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ON COMPUTATIONAL STRUCTURE AND DISPOSITION OF SOLUTION MATRICES OF SINGLE–DELAY LINEAR NEUTRAL SCALAR DIFFERENTIAL EQUATIONS

Ukwu Chukwunenye
Department of Mathematics, University of Jos, P.M.B 2084, Jos, Postal Code: 930001, Plateau State, Nigeria.

ABSTRACT: *This research article established the global computational structure of solution matrices for single–delay autonomous linear neutral equations. The development of the solution matrices exploited the continuity of these matrices for positive time periods, the method of steps, change of variables and theory of linear difference equations to obtain these matrices on successive intervals of length equal to the delay h.*

KEYWORDS: Computational, Equations, Matrices, Neutral, Solution, Structure.

INTRODUCTION

Solution matrices are integral components of variation of constants formulas in the computations of solutions of linear and perturbed linear functional differential equations. But quite curiously, no other author has made any serious attempt to investigate the existence or otherwise of their general expressions for various classes of these equations. Effort has usually focused on the single – delay model and the approach has been to start from the interval $[0, h]$, compute the solution matrices and solutions for given problem instances and then use the method of steps to extend these to the intervals $[kh, (k+1)h]$, for positive integral k, not exceeding 2, for the most part. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k. In other words such approach fails to address the issue of the structure of solution matrices and solutions quite vital for real-world applications.

THEORETICAL UNDERPINNING

It is a herculean task to compute the solution matrices of linear autonomous delay differential equations from the governing equations, from one interval to the next successive interval of size equal to the delay. Worse still, it is impossible to obtain these matrices on non-contiguous intervals using the afore-mentioned equations. This is a severe constraint imposed by method of steps. This can only be mitigated through the construction of optimal expressions for such matrices, which can be used to obtain such matrices on arbitrary intervals. Therefore, the need for this article is imperative. With a view to addressing such short-comings, Ukwu and Garba (2014w) blazed the trail by investigating the structure of the solution matrices of the class of double – delay scalar differential equations:

$$\dot{x}(t) = ax(t) + bx(t-h) + cx(t-2h),\ t \in \mathbf{R},$$

where a, b and c are arbitrary real constants.

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METHODOLOGY

By deploying ingenious combinations of summation notations, multinomial distribution, greatest integer functions, change of variables techniques, multiple integrals, as well as the method of steps, Ukwu and Garba (2014) derived the following optimal expressions for the solution matrices:

$$Y(t)=\begin{cases} e^{at},\ t\in J_0; \\ e^{at}+\sum_{i=1}^{k} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} + \sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]}\sum_{i=0}^{k-2j} \frac{b^i c^j}{i!\,j!}\left(t-[i+2j]h\right)^{i+j} e^{a(t-[i+2j]h))};\ t\in J_k, k\ge 1 \end{cases}$$

This article makes a positive contribution to knowledge by establishing the expressions for the solution matrices of linear neutral equations on $(-\infty,\ 5h]$; in the sequel the article derived the optimal computational structure of such matrices on the interval $(-\infty,\infty)$, laying the issue to rest once and for all.

RESULTS

Observe that the above piece-wise expressions for $Y(t)$ may be restated more compactly in the form

$$Y(t)=\sum_{j=0}^{\left[\left[\frac{k}{2}\right]\right]}\sum_{i=0}^{k-2j} d_{ij}\left(t-[i+2j]h\right)^{i+j} e^{a(t-[i+2j]h))}\,\mathrm{sgn}\left(\max\{k+1,0\}\right);\ t\in J_k, k\ge 0$$

$$\text{where } d_{ij}=\frac{b^i c^j}{i!\,j!},\ j\in\left\{0,\cdots,\left[\left[\frac{k}{2}\right]\right]\right\},\ i\in\{0,\cdots,k-2j\}$$

Let $Y_{k-i}(t-ih)$ be a solution matrix of

$$\dot{x}(t)=a_{-1}\dot{x}(t-h)+a_0x(t)+a_1x(t-h), \tag{1}$$

on the interval $J_{k-i}=[(k-i)h,(k+1-i)h], k\in\{0,1,\cdots\}, i\in\{0,1,\ 2\}$, where

$$Y(t)=\begin{cases}1, t=0\\ 0, t<0\end{cases} \tag{2}$$

Note that $Y(t)$ is a generic solution matrix for any $t\in\mathbf{R}$.

The coefficients $a_{-1}, a_0,\ a_1$ and the associated functions are all from the real domain.

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Theorem on $Y(t)$

For $t \in J_k, k \in \{0,1,2,3,4,5\}$,

$$Y(t) = e^{a_0 t} + \sum_{i=1}^{k} \frac{(a_{-1}a_0 + a_1)^i}{i!} (t-ih)^i e^{a_0(t-ih)} \operatorname{sgn}(\max\{0,k\})$$

$$+ \sum_{i=1}^{k-1} a_{-1}^i (a_{-1}a_0 + a_1)(t-[i+1]h) e^{a_0(t-[i+1]h)} \operatorname{sgn}(\max\{0,k-1\})$$

$$+ a_{-1}(a_{-1}a_0 + a_1)^2 (t-3h)^2 e^{a_0(t-3h)} \operatorname{sgn}(\max\{0,k-2\})$$

$$+ \left[a_{-1}(a_{-1}a_0 + a_1)^3 \frac{(t-4h)^3}{3} + \frac{3}{2} a_{-1}^2 (a_{-1}a_0 + a_1)^2 (t-4h)^2 \right] e^{a_0(t-4h)} \operatorname{sgn}(\max\{0,k-3\})$$

$$+ \left[\begin{aligned} & a_{-1}(a_{-1}a_0 + a_1)^4 \frac{(t-5h)^4}{8} \\ & + \frac{2}{3} a_{-1}^2 (a_{-1}a_0 + a_1)^3 (t-5h)^2 + 2a_{-1}^3 (a_{-1}a_0 + a_1)^2 (t-5h)^2 \end{aligned} \right] e^{a_0(t-5h)} \operatorname{sgn}(\max\{0,k-4\})$$

Proof

On $(0,h) \subset J_0, Y(t-h) = 0 \Rightarrow \dot{Y}(t) = a_0 Y(t)$ a.e. on $[0,h] \Rightarrow Y(t) \equiv Y_1(t) = e^{a_0 t} C; Y(0) = 1 \Rightarrow C = 1$

$$\Rightarrow Y(t) = e^{a_0 t} \text{ on } J_0 = [0,h], \text{ as in the single - and double - delay systems.}$$

