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On Cyclic Orthogonal Double Covers of Regular Circulant Graphs by Certain in…Nite Graph Classes

R. El-Shanawany, A. El-Mesady

Department of Physics and Engineering Mathematics, Faculty of Electronic Engineering, Menou…ya University, Menouf, Egypt.

ABSTRACT: *A collection G of isomorphic copies of a given subgraph G of a graph H is said to be an orthogonal double cover (ODC) of H by G; if every edge of H belongs to exactly two members of G and any two di¤erent elements from G share at most one edge. An ODC G of H is cyclic (CODC) if the cyclic group of order jV (H)j is a subgroup of the automorphism group of G. In this paper, the CODCs of regular circulant graphs by certain in…nite graph classes are considered.*

KEYWORDS: Orthogonal double covers, Orthogonal labelling, Circulant graphs, 2010 Mathematics Subject Classi…cation: 05C70; 05B30:

INTRODUCTION

A generalization of the notion of an orthogonal double cover (ODC) to undirected underlying graphs is as follows, see [1]. Let H be an undirected graph with n vertices and let G = fG_0; G_1; : : : ; $G_{n\ 1}$g be a collection of n spanning subgraphs of H; and V (H); E(H) refer to the vertices and the edges of the graph H respectively. G is called an orthogonal double cover (ODC) of H if there exists a bijective mapping : (V (H)) ! G such that:

(1) Every edge of H is contained in exactly two of the graphs G_0; G_1; : : : ; $G_{n\ 1}$:

(2) For every choice of di¤erent vertices a; b of H,

jE((a)) \ E((b))j = 1 if (a; b) 2 H; 0 otherwise.

An automorphism of an orthogonal double cover (ODC) G = f G_0; G_1; G_2; : : : ; $G_{n\ 1}$g of H is a permutation : V (H) ! V (H) such that f (G_0); (G_1); : : : ; ($G_{n\ 1}$)g = G, where for i 2 f0; 1; 2; : : : ; n 1g; (G_i) is a subgraph of H with V ((G_i)) = f (v) : v 2 V (G_i)g and E((G_i)) = f((u); (v)) : (u; v) 2

E(G_i)g: An ODC G of H is cyclic (CODC) if the cyclic group of order jV (H)j is a subgroup of the automorphism group of G, the set of all automorphisms of

G.

Let = f $_0$; $_1$; : : : ; $_{n\ 1}$g be an (additive) abelian group of order n. The vertices of $K_{n;n}$ will be labeled by the elements of Z_2. Namely, for (v; i) 2 Z_2 we will write v_i for the corresponding vertex and de…ne fw_i; u_j g 2 E($K_{n;n}$) if and only if i =6 j, for all w; u 2 and i; j 2 Z_2.

Let G be a spanning subgraph of $K_{n;n}$ and let a 2 . Then the graph G with E(G + a) = f(u + a; v + a) : (u; v) 2 E(G)g is called the a-translate of G. The length of an edge e = (u; v) 2 E(G) is de…ned by d(e) = v u:

G is called a half starter with respect to if jE(G)j = n and the lengths of all edges in G are di¤erent, i.e. fd(e) : e 2 E(G)g = : The following three results were established in ([1]).

Theorem 1 ([1]) If G is a half starter, then the union of all translates of G

[forms an edge decomposition of $K_{n;n}$;i.e. E(G + a) = E($K_{n;n}$):

a2

Here, the half starter will be represented by the vector: v(G) = (v $_0$; v $_1$; : : : ;

v $_{n\ 1}$); where v $_i$ 2 and (v $_i$)$_0$ is the unique vertex ((v $_i$; 0) 2 f0g) that belongs to the unique edge of length $_i$:

Two half starter vectors v(G_0) and v(G_1) are said to be orthogonal if fv (G_0) v (G_1) : 2 g = :

Theorem 2 ([1]) If two half starters v(G_0) and v(G_1) are orthogonal, then G = f$G_{a;i}$: (a; i) 2 Z_2g with $G_{a;i}$ = G_i + a is an orthogonal double cover (ODC) of $K_{n;n}$.

The subgraph G_s of $K_{n;n}$ with E(G_s) = ffu$_0$; v$_1$g : fv$_0$; u$_1$g 2 E(G)g is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter.

A half starter G is called a symmetric starter with respect to if v(G) and v(G_s) are orthogonal.

