -3, 3 \dots, -n, n\}$. The semigroup of T_n (the full transformation semigroup), P_n (the partial transformation semigroup) and I_n (the partial one – one transformation semigroup) serve the bedrock for these transformation semigroups, Let SO_n , SPO_n and SIO_n denotes the sets of signed full order – preserving, signed partial order – preserving and signed partial one – one order - preserving transformation semigroup respectively. The paper aims at investigate the sets of these subsemigroups.*

KEYWORDS: Order Preserving, Transformation Semigroup, Nilpotent, Idempotent, Self-Inverse, Chain Decomposition.

INTRODUCTION

The signed (partial) transformation semigroup, ST_n is the set of all mapping from set $\alpha: X_n \longrightarrow X_n^*$ such that $\alpha: dom(\alpha) \subseteq X_n \longrightarrow lm(\alpha) \subseteq X_n^*$ is said to full or total if $dom(\alpha) = X_n$ and $lm(\alpha) \subset X_n^*$. $Dom(\alpha)$ stands for the domain of α while the $lm(\alpha)$ denotes the image of α as define by Gamyushkim and Mazorchuk (2009), Howie (1995). Richard (2008) initiated the study of signed semigroups. Richard (2008), James and Kerber (1981) defined signed permutation groups on a set $X_n \longrightarrow X_n^*$ and the set ST_n is the semigroup analogue of T_n and SS_n(signed symmetric group) is the units of T_n.

A signed transformation α in SP_n that is $\alpha: dom(\alpha) \subseteq X_n \longrightarrow lm(\alpha) \subseteq X_n^*$ is order preserving if

$| i| \leq |j| \Longrightarrow |i\alpha| \leq |j\alpha| for\ i, j$ in $dom(\alpha)$.

Standard terms in semigroup theory are contained in Ganyushkim and Mazorchuk (2009), Laradji and Umar (2004c), Garba (1994). The semigroup of order – preserving full transformation of X_n is denoted by O_n. Howie (1971) investigated that the order and the number of idempotent of O_n are $|O_n| = \binom{2n-1}{n-1}$ and $|EO_n| = F_{2n}$ respectively. He consider the *Fibonacci Numbers* F_n defines by $F_1 = F_2 = 1$ and $F_0 = 0$ where

$F_n = F_{n-1} + F_{n-2}$ $(n \geq 3)$ and F_{2n} is the alternate Fibonacci numbers. Its integers sequence is 1,1.2,3,5, ... $(A00005)$ $Sloane$ (2010.) Gomes and Howie (1992), Garba (1994d) study the order of elements of PO_n and IO_n and showed that $|PO_n| = \sum_{r=0}^{n} \binom{n}{r}\binom{n+r-1}{r}$ and $|IO_n| = \binom{2n}{n}$ rrespectivey.

An element α of an arbitrary semigroup S is a nilpotent if $\alpha^n = 0$ (the zero mapping), for some integers $n > 0$. The set of all nilpotents elements is defined by $N(S)$. An element $\beta \in S$

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is self inverse denoted by $I(S)$ if $\beta^2 = i$ (the identity elements). Mogbonju (2015) studied chain decomposition of SS_n, ST_n (signed full transformation) and SP_n(signed partial one – one transformation semigroup) respectively. Also Adeniji (2012) studied the chain decomposition of identity difference of full and partial transformation semigroup.

Chain decompositions of elements of a semigroup is defined as the set of fragments of each $\beta \in ST_n$, for each $i \in dom(\beta), j \in Im(\beta),$ such that $i\beta = j \Longrightarrow \beta_k: i_k \longrightarrow j_k$ $where\ k = 1,2,3, \dots n$.

For example

Let $\beta = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ j_1 & j_2 & j_3 & j_4 & j_5 \end{pmatrix}$

Then the chain decomposition of $\beta = \left\{ \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \begin{pmatrix} i_2 \\ j_2 \end{pmatrix}, \begin{pmatrix} i_3 \\ j_3 \end{pmatrix}, \begin{pmatrix} i_4 \\ j_4 \end{pmatrix}, \begin{pmatrix} i_5 \\ j_5 \end{pmatrix} \right\}$

The following combinatorial notations are used:

$|S|$ = the cardinality of S

$|E(S)|$ = the total number of idempotent elements in S

$|N(S)|$ = the total number of nilpotent elements in S

$|I(S)|$ = the total number of self - inverse elements in S

$|H_n|$ = the total number of chains in the chain decomposition of all the elements of S

METHODOLOGY

The methods employed in carry out this research work are:

i. the elements in each of the semigroups were listed using matrix notations;
ii. the matrix notation was used in finding the product of elements of the semigroups.The approach aids us to study the structure of each semigroup.
iii. the elements were arranged, the pattern of the arrangement were studied and the combinatorial principles were applied to get the sequences.