Consider the interval $(h, 2h)$. Then

$$t-h \in [0,h] \Rightarrow Y(t-h) = e^{a_0(t-h)} \Rightarrow \dot{Y}(t-h) = A_0 e^{a_0(t-h)} \Rightarrow \dot{Y}(t) - a_0 Y(t) = (a_{-1}a_0 + a_1) e^{a_0(t-h)}$$

$$\Rightarrow \frac{d}{dt}\left[e^{-a_0 t} Y(t)\right] = e^{-a_0 t}\left[\dot{Y}(t) - a_0 Y(t)\right] = (a_{-1}a_0 + a_1) e^{a_0(t-h)}$$

$$\Rightarrow e^{-a_0 t} Y(t) - e^{-a_0 h} Y(h) = \int_h^t e^{-a_0 s} (a_{-1}a_0 + a_1) e^{a_0(s-h)} ds = \int_h^t e^{-a_0 h} (a_{-1}a_0 + a_1) ds$$

$$\Rightarrow Y(t) = e^{a_0(t-h)} Y(h) + (a_{-1}a_0 + a_1)(t-h) e^{-a_0(t-h)} = e^{a_0 t} + (a_{-1}a_0 + a_1)(t-h) e^{a_0(t-h)}, \text{ on } J_1.$$

$$t \in J_2 \Rightarrow t-h \in J_1 \Rightarrow Y(t-h) = e^{a_0(t-h)} + (a_{-1}a_0 + a_1)(t-2h) e^{a_0(t-2h)}, \text{ on } J_2.$$

On $J_k, k \geq 2$, the relation $\frac{d}{dt}\left[e^{-a_0 t} Y(t)\right] = e^{-a_0 t}\left[a_1 Y(t-h) + a_{-1}\dot{Y}(t-h)\right]$ a.e.

$$\Rightarrow Y(t) = e^{a_0(t-kh)} Y(kh) + \int_{kh}^t e^{a_0(t-s)} \left[a_1 Y(s-h) + a_{-1}\dot{Y}(s-h)\right] ds \tag{3}$$

$$t \in J_2 \Rightarrow t-h \in J_1, 2h \in J_1 \Rightarrow Y(t-h) = e^{a_0(t-h)} + (a_{-1}a_0 + a_1)(t-2h) e^{a_0(t-2h)}, \text{ on } J_2.$$

$$Y(2h) = e^{a_0 2h} + (a_{-1}a_0 + a_1) h e^{a_0 h}$$

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$$\Rightarrow Y(t)=e^{a_0(t-2h)}\left[e^{a_0 2h}+\left(a_{-1}a_0+a_1\right)he^{a_0h}\right]+\int_{2h}^{t}e^{a_0(t-s)}a_1\left[e^{a_0(s-h)}+\left(a_{-1}a_0+a_1\right)(s-2h)e^{a_0(s-2h)}\right]ds$$

$$+\int_{2h}^{t}e^{a_0(t-s)}\left[a_{-1}\frac{d}{ds}\left[e^{a_0(s-h)}+\left(a_{-1}a_0+a_1\right)(s-2h)e^{a_0(s-2h)}\right]\right]ds$$

$$\Rightarrow Y(t)=e^{a_0t}+\left(a_{-1}a_0+a_1\right)he^{a_0(t-h)}+(t-2h)a_1e^{a_0(t-h)}+a_1\int_{2h}^{t}\left(a_{-1}a_0+a_1\right)(s-2h)e^{a_0(t-2h)}ds$$

$$+a_{-1}a_0(t-2h)e^{a_0(t-h)}+a_{-1}\left(a_{-1}a_0+a_1\right)\left[(t-2h)+a_0\frac{\left(t-2h\right)^2}{2}\right]e^{a_0(t-2h)}$$

$$\Rightarrow Y(t)=e^{a_0t}+\left[a_{-1}a_0+a_1\right](t-h)e^{a_0(t-h)}+\frac{\left(a_{-1}a_0+a_1\right)^2}{2}\left(t-2h\right)^2e^{a_0(t-2h)}$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)e^{a_0(t-2h)}\left(t-2h\right);\ t\in J_2.$$

Thus the theorem is true for $t\in\bigcup_{k=0}^{2}J_k$.

Now consider the interval J_3; $s,t\in J_3\Rightarrow 3h\in J_2\cap J_3=\{3h\}$ and $s-h\in J_2$; hence

$$Y(3h)=e^{3a_0h}+\left[a_{-1}a_0+a_1\right]2he^{2a_0h}+\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0h}+a_{-1}\left(a_{-1}a_0+a_1\right)he^{a_0h}$$

$$=e^{3a_0h}+2\left[a_{-1}a_0+a_1\right]he^{2a_0h}+\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0h}+a_{-1}\left(a_{-1}a_0+a_1\right)he^{a_0h}$$

$$\Rightarrow Y(s-h)=e^{a_0(s-h)}+\left[a_{-1}a_0+a_1\right](s-2h)e^{a_0(s-2h)}+\frac{\left(a_{-1}a_0+a_1\right)^2}{2}\left(s-3h\right)^2e^{a_0(s-3h)}$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)e^{a_0(s-3h)}\left(s-3h\right);\ s\in J_3.$$

From the relation (3), we obtain

$$Y(t)=e^{a_0t}+2\left[a_{-1}a_0+a_1\right]he^{a_0(t-h)}+\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0(t-2h)}+a_{-1}\left(a_{-1}a_0+a_1\right)he^{a_0(t-2h)}$$

$$+a_1\int_{3h}^{t}e^{a_0(t-s)}\begin{pmatrix}e^{a_0(s-h)}+\left[a_{-1}a_0+a_1\right](s-2h)e^{a_0(s-2h)}+\dfrac{\left(a_{-1}a_0+a_1\right)^2}{2}\left(s-3h\right)^2e^{a_0(s-3h)}\\+a_{-1}\left(a_{-1}a_0+a_1\right)\left(s-3h\right)e^{a_0(s-3h)}\end{pmatrix}ds$$

$$+a_{-1}\int_{3h}^{t}e^{a_0(t-s)}\frac{d}{ds}\begin{pmatrix}e^{a_0(s-h)}+\left[a_{-1}a_0+a_1\right](s-2h)e^{a_0(s-2h)}+\dfrac{\left(a_{-1}a_0+a_1\right)^2}{2}\left(s-3h\right)^2e^{a_0(s-3h)}\\+a_{-1}\left(a_{-1}a_0+a_1\right)\left(s-3h\right)e^{a_0(s-3h)}\end{pmatrix}ds$$

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$$\Rightarrow Y(t) = e^{a_0 t} + 2\left[a_{-1}a_0 + a_1\right] h e^{a_0(t-h)} + \left(a_{-1}a_0 + a_1\right)^2 \frac{h^2}{2} e^{a_0(t-2h)} + a_{-1}\left(a_{-1}a_0 + a_1\right) h e^{a_0(t-2h)}$$
$$+ a_1(t-3h)e^{a_0(t-h)} + a_1\left[a_{-1}a_0 + a_1\right]\frac{(t-2h)^2 e^{a_0(t-2h)}}{2} - a_1\left[a_{-1}a_0 + a_1\right]\frac{h^2 e^{a_0(t-2h)}}{2}$$
$$+ a_1 \frac{\left(a_{-1}a_0 + a_1\right)^2}{3!}\left(t-3h\right)^3 e^{a_0(t-3h)} + \frac{a_{-1}a_1\left(a_{-1}a_0 + a_1\right)(t-3h)^2}{2} e^{a_0(t-3h)}$$
$$+ a_{-1}a_0(t-3h)e^{a_0(t-h)} + a_{-1}\left[a_{-1}a_0 + a_1\right](t-3h)e^{a_0(t-2h)} + \frac{a_{-1}a_0\left[a_{-1}a_0 + a_1\right](t-2h)^2}{2} e^{a_0(t-2h)}$$
$$- \frac{a_{-1}a_0\left[a_{-1}a_0 + a_1\right]h^2}{2} e^{a_0(t-2h)} + \frac{a_{-1}\left(a_{-1}a_0 + a_1\right)^2}{2}\left[\left(t-3h\right)^2 + a_0 \frac{\left(t-3h\right)^3}{3}\right] e^{a_0(t-3h)}$$
$$+ a_{-1}^2\left(a_{-1}a_0 + a_1\right)\left[t - 3h + a_0 \frac{(t-3h)^2}{2!}\right] e^{a_0(t-3h)}$$