Theorem 3 ([1]) Let n be a positive integer and let G be a half starter repre-sented by v(G) = (v $_0$; v $_1$; : : : ; v $_{n\ 1}$). Then G is symmetric starter if and only if fv v + : 2 g = :

De…nition 4 ([1]) Let G = (Z_2; E(G)) be a symmetric starter, let fa; ag be the edge in G with length zero. The graph F = (; E(F)) is called the corresponding graph of G, where fa; bg 2 E(F) if and only if (a; b) 2 E(G) with a =6 b:

Remark 5 ([1]) Note that jE(G) fa; agj = n 1 = jE(F)j the number of edges of the graph F:

j[$_{a2}$ E(G + a) [$_{a2}$ fa; agj = n^2 n = n(n 1) = [$_{a2}$ E(F +a) = jE(G)j the number of edges of an orthogonal double cover (ODC) of K_n group generated by F:

Theorem 6 ([1]) Let n be a positive integer. Let G be a symmetric starter of $K_{n;n}$ and let F be the corresponding graph of G. Then F is an orthogonal double cover (ODC) generating graph with respect to:

By using the previous theorem, we got some of the results introduced in sections 2, 3, and 4.

In this paper we make use of the usual notation: $K_{m;n}$ for the complete bipartite graph with partition sets of sizes m and n, K_n for the complete graph on n vertices, G [F for the disjoint union of G and F; $K_{m;n;k}$ for the complete tripartite graph with partition sets of sizes m; n and k, P_n for the path on n vertices; and GnF for the graph obtained by removing the edges of a subgraph F from the graph G; where the graphs G and F are spanning subgraphs of H: Let r 1; n_1; n_2; : : : ; n_r be positive integers, n_1; n_r 1 and n_i 0 for i 2 f2; 3; : : : ; r 1g; the caterpillar C_r(n_1; n_2; : : : ; n_r) is the tree obtained from the path P_r := x_1x_2 : : : x_r by joining vertex x_i to n_i new vertices, i 2 f1; 2; 3; : : : ; rg:

The authors of [2] introduced the notion of an orthogonal labelling. Given a graph G = (V; E) with n 1 edges and n vertices, a 1 1 mapping: V (G)!

Z_n is an orthogonal labelling of G if:
(1) For every l 2 f1; 2; : : : ; b (n$_2$ 1) cg; G contains exactly two edges of length l; and exactly one edge of length n=2 if n is even, and
(2) The rotation distance r; fr(l) : l 2 f1; 2; : : : ; b (n$_2$ 1) cgg= f1; 2; : : : ; b (n$_2$ 1) cg:
The following theorem of Gronau et al. [2] relates CODCs of K_n and the orthogonal labelling.

Theorem 7 ([2]). A CODC of K_n by a graph G exists if and only if there exists an orthogonal labelling of G.

The following theorem of Sampathkumar and Simaringa is a generalization of Theorem 7.

Theorem 8 ([3]): A CODC of Circ(n; fd_1; d_2; : : : ; d_kg) by a graph G exists if and only if there exists an orthogonal fd_1; d_2; : : : ; d_kg-labelling of G.

For results on orthogonal double cover of circulant graphs, see [3, 4, 5]. In [2, 6, 7, 8], other results of ODCs by di¤erent graph classes can be found.

Cyclic Orthogonal Double Covers of Regular Circulant Graphs by Complete Bipartite and Tripartite Graphs

In this Section, by Theorem 9, CODCs of 4n regular circulant graphs by complete bipartite graphs are constructed, and by Theorem 10, CODCs of (3n 1) regular circulant graphs by complete tripartite graphs are constructed.

Theorem 9 For all positive integers n > 1, there exists a CODC of 4n regular

Circ (4n + 2; f1; 2; : : : ; 2ng) by $K_{2;2n}$:

Proof. Let us de…ne : V ($K_{2;2n}$) ! Z_{4n+2} by (v_0) = 0; (v_1) = 2n + 1; (v_j) = 2n + j; where 2 j n + 1; and (v_j) = 4n + 1 + j; where n + 2 j 2n + 1: Then the edges of lengths l; where 1 l n; are ((v_1); (v_{l+1})); and ((v_1); (v_{n+l+1})); and the edges of lengths l; where n + 1 l 2n are ((v_0); ($v_{l\ n+1}$)); and ((v_0); (v_{l+1})); then for l 2 f1; 2; : : : ; 2ng; $K_{2;2n}$

contains exactly two edges of length l; and since every two edges of the same length are adjacent then fr(l) : l 2 f1; 2; : : : ; 2ngg = f1; 2; : : : ; 2ng; and hence $K_{2;2n}$ has an orthogonal labelling.