The following notations will be used in representing elements in transformation semigroups.

Let $\alpha \in SO_5$: this element can be represent in matrix form by placing ± 1 in $(i, j) -$ entry of an $n \times n$ matrix to indicate $j \longrightarrow \pm i$

Example 1

If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & 1 & -3 & 5 & 4 \end{pmatrix} \in SO_5$

written in matrix notation as

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Element in SO_n , SPO_n and SIO_n

The set of elements in SO_2 is as follows:

$$|SO_2| = \left\{ \begin{matrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}. \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \\ \left(\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}\right) \end{matrix} \right\} = 12$$

$$|Im(\alpha)| = h = 1 = \left\{ \begin{matrix} \left(\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \right) \end{matrix} \right\} = 8$$

$$|Im(\alpha)| = h = 2 = \left\{\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}. \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\right\} = 4$$

Table 1 : Values of Elements in SO_n for each n.

$n/\|Im(\alpha)\| = h$	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$\|SO_n\| = 2^n\binom{2n-1}{n-1}$
1	2	-	-	-	-	2
2	8	4	-	-	-	12
3	24	48	8	-	-	80
4	64	288	192	16	-	560
5	160	1280	1920	640	32	4032

Theorem 3.1.1: Let $S = SO_n$. Then for $n \geq 1, |S| = 2^n \binom{2n-1}{n-1}$

Proof: Let $\alpha \in S$ and $X_n^* = \{-1, 1, -2, 2, -3,3 \dots, -n, n\}$. such that $dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$ such that the $dom(\alpha) = X_n$ and $lm(\alpha) \subset X_n^*$. Since the $|Im(\alpha)| = i$ or $-i$ for $i = 1,2,3,4, \dots n$ then there are 2^nelements and is equivalent to $\sum_{k=0}^{n} \binom{n}{k}$. Also if the $|Im(\alpha)| = h, h = 1,2,3, \dots and\ n = 1,2,3, \dots$ there are $|\alpha(S)| = 2^n$ elements. Since the elements of $dom(\alpha)$ can be chosen from X_n and the elements of $Im(\alpha)$ can be chosen $Im(\alpha) \in X_n^*$ then the choices of n is independently, it follows by applying the product rule

Table 2 : Values of Elements in SPO_n for each n

$n/\|Im(\alpha)\| = h$	$h = \emptyset$	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$\|SPO_n\| = \sum_{k=0}^{n} \binom{n}{k}^3 2^k$
1	1	2	-	-			3
2	1	16	4				21
3	1	78	84				171
4	1	320	832				1521
5	1	1320	6140				14283

Theorem 3.1.2: Let $S = SPO_n$, then $|S| = \sum_{k=0}^{n} \binom{n}{k}^3 2^k$

Proof: Let $\alpha \in S$. The semigroup has an empty map since it's a partial transformation. Since α is a bijection, k elements of domain can be chosen from X_n in $\binom{n}{k}$ ways. Let $Im(\alpha) \subset X_n^*$, if $|Im(\alpha)| = 0$, then $|\alpha S| = 1$ and if $|Im(\alpha)| = i \; for \; i = 1,2,3, \ldots$ then $|\alpha S| = 2^n$ for each n and 2^k where $k = 1,2,3, \ldots, n$. Then sum the product rule hence the proof. (Note that the number of elements in the image is equal to h that is $Im(\alpha) = h$)

Theorem 3.1.3: Let $S = SIO_n$, then $|S| = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}$

Proof : First observed that $\alpha \in S$. If the $|Im(\alpha)| = 0$, then $|\alpha S| = 1$ for each n. Let $Im(\alpha) = \{\,\}$ denotes an empty map, and we observed that $|\alpha S| = 2^n$ for $|Im(\alpha)| = i \; for \; i = 1,2,3, \ldots$ and $k = 1,2,3, \ldots n$which is equivalent to $\sum_{k=0}^{n} \binom{n}{k}$ since $k$elements of the of the $dom(\alpha)$ can be chosen from X_n^* in $\binom{n}{k}$ ways and since the empty map is an elements of SIO_n and $n \geq k \; where \; k$ is the maximum elements in the images, then k elements of $Im(\alpha)$ can be chosen from X_n^* in $\binom{n+k}{k}$ ways.

Idempotent in SO_n , $SPO_n$and SIO_n

Table 3: Values of idempotents elements in signed order – preserving transformation semigroup.

n/S	$\|E(SO_n)\|$	$\|E(SPO_n)\|$	$\|E(SIO_n)\|$
1	1	2	2
2	4	8	4
3	22	18	8
4	140	200	16
5	969	1010	32

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Proposition 3.2.1: Let $S = SO_{n,}$ and $\in S$. Then $|E(S)| = \frac{1}{3n+1}\binom{4n}{n}$.