The evaluation of the integrals and skillful collection of like terms result in the following expression for $Y(t)$:

$$Y(t) = e^{a_0 t} + \left[a_{-1}a_0 + a_1\right](t-h)e^{a_0(t-h)} + \frac{\left(a_{-1}a_0 + a_1\right)^2}{2!}\left(t-2h\right)^2 e^{a_0(t-2h)}$$
$$+ \frac{\left(a_{-1}a_0 + a_1\right)^3}{3!}\left(t-3h\right)^3 e^{a_0(t-3h)} + a_{-1}\left(a_{-1}a_0 + a_1\right)\left(t-2h\right)e^{a_0(t-2h)} + a_{-1}^2\left(a_{-1}a_0 + a_1\right)(t-3h)e^{a_0(t-3h)}$$
$$+ a_{-1}\left(a_{-1}a_0 + a_1\right)^2\left(t-3h\right)^2 e^{a_0(t-3h)}$$

Observe that for $k \in \{0,1,2,3\}$ and $t \in J_k$,

$$Y(t) = e^{a_0 t} + \left(\sum_{i=1}^{k} \frac{\left(a_{-1}a_0 + a_1\right)^i}{i!}\left(t - ih\right)^i e^{a_0(t-ih)}\right)\operatorname{sgn}\left(\max\left\{0, k\right\}\right)$$
$$+ \sum_{i=1}^{k-1} a_{-1}^i\left(a_{-1}a_0 + a_1\right)(t - [i+1]h)e^{a_0(t-[i+1]h)}\operatorname{sgn}\left(\max\left\{0, k-1\right\}\right)$$
$$+ a_{-1}\left(a_{-1}a_0 + a_1\right)^2 (t-3h)^2 e^{a_0(t-3h)}\operatorname{sgn}\left(\max\left\{0, k-2\right\}\right) \qquad (4)$$

The process continues.

Consider the interval J_4; $s,t \in J_4 \Rightarrow 4h \in J_3 \cap J_4 = \{4h\}$ and $s - h \in J_3$; hence the relation (3) implies that

$$Y(t) = e^{a_0(t-4h)}\left[\begin{array}{l} e^{4a_0 h} + \sum_{i=1}^{3} \frac{\left(a_{-1}a_0 + a_1\right)^i}{i!}\left([4-i]h\right)^i e^{a_0[4-i]h} + \sum_{i=1}^{2} a_{-1}^i\left(a_{-1}a_0 + a_1\right)[3-i]h e^{a_0([3-i]h)} \\ \quad + a_{-1}\left(a_{-1}a_0 + a_1\right)^2 h^2 e^{a_0 h} \end{array}\right]$$

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$$+\int_{4h}^{t} a_1 e^{a_0(t-s_4)}\left[\begin{array}{l} e^{a_0(s_4-h)}+\left(\sum_{i=1}^{3}\frac{(a_{-1}a_0+a_1)^i}{i!}\left(s_4-[i+1]h\right)^i e^{a_0(s_4-[i+1]h)}\right)\\ +\sum_{i=1}^{2}a_{-1}^{i}(a_{-1}a_0+a_1)(s_4-[i+2]h)e^{a_0(s_4-[i+2]h)}+a_{-1}(a_{-1}a_0+a_1)^2(s_4-4h)^2e^{a_0(s_4-4h)}\end{array}\right]ds_4$$

$$+\int_{4h}^{t} a_{-1} e^{a_0(t-s_4)}\frac{d}{ds_4}\left[\begin{array}{l} e^{a_0(s_4-h)}+\left(\sum_{i=1}^{3}\frac{(a_{-1}a_0+a_1)^i}{i!}\left(s_4-[i+1]h\right)^i e^{a_0(s_4-[i+1]h)}\right)\\ +\sum_{i=1}^{2}a_{-1}^{i}(a_{-1}a_0+a_1)(s_4-[i+2]h)e^{a_0(s_4-[i+2]h)}\\ +a_{-1}(a_{-1}a_0+a_1)^2(s_4-4h)^2e^{a_0(s_4-4h)}\end{array}\right]ds_4$$

$$\Rightarrow Y(t)=e^{a_0t}+\sum_{i=1}^{3}\frac{(a_{-1}a_0+a_1)^i}{i!}\left([4-i]h\right)^i e^{a_0(t-ih)}+\sum_{i=1}^{2}a_{-1}^{i}(a_{-1}a_0+a_1)[3-i]he^{a_0(t-[i+1]h)}$$
$$+a_{-1}(a_{-1}a_0+a_1)^2h^2e^{a_0(t-3h)}$$

$$+a_1(t-4h)e^{a_0(t-h)}+\sum_{i=1}^{3}\frac{a_1(a_{-1}a_0+a_1)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1}e^{a_0(t-[i+1]h)}$$

$$-\sum_{i=1}^{3}\frac{a_1(a_{-1}a_0+a_1)^i}{(i+1)!}\left([3-i]h\right)^{i+1}e^{a_0(t-[i+1]h)}+\sum_{i=1}^{2}a_1a_{-1}^{i}(a_{-1}a_0+a_1)\frac{(t-[i+2]h)^2e^{a_0(t-[i+2]h)}}{2}$$

$$-\sum_{i=1}^{2}a_1a_{-1}^{i}(a_{-1}a_0+a_1)\frac{([2-i]h)^2e^{a_0(t-[i+2]h)}}{2}+a_1a_{-1}(a_{-1}a_0+a_1)^2\frac{(t-4h)^3}{3}e^{a_0(t-4h)}$$

$$+a_{-1}a_0(t-4h)e^{a_0(t-h)}+\sum_{i=1}^{3}\frac{a_{-1}(a_{-1}a_0+a_1)^i}{i!}\left(t-[i+1]h\right)^i e^{a_0(t-[i+1]h)}-\sum_{i=1}^{3}\frac{a_{-1}(a_{-1}a_0+a_1)^i}{i!}\left([3-i]h\right)^i e^{a_0(t-[i+1]h)}$$

$$+\sum_{i=1}^{3}a_{-1}a_0\frac{(a_{-1}a_0+a_1)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1}e^{a_0(t-[i+1]h)}-\sum_{i=1}^{3}a_{-1}a_0\frac{(a_{-1}a_0+a_1)^i}{(i+1)!}\left([3-i]h\right)^{i+1}e^{a_0(t-[i+1]h)}$$