As a direct construction for Theorem 9, let n = 2; then there exists a CODC of 8 regular Circ (10; f1; 2; 3; 4g) by $K_{2;4}$; as shown in Figure 1.

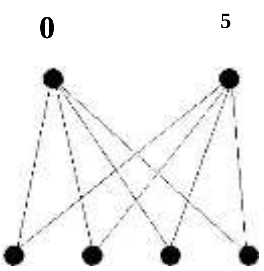

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Figure 1: CODC generating graph of 8 regular Circ (10; f1; 2; 3; 4g) by $K_{2;4}$:

Theorem 10 For any positive integer n 5, there exists a CODC of (3n

1) regular Circ (3n; f1; 2; : : : ; $^3{}_2{}^n$ g) by $K_{1;2;n\ 1}$:

Proof. Firstly, $K_{m;n;k}$ refers to the complete tripartite graph with partition sets of sizes m; n and k; where any vertex in a certain partition set is adjacent to the all vertices of the other two partition sets:

Secondly, let us de…ne : V ($K_{1;2;n\ 1}$) ! Z_{3n} by (v_0) = 0; (v_1) = 2; (v_2) = 4; and (v_j) = 3j 6; where 3 j n + 1: Then from the edges of

E($K_{1;2;n\ 1}$) = ff((v_0); (v_1))g [f((v_0); (v_2))g [f((v_0); (v_j)) : 3 j n + 1g [f((v_1); (v_j)) : 3 j n + 1g [f((v_2); (v_j)) : 3 j n + 1gg:

Case 1. For n = 2m + 1; m 2.

the edges of length 1 are ((v_1); (v_3)) and ((v_2); (v_3)); the edges of length 2 are ((v_0); (v_1)) and ((v_2); (v_4)); the edges of length 4 are ((v_0); (v_2)) and ((v_1); (v_4)); the edges of length 3i where 1 i m are ((v_0); (v_{i+2})) and ((v_0); ($v_{2m+3\ i}$)); the edges of length 3i + 2 where 1 i m 1 are ((v_2); (v_{i+4})) and ((v_1); ($v_{2m+3\ i}$)); the edges of length 3i + 4 where

1 i m $_{3n}$1 are ((v_1); (v_{i+4})) and ((v_2); ($v_{2m+3\ i}$)); and hence for l 2 f1; 2; : : : ; $_2$ g; $K_{1;2;n\ 1}$ contains exactly two edges of length f1; 2; : : : ; $\frac{3n}{2}$ g; and fr(l) : f 1; 2; : : : ; $\frac{3n}{2}$ g = f 1; 2; : : : ; $\frac{3n}{2}$ g ; and hence K 1;2;n 1 has an orthogonal labelling.

Case 2. For n = 2m; m 3.

the edge of length 3m is ((v_0); (v_{m+2})); the edges of length 1 are ((v_1); (v_3)) and ((v_2); (v_3)); the edges of length 2 are ((v_0); (v_1)) and ((v_2); (v_4)); the edges of length 4 are ((v_0); (v_2)) and ((v_1); (v_4)); the edges of length 3i where 1 i m 1 are ((v_0); (v_{i+2})) and ((v_0); ($v_{2m+2\ i}$)); the edges of length 3i+2 where 1 i m 1 are ((v_2); (v_{i+4})) and ((v_1); ($v_{2m+2\ i}$));

the edges of length 3i + 4 where 1 i m $_{3n}$ 2 are ((v_1); (v_{i+4})) and ((v_2); ($v_{2m+2\ i}$)); and hence for l $_3{}^2{}_n$ f1; 2; : : : ; 2 1g; $K_{1;2;n\ 1}$ contains exactly two edges of length f 1; 2; : : : ; $\frac{3n}{2}$ 1 g ; and f r(l) : f 1; 2; : : : ; $\frac{3n}{2}$ 1 gg = f1; 2; : : : ; $\frac{3n}{2}$ 1g; and hence $K_{1;2;n\ 1}$ has an orthogonal labelling.

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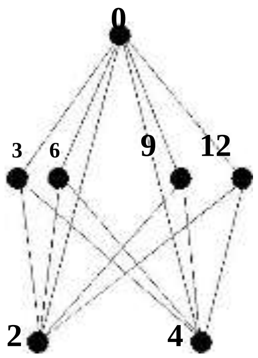

Figure 2: CODC generating graph of 14 regular Circ (15; f1; 2; 3; 4; 5; 6; 7g) by $K_{1;2;4}$

As a direct construction for Theorem 10, let n = 5; then there exists a CODC of 14 regular Circ (15; f1; 2; 3; 4; 5; 6; 7g) by $K_{1;2;4}$; as shown in Figure 2.