Proof : $X_n = \{1, 2, 3, .., \text{n}\}, X_n \longrightarrow X_n^*$ and the $Im(\alpha) = \{i, -i\}$ and $i = 1,2,3, \dots, n$ and let α be transformation in S and observed that n elements of $dom(\alpha)$ can be chosen from X_n in $\binom{4n}{n}$ ways and if the $Im(\alpha)$ can be chosen in X_n^* in one – one fashions, then the number of idempotent elements in $\alpha \in S$ is $\frac{1}{3n+1}\binom{4n}{n}$.

Theorem 3.2.1: Let $S = SIO_{n,}$, then $|E(S)| = 2^n$

Proof : Let $\alpha \in S$. Idempotent are special case of binomial theorem which say $\sum_{k=0}^{n}\binom{n}{r}x^r y^{n-r} = (x+y)^n = 2^n$ if $x = y = 1$ $i.e$ $\sum_{r=0}^{n}\binom{n}{r} = 2^n \Longrightarrow E(SIO_n) = 2^n$

Self – inverse elements in signed order - - preserving transformation semigroups:

For $S = SO_{3,}$, the set of all self – inverse elements is;

$$|I(\boldsymbol{SO_3})| = \left\{\begin{matrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}\right\}$$

Example 3.3.1

Consider the element

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \end{pmatrix} \in SO_{3,}$$

This elements can be written in matrix form as:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and this element is self – inverse via matrix multiplication:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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The following theorem prove the number of self – inverse elements in order – preserving transformation semigroup'

Theorem 3.3.1: Let $S = SO_{n,}$ then $|I(S)| = 2^n \; for \; n > 1$.

Proof ; Let $dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$. and $\beta \in SO_n$. An elements β is self inverse if $\beta^2 = i$ (identity elements). If $|Im(\alpha)| = 1$, $|\alpha S| = 0$ and $|\alpha S| = 2^n$ if $|Im(\alpha)| = n$ for each n $(n > 1)$. Hence, the prove.

Proposition 3.3.1. : $|S| = I(SO_{n,}) = I(SPO_{n,}) = I(SIO_{n,}) = 2^n$

Proof: The proof is similar to the proof of theorem 4.1

Chain decomposition of element of $SO_{n,}$

The total number of chains in the chain decomposition of all elements of S is any of the SO_n , $SPO_n and \; SIO_{n,}$ are described below.

Theorem 3.4.1: Let $S = SO_{n,}$, then $|H_n| = n(n+1)$ for all n.

Proof : If $\alpha \in S$ for each $i \in dom(\alpha)$, $\exists \; j \in Im(\alpha) \; such \; that \alpha: i_k \longrightarrow j_k$ where $i = 1,2,3, \dots, and \; i\alpha = j$. Since the $Im(\alpha) \subset X_n^*$, then each element taken from the domain could occur in $n -$ times in $(n+1)$ ways. Hence

$|H_n| = n^2 + n$.

Theorem 3.4.2: Let $S = SPO_{n,}$, then $|H_n| = 2n^2 + n$ for all n.

Proof : Empty map is the subsets of partial transformation. Since the $m(\alpha) \subset X_n^*$, then for each i maps all j including the empty set which n times in $2n + 1$ ways.

Theorem 3.4.3: Let $S = SIO_{n,}$, the total number of chains, $|H_n|$ in the chains decompositions of all elements of $SIO_{n,}$ is $2n^2 + n$.

Proof : . Let the $Im(\alpha) \subset X_n^*$, and empty map is the subsets of partial transformation each each i maps all j including the empty set with n times in $2n + 1$ ways . Hence $|H_n| = 2n^2 + n$

SUMMARY OF THE RESULTS

Signed semigroups are a new class of transformation semigroup and much research work has not been done.

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Table 4: The summary of various results obtained.

S	$\lvert S \rvert$	$\lvert E(S) \rvert$	$\lvert N(S) \rvert$	$\lvert I(S) \rvert$	$\lvert H_n \rvert$
SO_n	$2^n \binom{2n-1}{n-1}$	$\frac{1}{3n+1}\binom{4n}{n}$	$-$	2^n	n^2+n
SPO_n	$\sum_{k=0}^{n} \binom{n}{k}^3 2^k$	?	?	2^n	$2n^2+n$
SIO_n	$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$	2^n	?	2^n	$2n^2+n$

CONCLUSION

The sequences generated have application in coding theory, combinatorial algebra and computational theory. The total number of nilpotent elements of SPO_n and SIO_n has not been known.

Future Research

Further study on signed transformation semigroup can be carried out on order decreasing transformation semigroups.

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