$$+\sum_{i=1}^{2}a_{-1}^{i+1}(a_{-1}a_0+a_1)(t-4h)e^{a_0(t-[i+2]h)}+\sum_{i=1}^{2}a_{-1}^{i+1}a_0(a_{-1}a_0+a_1)\frac{\left(t-[i+2]h\right)^2}{2}e^{a_0(t-[i+2]h)}$$

$$-\sum_{i=1}^{2}a_{-1}^{i+1}a_0(a_{-1}a_0+a_1)\frac{\left([2-i]h\right)^2}{2}e^{a_0(t-[i+2]h)}+a_{-1}^{2}(a_{-1}a_0+a_1)^2(t-4h)^2e^{a_0(t-4h)}$$

$$+a_{-1}^{2}a_0(a_{-1}a_0+a_1)^2\frac{(t-4h)^3}{3}e^{a_0(t-4h)}$$

It is evident from change of variables and grouping techniques that

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$$e^{a_0 t}+\sum_{i=1}^{3}\frac{a_1\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1}e^{a_0\left(t-[i+1]h\right)}+\sum_{i=1}^{3}a_{-1}a_0\frac{\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1}e^{a_0\left(t-[i+1]h\right)}$$

$$+a_1\left(t-4h\right)e^{a_0(t-h)}+a_{-1}a_0\left(t-4h\right)e^{a_0(t-h)}+\sum_{i=1}^{3}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left([4-i]h\right)^i e^{a_0(t-ih)}$$

$$-\sum_{i=1}^{3}\frac{a_1\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}([3-i]h)^{i+1}e^{a_0(t-[i+1]h)}-\sum_{i=1}^{3}a_{-1}a_0\frac{\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left([3-i]h\right)^{i+1}e^{a_0\left(t-[i+1]h\right)}$$

$$=e^{a_0 t}+\sum_{i=1}^{4}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left(t-ih\right)^i e^{a_0\left(t-ih\right)}; \qquad (5)$$

$$\sum_{i=1}^{2}a_1a_{-1}^i\left(a_{-1}a_0+a_1\right)\frac{(t-[i+2]h)^2e^{a_0\left(t-[i+2]h\right)}}{2}+\sum_{i=1}^{2}a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)\frac{\left(t-[i+2]h\right)^2}{2}e^{a_0\left(t-[i+2]h\right)}$$

$$=a_{-1}\left(a_{-1}a_0+a_1\right)^2\frac{\left(t-3h\right)^2}{2}e^{a_0(t-3h)}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^2\frac{\left(t-4h\right)^2}{2}e^{a_0(t-4h)} \qquad (6)$$

$$=\sum_{i=1}^{2}a_{-1}^i\left(a_{-1}a_0+a_1\right)^2\frac{(t-[i+2]h)^2e^{a_0\left(t-[i+2]h\right)}}{2};$$

Also,

$$a_{-1}\left(a_{-1}a_0+a_1\right)^2h^2e^{a_0(t-3h)}-\sum_{i=1}^{2}a_1a_{-1}^i\left(a_{-1}a_0+a_1\right)\frac{([2-i]h)^2e^{a_0\left(t-[i+2]h\right)}}{2}$$

$$-\sum_{i=1}^{2}a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)\frac{\left([2-i]h\right)^2}{2}e^{a_0\left(t-[i+2]h\right)}=a_{-1}\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0(t-3h)} \qquad (7)$$

Furthermore,

$$\sum_{i=1}^{2}a_{-1}^i\left(a_{-1}a_0+a_1\right)[3-i]he^{a_0\left(t-[i+1]h\right)}+a_1a_{-1}\left(a_{-1}a_0+a_1\right)^2\frac{(t-4h)^3}{3}e^{a_0(t-4h)}$$

$$+\sum_{i=1}^{2}a_{-1}^{i+1}\left(a_{-1}a_0+a_1\right)(t-4h)e^{a_0\left(t-[i+2]h\right)}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^2\left(t-4h\right)^2e^{a_0(t-4h)}$$

$$+a_{-1}^2a_0\left(a_{-1}a_0+a_1\right)^2\frac{\left(t-4h\right)^3}{3}e^{a_0(t-4h)}+\sum_{i=1}^{3}\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^i}{i!}\left(t-[i+1]h\right)^i e^{a_0\left(t-[i+1]h\right)}$$

$$-\sum_{i=1}^{3}\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^i}{i!}\left([3-i]h\right)^i e^{a_0\left(t-[i+1]h\right)}$$

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$$= \sum_{i=1}^{4-1} a_{-1}^{i}\left(a_{-1}a_{0}+a_{1}\right)\left(t-[i+1]h\right)e^{a_0\left(t-[i+1]h\right)}+\frac{1}{2}a_{-1}\left(a_{-1}a_{0}+a_{1}\right)^{2}\left(t-3h\right)^{2}e^{a_0(t-3h)}$$

$$+a_{-1}^{2}\left(a_{-1}a_{0}+a_{1}\right)^{2}\left(t-4h\right)^{2}e^{a_0(t-4h)}+a_{-1}\left(a_{-1}a_{0}+a_{1}\right)^{3}\frac{(t-4h)^{3}}{2}e^{a_0(t-4h)}$$

$$-a_{-1}\left(a_{-1}a_{0}+a_{1}\right)^{2}\frac{h^{2}}{2}e^{a_0(t-3h)} \quad (8)$$

Adding up expressions (5), (6), (7) and (8) yields

$$Y(t)=e^{a_0 t}+\left(\sum_{i=1}^{4}\frac{\left(a_{-1}a_{0}+a_{1}\right)^{i}}{i!}\left(t-ih\right)^{i}e^{a_0(t-ih)}\right)$$

$$+\sum_{i=1}^{4-1} a_{-1}^{i}\left(a_{-1}a_{0}+a_{1}\right)(t-[i+1]h)e^{a_0\left(t-[i+1]h\right)}+a_{-1}\left(a_{-1}a_{0}+a_{1}\right)^{2}(t-3h)^{2}e^{a_0(t-3h)}$$

$$+\left[a_{-1}\left(a_{-1}a_{0}+a_{1}\right)^{3}\frac{(t-4h)^{3}}{2}+\frac{3}{2}a_{-1}^{2}\left(a_{-1}a_{0}+a_{1}\right)^{2}(t-4h)^{2}\right]e^{a_0(t-4h)} \quad (9)$$

Hence for $t\in J_k, k\in\{0, 1, 2, 3, 4\}$,

$$Y(t)=e^{a_0 t}+\left(\sum_{i=1}^{k}\frac{\left(a_{-1}a_{0}+a_{1}\right)^{i}}{i!}\left(t-ih\right)^{i}e^{a_0(t-ih)}\right)\operatorname{sgn}\left(\max\{0,k\}\right)$$

$$+\sum_{i=1}^{k-1} a_{-1}^{i}\left(a_{-1}a_{0}+a_{1}\right)(t-[i+1]h)e^{a_0\left(t-[i+1]h\right)}\operatorname{sgn}\left(\max\{0,k-1\}\right)$$

$$+a_{-1}\left(a_{-1}a_{0}+a_{1}\right)^{2}(t-3h)^{2}e^{a_0(t-3h)}\operatorname{sgn}\left(\max\{0,k-2\}\right)$$

$$+\left[a_{-1}\left(a_{-1}a_{0}+a_{1}\right)^{3}\frac{(t-4h)^{3}}{2}+\frac{3}{2}a_{-1}^{2}\left(a_{-1}a_{0}+a_{1}\right)^{2}(t-4h)^{2}\right]e^{a_0(t-4h)}\operatorname{sgn}\left(\max\{0,k-3\}\right) \quad (10)$$