Cyclic Orthogonal Double Covers of Regular Circulant Graphs by H^n

In this Section, by Theorem 11, Corollary 12, and Corollary 13, CODCs of (2n 1) regular circulant graphs, (2n 3) regular circulant graphs and (2n 5) regular circulant graphs are constructed respectively.

For this section, let H^n (see Figure 3) be a graph with the edge set: $E(H^n)$ = f((v_1); (v_3)); ((v_0); (v_2)); ((v_3); (v_0)); ((v_3); (v_2));

((v_0); (v_4)); ((v_5); (v_4)); ((v_0); (v_5)); ((v_6); (v_5)); ((v_6); (v_2))g[

f ((v_6); (v_{+4})) : 3 n 3g [f ((v_6); (v_1)) : n +3 2n 3g; where : V (H^n) ! Z_{2n} is de…ned by (v_0) = 0; (v_1) = 1;

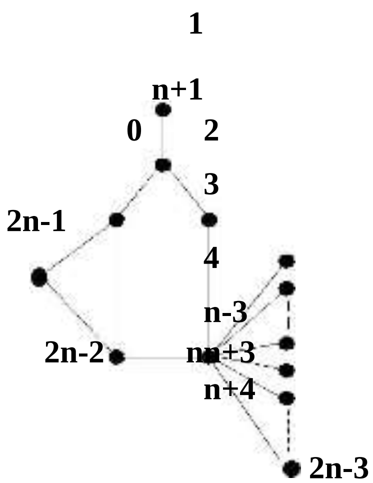

Figure 3: CODC generating graph of (2n 1) regular Circ (2n; f1; 2; : : : ; ng) by H^n

(v_2) = 2; (v_3) = n +1; (v_4) = 2n 1; (v_5) = 2n 2; (v_6) = n; (v $_{+4}$) =

; where 3 n 3; (v $_1$) = ; where n + 3 2n 3:

Theorem 11 For all positive integers n 6, there exists a CODC of (2n

1) regular Circ (2n; f1; 2; : : : ; ng) by H^n:

Proof. From the edges of H^n; the edge of length n is ((v_1); (v_3)); the edges of length 1 are ((v_0); (v_4)) and ((v_5); (v_4)); the edges of length 2 are ((v_0); (v_2)) and ((v_0); (v_5)); the edges of length l where 3 l n 3 are ((v_6); ($v_{n\ l+4}$)) and ((v_6); ($v_{l+n\ 1}$)); the edges of length n 2 are ((v_6); (v_5)) and ((v_6); (v_2)); the edges of length n 1 are ((v_3); (v_0)) and ((v_3); (v_2)), then for every l 2 f1; 2; : : : ; ng; H^n contains exactly two edges of length f1; 2; : : : ; n 1g; and exactly one edge of length n; and fr(l_1) : l_1 2 f1; 2; : : : ; n 1gg = f1; 2; : : : ; n 1g; and hence H^n has an orthogonal labelling.

As a direct construction for Theorem 11, let n = 8; then there exists a CODC of 15 regular Circ (16; f1; 2; 3; 4; 5; 6; 7; 8g) by H^8; as shown in Figure 4.

Corollary 12 For all positive integers n 6, there exists a CODC of (2n

3) regular Circ (2n; ff1; 2; : : : ; ngnfn 2gg) by H^nnf((v_6); (v_5)); ((v_6); (v_2)

Proof. The result follows from Theorem mapping : V (H^nnf((v_6); (v_5))) ! H^nnf((v_6); (v_5));

((v_6); (v_2))g.

11, and the fact that the 1 1 Z_{2n} is an orthogonal labelling of

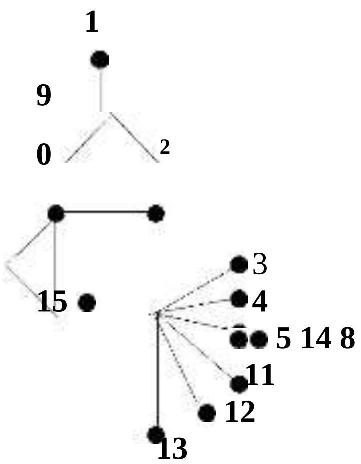

Figure 4: CODC generating graph of 13 regular Circ (16; f1; 2; 3; 4; 5; 7; 8g) by H^8:

As a direct construction for Corollary 12, let n = 8; then there exists a

CODC of 13 regular Circ (16; f1; 2; 3; 4; 5; 7; 8g) by H^8nf((v_6); (v_5)); ((v_6); (v_2))g; as shown in Figure 4.