Observe that $k\in\{0,1,\cdots,4\}, t\in J_k \Rightarrow$

$$Y(t)=e^{a_0 t}\max\{k+1,0\}+\left(\sum_{i=0}^{k-1}\sum_{j=1}^{k-i}c_{ij}a_{-1}^{i}\left(a_{-1}a_{0}+a_{1}\right)^{j}\left(t-[i+j]h\right)^{j}e^{a_0\left(t-[i+j]h\right)}\right)\operatorname{sgn}\left(\max\{0,k\}\right)$$

where

$$c_{0j}=\frac{1}{j!}, j\in\{0,\cdots,k\}; c_{i1}=1, i\in\{1,\cdots,k-1\}; c_{1j}=\frac{1}{(j-1)!}, j\in\{1,\cdots,k-1\}; c_{22}=\frac{3}{2}, k=4$$

It is also instructive to recognize that $\frac{1}{2}c_{21}+c_{12}=\frac{1}{2}+1=\frac{3}{2}=c_{22}$. Further involved investigation suggests the emerging relationship $c_{ij}=\frac{1}{j}c_{i\,j-1}+c_{i-1\,j}$. This pattern will be validated by an inductive proof.

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Alternatively, $k \in \{0,1,\cdots,4\}$, $t \in J_k \Rightarrow$

$$Y(t) = e^{a_0 t} \max\{k+1,0\} + \sum_{j=1}^{k} \frac{(a_{-1}a_0 + a_1)^j}{j!}(t - jh)^j e^{a_0(t-jh)} \max\{k,0\}$$
$$+ \sum_{j=1}^{k-1} a_{-1}^i (a_{-1}a_0 + a_1)(t - [i+1]h) e^{a_0(t-[i+1]h)} \max\{k-1,0\}$$
$$+ \left(\sum_{i=1}^{k-2} \sum_{j=2}^{k-i} c_{ij} a_{-1}^i (a_{-1}a_0 + a_1)^j (t - [i+j]h)^j e^{a_0(t-[i+j]h)} \right) \mathrm{sgn}(\max\{0, k-2\}) \quad (11)$$

where

$$c_{1j} = \frac{1}{(j-1)!}, \, j \in \{1,\cdots,k-1\}; c_{22} = \frac{3}{2}, \text{for } k = 4.$$

It is also instructive to recognize that $\frac{1}{2}c_{21} + c_{12} = \frac{1}{2} + 1 = \frac{3}{2} = c_{22}$. Further investigation-quite involved-suggests the emerging relationship: $c_{ij} = \frac{1}{j} c_{i\,j-1} + c_{i-1\,j}, i \in \{1,2,\cdots,k-2\}, j \in \{1,2,\cdots,k-i\}$.

This pattern will be validated using an inductive proof.

The next result gives the optimal computational structure of the solution matrices.

Theorem On The Optimal Computational Structure of the Solution Matrices

Let $t \in J_k$, $i, j \in \{1,2,\cdots,k\}$ and let $c_{0j} = \frac{1}{j!}, c_{i1} = 1$. Suppose that $a_{-1}(a_{-1}a_0 + a_1) \neq 0$. Then

$$Y(t) = e^{a_0 t} + \left(\sum_{j=1}^{k} \frac{(a_{-1}a_0 + a_1)^j}{j!}(t - jh)^j e^{a_0(t-jh)} \right) \mathrm{sgn}(\max\{0, k\})$$
$$+ \sum_{i=1}^{k-1} a_{-1}^i (a_{-1}a_0 + a_1)(t - [i+1]h) e^{a_0(t-[i+1]h)} \mathrm{sgn}(\max\{0, k-1\})$$
$$+ \sum_{i=1}^{k-2} \sum_{j=2}^{k-i} c_{ij} a_{-1}^i (a_{-1}a_0 + a_1)^j (t - [i+j]h)^j e^{a_0(t-[i+j]h)} \mathrm{sgn}(\max\{0, k-2\}) \quad (12)$$

for some real positive constants c_{ij} such that $c_{1j} = \frac{1}{(j-1)!}$, $c_{i2} = \frac{1}{2}(i+1)$, $i \in \{1,2,\cdots k-2\}$

and $c_{ij} - \frac{1}{j} c_{i\,j-1} - c_{i-1\,j} = 0, \forall i \in \{2,3,\cdots k-2\}, j \in \{2,3,\cdots k-i\}$. There are exactly $\max\left\{\frac{1}{2}(k-3)(k-2)\mathrm{sgn}(k-1), 0\right\}$ such unknown c_{ij} values to be determined quite easily; this number is pruned to a mere $\max\{k-2, 0\}$, in any transition from $Y_k(t)$ to $Y_{k+1}(t)$.

Proof

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It is clear from theorem 2.1 that the theorem is valid for $1 \le i \le 3,\ 2 \le j \le 4-i$. Assume the validity of the theorem for all $1 \le i \le k-2,\ 2 \le j \le k-i,$ for some integer $k \ge 5$. Then for $t \in J_{k+1}$, the relation (3) implies that

$$
\begin{aligned}
Y(t) &= e^{a_0 t} + \sum_{i=1}^{k} \frac{(a_{-1}a_0 + a_1)^i}{i!}\left([k+1-i]h\right)^i e^{a_0(t-ih)} + \sum_{i=1}^{k-1} a_{-1}^i (a_{-1}a_0 + a_1)([k-i]h) e^{a_0(t-[i+1]h)} \\
&+ \sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^i (a_{-1}a_0 + a_1)^j \left([k+1-i-j]h\right)^j e^{a_0(t-[i+j]h)} \\
&+ a_1\left(t-[k+1]h\right) e^{a_0(t-h)} + \sum_{i=1}^{k} a_1 \frac{(a_{-1}a_0 + a_1)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1} e^{a_0(t-[i+1]h)} \\
&- \sum_{i=1}^{k} a_1 \frac{(a_{-1}a_0 + a_1)^i}{(i+1)!}\left([k-i]h\right)^i e^{a_0(t-[i+1]h)} + \sum_{i=1}^{k-1} a_1 a_{-1}^i (a_{-1}a_0 + a_1) \frac{(t-[i+2]h)^2 e^{a_0(t-[i+2]h)}}{2} \\
&- \sum_{i=1}^{k-1} a_1 a_{-1}^i (a_{-1}a_0 + a_1) \frac{([k-i-1]h)^2 e^{a_0(t-[i+2]h)}}{2} \\
&+ \sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_1 a_{-1}^i (a_{-1}a_0 + a_1)^j \frac{\left(t-[i+j+1]h\right)^{j+1} e^{a_0(t-[i+j+1]h)}}{j+1} \\
&- \sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_1 a_{-1}^i (a_{-1}a_0 + a_1)^j \frac{\left([k-i-j]h\right)^{j+1} e^{a_0(t-[i+j+1]h)}}{j+1} \\
&+ a_{-1}a_0\left(t-[k+1]h\right) e^{a_0(t-h)} + \sum_{i=1}^{k} \frac{a_{-1}(a_{-1}a_0 + a_1)^i}{i!}\left(t-[i+1]h\right)^i e^{a_0(t-[i+1]h)} \\
&- \sum_{i=1}^{k} \frac{a_{-1}(a_{-1}a_0 + a_1)^i}{i!}\left([k-i]h\right)^i e^{a_0(t-[i+1]h)} + \sum_{i=1}^{k} \frac{a_{-1}a_0(a_{-1}a_0 + a_1)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1} e^{a_0(t-[i+1]h)} \\
&- \sum_{i=1}^{k} \frac{a_{-1}a_0(a_{-1}a_0 + a_1)^i}{(i+1)!}\left([k-i]h\right)^{i+1} e^{a_0(t-[i+1]h)} + \sum_{i=1}^{k-1} a_{-1}^{i+1}(a_{-1}a_0 + a_1)(t-[i+2]h) e^{a_0(t-[i+2]h)} \\
&- \sum_{i=1}^{k-1} a_{-1}^{i+1}(a_{-1}a_0 + a_1)([k-i-1]h) e^{a_0(t-[i+2]h)} + \sum_{i=1}^{k-1} a_{-1}^{i+1} a_0 (a_{-1}a_0 + a_1) \frac{(t-[i+2]h)^2}{2} e^{a_0(t-[i+2]h)} \\
&- \sum_{i=1}^{k-1} a_{-1}^{i+1} a_0 (a_{-1}a_0 + a_1) \frac{([k-i-1]h)^2}{2} e^{a_0(t-[i+2]h)} \\
&+ \sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i+1} (a_{-1}a_0 + a_1)^j (t-[i+j+1]h)^j e^{a_0(t-[i+j+1]h)}
\end{aligned}
$$

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$$-\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i+1}\left(a_{-1}a_0+a_1\right)^j ([k-i-j]h)^j e^{a_0(t-[i+j+1]h)}$$

$$+\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)^j \frac{(t-[i+j+1]h)^{j+1}e^{a_0(t-[i+j+1]h)}}{j+1}$$

$$-\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)^j \frac{([k-i-j]h)^{j+1}e^{a_0(t-[i+j+1]h)}}{j+1}$$

Use of change of variables, as well as appropriate groupings yields the following relations:

$$\sum_{i=1}^{k}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left(\left[k+1-i\right]h\right)^i e^{a_0(t-ih)} + \sum_{i=1}^{k} a_1\frac{\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(t-\left[i+1\right]h\right)^{i+1} e^{a_0(t-[i+1]h)}$$

$$+a_1\left(t-\left[k+1\right]h\right)e^{a_0(t-h)} - \sum_{i=1}^{k} a_1\frac{\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(\left[k-i\right]h\right)^i e^{a_0(t-[i+1]h)} + a_{-1}a_0\left(t-\left[k+1\right]h\right)e^{a_0(t-h)}$$

$$+\sum_{i=1}^{k}\frac{a_{-1}a_0\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1} e^{a_0(t-[i+1]h)} - \sum_{i=1}^{k}\frac{a_{-1}a_0\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left([k-i]h\right)^{i+1} e^{a_0(t-[i+1]h)}$$

$$= \sum_{i=1}^{k+1}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left(t-ih\right)^i e^{a_0(t-ih)}; \qquad (13)$$

$$\sum_{i=1}^{k-1} a_{-1}^{i+1}\left(a_{-1}a_0+a_1\right)(t-[i+2]h)e^{a_0(t-[i+2]h)} + \sum_{i=1}^{k}\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^i}{i!}\left(t-[i+1]h\right)^i e^{a_0(t-[i+1]h)} \qquad (14)$$

$$= \sum_{i=1}^{[k+1]-1} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)(t-[i+1]h)e^{a_0(t-[i+1]h)} + \sum_{j=2}^{[k+1]-1}\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^j}{j!}\left(t-[1+j]h\right)^j e^{a_0(t-[1+j]h)}$$

Furthermore,

$$\sum_{i=1}^{k-1} a_1 a_{-1}^{i}\left(a_{-1}a_0+a_1\right)\frac{(t-\left[i+2\right]h)^2 e^{a_0(t-[i+2]h)}}{2} + \sum_{i=1}^{k-1} a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)\frac{(t-[i+2]h)^2}{2}e^{a_0(t-[i+2]h)}$$

$$= \sum_{i=1}^{[k+1]-2} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^2\frac{(t-\left[i+2\right]h)^2 e^{a_0(t-[i+2]h)}}{2} \qquad (15)$$

All the other five summations free of c_{ij}, of the terms in $e^{a_0(t-[i+1]h)}$ and $e^{a_0(t-[i+2]h)}$ add up to

$$-\sum_{i=1}^{k-1} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^2\frac{\left(\left[k-1-i\right]h\right)^2}{2}e^{a_0(t-[i+2]h)} - \sum_{j=2}^{k-1} a_{-1}\left(a_{-1}a_0+a_1\right)^j\frac{\left(\left[k-j\right]h\right)^j}{j!}e^{a_0(t-[j+1]h)} \qquad (16)$$

$$= -\sum_{i=1}^{[k+1]-2} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^2\frac{\left(\left[k-1-i\right]h\right)^2}{2}e^{a_0(t-[i+2]h)} - \sum_{j=2}^{[k+1]-1} a_{-1}\left(a_{-1}a_0+a_1\right)^j\frac{\left(\left[k-j\right]h\right)^j}{j!}e^{a_0(t-[j+1]h)}$$

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The seven summations involving c_{ij} add up to

$$\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{j+1}\frac{(t-[i+j+1]h)^{j+1}e^{a_0(t-[i+j+1]h)}}{j+1}$$
$$-\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{j+1}\frac{([k-i-j]h)^{j+1}e^{a_0(t-[i+j+1]h)}}{j+1}$$
$$+\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{j}([k+1-i-j]h)^{j}e^{a_0(t-[i+j]h)}$$
$$+\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i+1}\left(a_{-1}a_0+a_1\right)^{j}(t-[i+j+1]h)^{j}e^{a_0(t-[i+j+1]h)}$$
$$-\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i+1}\left(a_{-1}a_0+a_1\right)^{j}([k-i-j]h)^{j}e^{a_0(t-[i+j+1]h)} \qquad (17)$$