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Corollary 13 For all positive integers n 6, there exists a CODC of (2n

5) regular Circ (2n; ff1; 3; 4; : : : ; ngnfn 2gg) by C_3(1; 0; 2) [$K_{1;2n\ 10}$ = H^nn f((v_6); (v_5)); ((v_6); (v_2)); ((v_0); (v_2)); ((v_0); (v_5))g:

Proof. The result follows from Theorem 11, and the fact that the 1 1 mapping

: V (C_3(1; 0; 2)[$K_{1;2n\ 10}$) = H^nnf((v_6); (v_5)); ((v_6); (v_2)); ((v_0); (v_2));

((v_0); (v_5))g) ! Z_{2n} is an orthogonal labelling of C_3(1; 0; 2) [$K_{1;2n\ 10}$ =

H^nnf((v_6); (v_5)); ((v_6); (v_2)); ((v_0); (v_2)); ((v_0); (v_5))g.

As a direct construction for Corollary 13, let n = 8; then there exists a CODC of 11 regular Circ (16; f1; 3; 4; 5; 7; 8g) by C_3(1; 0; 2) [$K_{1;6}$; as shown in Figure 5.

Cyclic Orthogonal Double Covers of Regular Circulant Graphs by G^n

In this Section, by Theorem 14, and Corollary 15, CODCs of (2n 4) regular circulant graphs and (2n 6) regular circulant graphs are constructed respec-tively.

For this Section, let G^n(see Figure 6) be a graph with the edge set E(G^n) = f((v_2); (v_0)); ((v_2); (v_1)); ((v_0); (v_4)); ((v_0); (v_1));

((v_0); (v_3)); ((v_4); (v_3))g [f ((v_5); (v_{+3})) : 3 n 3g [f

((v_5); ($v_{\ 2}$)) : n + 3 2n 3g; where : V (G^n) ! Z_{2n} is de…ned by

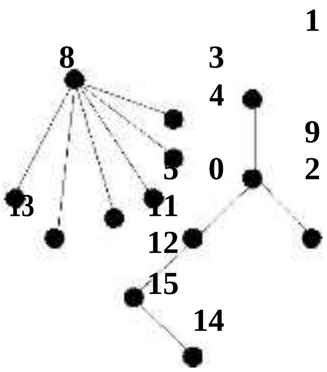

Figure 5: CODC generating graph of 11 regular Circ ^{11}(16; f1; 3; 4; 5; 7; 8g) by C_3(1; 0; 2) [$K_{1;6}$:

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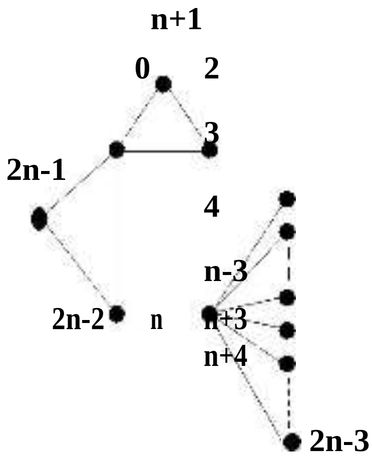

Figure 6: CODC generating graph of (2n 4) regular Circ(2n; ff1; 2; : : : ; n 1gnfn 2gg) by G^n:

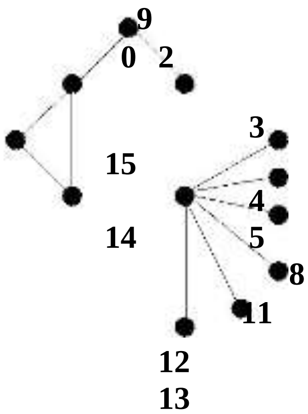

Figure 7: CODC generating graph of 12 regular Circ (16; 1; 2; 3; 4; 5;) by G^8: f8

8

3

4

9

5 0 2

13 11

12

15

14

11

Figure 8: CODC generating graph of 10 regular Circ (16; f1; 3; 4; 5; 7g) by $^{P}5^{[\ K}1;6$:
(v_0) = 0; (v_1) = 2; (v_2) = n + 1; (v_3) = 2n 1; (v_4) = 2n 2; (v_5) = n; (v_{+3}) = ; where 3 n 3; (v_2) =; where n + 3 2n 3:

Theorem 14 For all positive integers n 6, there exists a CODC of (2n 4) regular Circ (2n; ff1; 2; : : : ; n 1gnfn 2gg) by G^n:

Proof. From the edges of G^n; the edges of length 1 are ((v_0); (v_3)) and ((v_4); (v_3)); the edges of length 2 are ((v_0); (v_1)) and ((v_0); (v_4)); the edges of length l where 3 l n 3 are ((v_5); $(v_{n\ l+3})$) and ((v_5); $(v_{l+n\ 2})$); the edges of length n 1 are ((v_2); (v_0)) and ((v_2); (v_1)), then for every l 2 ff1; 2; : : : ; n 1gnfn 2gg; G^n contains exactly two edges of length ff1; 2; : : : ; n

1gnfn 2gg; and fr(l) : l 2 ff1; 2; : : : ; n 1gnfn 2ggg = ff1; 2; : : : ; n 1gnfn

2gg; and hence G^n has an orthogonal labelling.

As a direct construction for Theorem 14, let n = 8; then there exists a CODC of 12 regular Circ (16; f1; 2; 3; 4; 5; 7g) by G^8; as shown in Figure 7.

Corollary 15 For all positive integers n 6, there exists a CODC of (2n 6) regular Circ (2n; ff1; 3; 4; : : : ; n 1gnfn 2gg) by P_5 [$K_{1;2n\ 10}$ = G^nn f((v_0); (v_1)); ((v_0); (v_4))g:

Proof. The result follows from Theorem 14, and the fact that the 1 1 mapping

: V (P_5 [$K_{1;2n\ 10}$ = G^nnf((v_0); (v_1)); ((v_0); (v_4))g) ! Z_{2n} is an orthogonal labelling of P_5 [$K_{1;2n\ 10}$ = G^nnf((v_0); (v_1)); ((v_0); (v_4))g.

As a direct construction for Corollary 15, let n = 8; then there exists a CODC of 10 regular Circ (16; f1; 3; 4; 5; 7g) by P_5 [$K_{1;6}$; as shown in Figure 8.\

CONCLUSION

In conclusion, in this paper we got the orthogonal double covers of circulant graphs by new graph classes as follows, in Section 2, by Theorem 9, CODCs of 4n regular circulant graphs by complete bipartite graphs are constructed, and by Theorem 10, CODCs of (3n 1) regular circulant graphs by complete tri-partite graphs are constructed. In Section 3, by Theorem 11, Corollary 12, and Corollary 13, CODCs of (2n 1) regular circulant graphs, (2n 3) regular circulant graphs and (2n 5) regular circulant graphs are constructed respectively; and in Section 4, by Theorem 14, and Corollary 15, CODCs of (2n 4) regular circulant graphs and (2n 6) regular circulant graphs are constructed respec-tively.

REFERENCES

R. El-Shanawany, “Orthogonal double covers of complete bipartite graphs”, [Ph.D. Thesis], Universita•t Rostock (2002).

H.-D.O.F. Gronau, R.C. Mullin, A. Rosa, “Orthogonal double covers of complete graphs by trees”, Graphs Combin. Vol. 13, pp:251-262, 1997.

R. Sampathkumar, S .Srinivasan, “Cyclic orthogonal double covers of 4-regular circulant graphs”, Discrete Mathematics. 2011 ;311 2417-2422.

M. Higazy, “On Cyclic Orthogonal Double Covers of Circulant Graphs us-ing In…nite Graph

ISSN 2053-2229 (Print), ISSN 2053-2210 (Online)

Classes", British Journal of Mathematics & Computer Science. 3(3): 425-436, 2013.
R. El Shanawany, and H. Shabana, "On Orthogonal Double Covers of Cir-culant Graphs", British Journal of Mathematics & Computer Science. 4(3): 394-401, 2014.
H.-D.O.F. Gronau, S. Hartmann, M. Grüttmüller, U. Leck, and V. Leck,"On orthogonal double covers of graphs", Des. Codes Cryptogr.27,49-91(2002).
R. Scapellato, R. El Shanawany, and M. Higazy, "Orthogonal double covers of Cayley graphs", Discrete Appl. Math. 157, 3111–3118 (2009).
R. El Shanawany, M. Higazy, and A. El Mesady, "On Cartesian Products of Orthogonal Double Covers", International Journal of Mathematics and Mathematical Sciences, Vol. 2013, 4 Pages, 2013. doi:1155/2013/265136.

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