All summations with constant coefficients add up to zero. Therefore appropriate changes of variables transform above expressions to the following equivalent equation:

$$-\sum_{i=1}^{k-2}\sum_{j=3}^{1+k-i} c_{i\,j-1} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{j}\frac{([k+1-i-j]h)^{j}e^{a_0(t-[i+j]h)}}{j}$$
$$+\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{ij} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{j}([k+1-i-j]h)^{j}e^{a_0(t-[i+j]h)}$$

$$-\sum_{i=2}^{k-1}\sum_{j=2}^{k-i} c_{i-1\,j} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{j}([k+1-i-j]h)^{j}e^{a_0(t-[i+j]h)}$$
$$-\sum_{i=1}^{k-2} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{2}\frac{\left(\left[k-1-i\right]h\right)^{2}}{2}e^{a_0(t-[i+2]h)}-\sum_{j=2}^{k-1} a_{-1}\left(a_{-1}a_0+a_1\right)^{j}\frac{\left(\left[k-j\right]h\right)^{j}}{j!}e^{a_0(t-[j+1]h)}=0$$

$$\Rightarrow\left(c_{12}-1\right)a_{-1}\left(a_{-1}a_0+a_1\right)^{2}\left(\left[k-j\right]h\right)^{2}e^{a_0(t-3h)}$$
$$+\sum_{j=3}^{k-1}\left[c_{1j}-\frac{1}{j}c_{1\,j-1}-\frac{1}{j!}\right]a_{-1}\left(a_{-1}a_0+a_1\right)^{j}\left(\left[k-j\right]h\right)^{j}e^{a_0(t-[j+1]h)}$$
$$-\sum_{i=2}^{k-2}\left(c_{i2}-c_{i-1\,2}-\frac{1}{2}\right)a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{2}\left(\left[k-1-i\right]h\right)^{2}e^{a_0(t-[i+2]h)}$$
$$+\sum_{i=2}^{k-2}\sum_{j=3}^{k-i}\left[c_{ij}-\frac{1}{j}c_{i\,j-1}-c_{i-1\,j}\right]a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^{j}([k+1-i-j]h)^{j}e^{a_0(t-[i+j]h)}=0$$

These, combined with $c_{0j}=\frac{1}{j!}, c_{i1}=1\Rightarrow c_{12}=1;\ c_{1j}-\frac{1}{j}c_{1\,j-1}=\frac{1}{j!},\ j\in\{2,4,\cdots,k-1\};$

$c_{i2}-c_{i-1\,2}=\frac{1}{2}, i\in\{1,2,\cdots k-2\};\ c_{ij}-\frac{1}{j}c_{i\,j-1}-c_{i-1\,j}=0, \forall i\in\{1,2,\cdots k-2\},\ j\in\{2,3,\cdots k-i\}.$

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Consider the difference equation

$$c_{1j} - \frac{1}{j}c_{1j-1} = \frac{1}{j!},\ j \in \{2,\cdots,k-1\} \tag{18}$$

Assertion 1

$$c_{1j} = \frac{1}{(j-1)!},\ j \in \{2,3,\cdots,k-1\} \tag{19}$$

Proof

The proof is by mathematical induction on j. For $t \in J_5$, it is clear from (11) that,

$c_{1j} = \frac{1}{(j-1)!},\ j \in \{2,3,4\}$; these c_{1j}s satisfy (18) for $j \in \{2,3,4\}$. Assume that the assertion is valid for $j \in \{5,\cdots,k-1\}$, for some integer $k \ge 6$. Then from (18) and the induction hypothesis,

$$c_{1k} = \frac{1}{k}c_{1k-1} + \frac{1}{k!} = \frac{1}{k(k-2)!} + \frac{1}{k!} = \frac{1}{k(k-2)!}\left(1+\frac{1}{k-1}\right) = \frac{1}{(k-1)!} \Rightarrow c_{1[k+1]-1} = \frac{1}{([k+1-1]-1)!}$$

$\Rightarrow c_{1j} = \frac{1}{(j-1)!},\ j \in \{2,3,\cdots,[k+1]-1\}$. Hence the assertion is valid.

Consider the difference equation

$$c_{i\,2} - c_{i-1\,2} = \frac{1}{2}, i \in \{2,3,\cdots k-2\}; \tag{20}$$

Assertion 2

$$c_{i\,2} = \frac{1}{2}(i+1),\ i \in \{2,3,\cdots k-2\} \tag{21}$$

Proof

From (11), (20) and (21), it is clear that the assertion is valid for $i \in \{2,3\}$. Assume the validity of the assertion for $i \in \{4,\cdots,k-2\}$, for some integer k. Then for $t \in J_{k+1}, c_{k-1,2} - c_{k-2,2} = \frac{1}{2}$

$\Rightarrow c_{k-1,2} = c_{k-2,2} + \frac{1}{2} = \frac{1}{2}(k-2+1) + \frac{1}{2} = \frac{1}{2}([k-1]+1) \Rightarrow$ the assertion is valid for $i = [k+1]-2$ if $t \in J_{k+1}$, completing the proof of the asssertion.

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DISCUSSION

Remarks on Solutions of (22)

Certainly, there is no general solution for the equation (22) below,for all $i \in \{2,3,\cdots,k-2\}$, $j \in \{1,2,\cdots,k-i\}$ taken simultaneously. The good news is that thus far we know the values of $c_{0\,j}$, $c_{i\,1}, c_{1\,4}, c_{4\,1}, c_{i\,j}, i, j \in \{1,2,3\} : i+j \le 5$; see expression (11). Furthermore for each fixed $i \in \{2,\cdots,k-2\}$,

$$c_{i\,j} - \frac{1}{j} c_{i\,j-1} - c_{i-1\,j} = 0, \tag{22}$$

is a difference equation that can be solved quite easily , for $c_{i\,j}$, $j \in \{2,3,\cdots k-i\}$, in one fell swoop or alternatively, for each pair i and j, (22) is a very simple equation, that depends on previously determined values of $c_{i-1\,j}$ and $c_{i\,j-1}$ - the iterative process is recursive for incremental i or j - and each $c_{i\,j}$ can be quite easily obtained in less than one minute.

5.2 Concluding Part of the Proof of Theorem 4.2

Applications of the relations (18) through (22) and changes of variables on the remaining summations in (17) yield

$$\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{i\,j} a_{-1}^{i+1}\left(a_{-1}a_0 + a_1\right)^j (t-[i+j+1]h)^j e^{a_0(t-[i+j+1]h)}$$

$$+\sum_{i=1}^{k-2}\sum_{j=2}^{k-i} c_{i\,j} a_{-1}^{i}\left(a_{-1}a_0 + a_1\right)^{j+1} \frac{(t-[i+j+1]h)^{j+1} e^{a_0(t-[i+j+1]h)}}{j+1}$$

$$= \sum_{i=1}^{[k+1]-2}\ \sum_{j=2}^{[k+1]-i} c_{i-1\,j} a_{-1}^{i}\left(a_{-1}a_0 + a_1\right)^j (t-[i+j]h)^j e^{a_0(t-[i+j]h)}$$

$$-\sum_{j=2}^{[k+1]-1} \frac{1}{j!} a_{-1}\left(a_{-1}a_0 + a_1\right)^j (t-[j+1]h)^j e^{a_0(t-[j+1]h)}$$

$$-\sum_{i=1}^{[k+1]-2} c_{\,i\,1}\, a_{-1}^{i}\left(a_{-1}a_0 + a_1\right)^2 \frac{(t-[i+2]h)^2 e^{a_0(t-[i+2]h)}}{2}$$

Add these to $e^{a_0 t}$, expressions (13), (14) and (15) to obtain the following result:

$$t \in J_{k+1}, k \in \{3,4,\cdots\} \Rightarrow$$

$$Y(t) \equiv Y_{k+1}(t) = e^{a_0 t} + \sum_{j=1}^{k+1} \frac{\left(a_{-1}a_0 + a_1\right)^j}{j!}\left(t - jh\right)^j e^{a_0(t-jh)}$$

$$+ \sum_{i=1}^{[k+1]-1} a_{-1}^{i}\left(a_{-1}a_0 + a_1\right)(t-[i+1]h) e^{a_0(t-[i+1]h)}$$

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$$+\sum_{i=1}^{[k+1]-2}\sum_{j=2}^{[k+1]-i} c_{ij}a_{-1}^{i}\left(a_{-1}a_0+a_1\right)^j (t-[i+j]h)^j e^{a_0(t-[i+j]h)}$$

for some real positive constants c_{ij} such that $c_{1j}=\dfrac{1}{(j-1)!}$, $c_{i2}=\dfrac{1}{2}(i+1)$, $i\in\{1,2,\cdots[k+1]-2\}$

and $c_{ij}-\dfrac{1}{j}c_{ij-1}-c_{i-1j}=0, \forall\, i\in\{1,2,\cdots[k+1]-2\}$, $j\in\{2,3,\cdots,[k+1]-i\}$.

The statements about the unknown c_{ij} values to be determined follow respectively from the feasible ranges of values for i and j and from the fact that $\sum_{p=1}^{k-3} p=\frac{1}{2}(k-3)(k-2)$; needles to say that the number of integers from 2 to $([k+1]-2)$ is $1+([k+1]-2)-2=k-2$. This completes the inductive proof, proves the cardinality statements and hence establishes the theorem.

For the case $a_{-1}\left(a_{-1}a_0+a_1\right)=0$, $Y(t)=e^{a_0t}\max\{k+1,0\}$.

IMPLICATION TO RESEARCH AND PRACTICE

The results of this article have wide-ranging implications to research and practice. First, they obviate the need for the dependence on (1) for the computations of the solution matrices, with the associated computational complexity and proneness to severe errors. Furthermore, the extension of the solution matrices from one interval to the next contiguous interval of length equal to the delay can be achived effortlessly-only very few coefficients need to be determined. The implication is that solution trajectories can be easily obtained for any initial function specification.

Illustrative Examples

$$Y(t)=e^{a_0t}+\sum_{j=1}^{\max\{k,5\}}\frac{\left(a_{-1}a_0+a_1\right)^j}{j!}(t-jh)^j e^{a_0(t-jh)}+\sum_{i=1}^{\max\{k-1,4\}} a_{-1}^{i}\left(a_{-1}a_0+a_1\right)(t-[i+1]h)e^{a_0(t-[i+1]h)}$$

$$+\,a_{-1}\left(a_{-1}a_0+a_1\right)^2(t-3h)^2 e^{a_0(t-3h)}+\left[a_{-1}\left(a_{-1}a_0+a_1\right)^3\frac{(t-4h)^3}{2}+\frac{3}{2}a_{-1}^2\left(a_{-1}a_0+a_1\right)^2(t-4h)^2\right]e^{a_0(t-4h)}$$

$$+\left[\begin{array}{l} a_{-1}\left(a_{-1}a_0+a_1\right)^4\dfrac{(t-5h)^4}{3!}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^3(t-5h)^3 \\ +2a_{-1}^3\left(a_{-1}a_0+a_1\right)^2(t-5h)^2 \end{array}\right]e^{a_0(t-5h)}\operatorname{sgn}\left(\max\{k-4,0\}\right)$$

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$$+\begin{bmatrix} a_{-1}\left(a_{-1}a_0+a_1\right)^5\dfrac{\left(t-6h\right)^5}{4!}+\dfrac{5}{12}a_{-1}^2\left(a_{-1}a_0+a_1\right)^4\left(t-6h\right)^4 \\ +\dfrac{5}{2}a_{-1}^4\left(a_{-1}a_0+a_1\right)^2\left(t-6h\right)^2+\dfrac{5}{3}a_{-1}^3\left(a_{-1}a_0+a_1\right)^3\left(t-6h\right)^3 \end{bmatrix} e^{a_0(t-6h)}\operatorname{sgn}\left(\max\{k-5,0\}\right)$$

Therefore on the interval $(-\infty,\ 7h]$, the general expression for the solution matrices, $Y(t)$ can be stated as follows: Let $J_k=[kh,(k+1)h]$, for any integer $k\le 6$. Let $t\in J_k$. Then

$$\begin{aligned} Y(t) &= e^{a_0 t}\max\{k+1,0\}+\sum_{j=1}^{k}\frac{\left(a_{-1}a_0+a_1\right)^j}{j!}(t-jh)^j e^{a_0(t-jh)}\max\{k,0\} \\ &+\sum_{i=1}^{k-1}a_{-1}^i\left(a_{-1}a_0+a_1\right)(t-[i+1]h)e^{a_0(t-[i+1]h)}\max\{k-1,0\} \\ &+a_{-1}\left(a_{-1}a_0+a_1\right)^2(t-3h)^2 e^{a_0(t-3h)}\max\{k-2,0\} \\ &+\left[a_{-1}\left(a_{-1}a_0+a_1\right)^3\frac{(t-4h)^3}{2}+\frac{3}{2}a_{-1}^2\left(a_{-1}a_0+a_1\right)^2(t-4h)^2\right]e^{a_0(t-4h)}\max\{k-3,0\} \\ &+\begin{bmatrix} a_{-1}\left(a_{-1}a_0+a_1\right)^4\dfrac{(t-5h)^4}{3!}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^3(t-5h)^3 \\ +2a_{-1}^3\left(a_{-1}a_0+a_1\right)^2(t-5h)^2 \end{bmatrix} e^{a_0(t-5h)}\max\{k-4,0\} \\ &+\begin{bmatrix} a_{-1}\left(a_{-1}a_0+a_1\right)^5\dfrac{(t-6h)^5}{4!}+\dfrac{5}{12}a_{-1}^2\left(a_{-1}a_0+a_1\right)^4(t-6h)^4 \\ +\dfrac{5}{2}a_{-1}^4\left(a_{-1}a_0+a_1\right)^2(t-6h)^2+\dfrac{5}{3}a_{-1}^3\left(a_{-1}a_0+a_1\right)^3(t-6h)^2 \end{bmatrix} e^{a_0(t-6h)}\max\{k-5,0\} \end{aligned}$$

CONCLUSION

This article obtained the structure of the solution matrices of single-delay neutral differential equations with the determination of the expressions for certain coefficients of the matrices and an easily solvable recursive difference equation for the remaining coefficients, proving conclusively that there is no general expression for such coefficients. This contrasts quite sharply with the coefficients of solution matrices of single-delay and the class of double-delay differential equations whose expressions are clearly established, as in the observation in Ukwu and Garba (2014).

FUTURE RESEARCH

The investigations carried in this article will be extended to the system counterpart of equation (1). If successful, the results will be applied to the corresponding variation of parameters formula with a view to explicitly solving associated initial function problems. See Dauer and Gahl (1977) for solution trajectories, subject to initial function specifications.

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