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NUMBER LINE PROPOSITION: THE NUMBER LINE IS COUNTABLE

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ABSTRACT: *Where mathematics often views the number line as infinitely dense and uncountable, the proposition is offered that the number line, while infinite in extent, is countable.*

KEYWORDS: density, enumerate, Euclid, infinite, natural numbers, number line, order, precision, set theory, Zeno's Paradox.

INTRODUCTION

Highly useful, the number line is taught to children at an early age as they learn to count using the set of natural numbers. The set of natural numbers begins with the number one, and then generates itself in an ordered sequence by adding one to itself, and repeating the addition of one to the sum obtained.

Enumerable, uniform in composition, and precise, the number line allows the mind to count to very large numbers such as the number of stars in the sky or grains of sand on a beach without becoming vague, imprecise, or repeating itself. Each element of the set is unique, and can be compared with another element to establish whether it is greater or lesser, and the difference calculated precisely.

The number line first appears as a line with regularly spaced points representing the natural numbers. Later, it is expanded to include fractions, zero, negative numbers, and irrational numbers such as π and e and algebraic expressions such as square roots. Later, students are sometimes exposed to the idea that the number line contains such a great density of numbers that it is uncountable.

In the 1600's René Descartes gave an important advance using the number line, placing two number lines in a plane at a right angle to each other, to form a horizontal x axis and vertical y axis, representing a system of (x, y) coordinates or points on a plane known as Cartesian coordinates. Using points defined by (x, y), his system of analytic geometry made it easier to graph algebraic functions in the form of $y = f(x)$. Extensions later incorporated multiple variables and dimensions.

To help introduce the idea of a countable number line, it may be helpful to suggest that the number line operates at different levels similar to how the universe appears to operate at different levels, seen in the very large numbers found in astronomy, the very small numbers found in atomic physics, and the numbers used in everyday life.

In astronomy, for example, the numbers used to measure distance undergo a radical change. Since powers of ten seem inadequate to describe the changes at the scale of the very large, new units of measurement are introduced. For example, to describe the vast distances between stars and galaxies, scientists use the light year rather than the mile or kilometer. In contrast, atomic physics uses tiny units of measurement such as the nanometer, much smaller than the mile or kilometer, or foot or meter.

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The operation of a number line at different levels is also suggested by how physics adjusts according to scale. For example, Einstein's theory of relativity is used to adjust Newtonian physics for certain problems in astronomy involving gravity, and quantum mechanics is used in addition to Newtonian physics to describe the operation of the electron and certain features of light.

Different forces dominate the natural universe at different scales. In astronomy, the force of gravity dominates the motion of the planets and galaxies. In everyday life, mechanical, electrical, and chemical forces dominate. Within the atom, strong and weak forces are found in addition to the electric charge of the electron and proton.

The natural universe seems to contain limits on its use of precision. Astronomy measures distance in light years, not light years plus inches. Highway distances are given in miles, not miles plus inches. Within the atom, distances are often measured in nanometers or other tiny units, not nanometers plus kilometers. In other words, systems of physical measurement seem to contain natural limits on their use of precision.

An interesting example appears in the definition of the Astronomical Unit that represents the average distance between the Earth and Sun. Recently, astronomers redefined it to be exactly 149,597,870,700 meters or 92,955,807.27 miles, or about 9 meters longer than before due to changes in the size of the Sun, which advances in technology enabled astronomers to detect.

Devised to let people grasp orbital relationships quickly, the Astronomical Unit uses a few digits of precision to relate the orbit of a planet or asteroid to the Earth's orbit. It is a convenient yardstick for measuring distance within the solar system, and is now used in describing other solar systems. But it is not a basic unit of measurement like the meter, which was carefully defined with a high degree of precision.

In other words, the new definition of the Astronomical Unit illustrates how advances in technology can obtain such precise measurements that adjustments are seemingly needed for changes in the size of the objects used as endpoints in determining its length. This suggests how there are practical limits in the value of more precision for some applications.

Many practical applications involve numbers with just a few digits of precision. For years, engineers used slide rules to make many important calculations rapidly with just three significant figures.

Understanding that there are practical limits on the measurement of physical quantities, and the value of added precision for some applications, may suggest how the number line contains natural limits in its enumeration of numbers.

One

The natural universe requires only enough numbers to describe it. In other words, to describe the natural universe, the number line requires only enough numbers to measure, calculate, and convey meaning. There is no evident requirement that the number line consist of mathematical abstractions with an infinite density.

These conditions for a number line, to measure, calculate, and convey meaning, can be satisfied by a countable set of numbers. This is seen in how scientific reports are written using a countable set of numbers, and describe a range of numbers by using two endpoints.

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Suggesting how the natural universe only requires a countable number line to describe it, the universe encompasses a very large but finite space measured in billions of light years, with an age estimated at twelve to fifteen billions years. A countable number line thus appears adequate for describing the natural universe, at least in astronomical terms of distance, age, and mass.

Since a countable number line also appears adequate for describing the numbers commonly used in everyday life and world of the atom, the natural universe appears to be consistent in requiring only a countable number line to use in its description.

Interestingly, in the world of finance, natural limits appear in both the use of large numbers, such as used to measure national budgets, often expressed in trillions of dollars, and in the use of small numbers, such as cents. In other words, the financial world shows how a countable number line is enough for counting both large and small numbers, with natural limits appearing at both ends. It also suggests that the need for a number line to include fractional numbers is easily satisfied since financial transactions are almost always rounded to the dollar or cent.

One way to build a number line starts with the set of natural numbers, which gives an orderly and precise means to count from one to infinity in uniform steps of one. While a number line based on the natural numbers does not include fractions, fractional numbers can be added to it as needed in an indirect manner, as follows.

From a vast reservoir of fractional numbers between zero and one, the fractional part of a number can be drawn, and added to a natural number to express a number to any desired level of precision, at least in terms of decimal places. Such a vast reservoir of fractional numbers can be obtained by placing a decimal point in front of each natural number, turning it into a number between zero and one.

Since the fractional parts are drawn from the reservoir one at a time, this operation may be simplified to draw from a smaller reservoir, whose only requirement is to include the fractional parts actually used. This allows the reservoir to be drained of all its excess numbers.

This indirect approach provides a countable number line at any desired level of precision, at least in terms of decimal places. Supplying fractional numbers as needed, it represents a different approach to the construction of a number line than a direct enumeration of every conceivable number in a seeming infinite and unordered density.

As an aside, just as some natural numbers draw more interest than others, like the primes, the irrational numbers seem to be dominated by a relatively few numbers such as π, e and other physical constants, as well as algebraic expressions such as square roots. However, the set of irrational numbers is potentially vast since it includes numbers obtained from calculations involving physical constants and algebraic expressions.

This observation suggests how a number line has numbers like building blocks and does not require an infinite density, letting it be built using a relatively few numbers, just as the natural numbers can be generated using a single number.

Regarding the rational and irrational numbers, a question appears of how to describe a rational number that repeats as a decimal and given a variation by changing a single digit. Is

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that number irrational? Irrational numbers are generally thought not to include blocks of repeating decimals like a rational number.

Two

Where the first argument for a countable number line focused on its use in describing the natural universe, a second argument appears in describing its functionality, attempting to answer the question of what a number line should do, to help define its basic functions and features, including composition and density.

One reply to the question "What is the number line supposed to do?" would say that a number line sorts, enumerates, or orders sets of elements related to the natural universe. Since the natural universe is finite in composition, a number line only needs a countable number of elements to establish a one-to-one correspondence between itself and the various sets of objects found in the natural universe.

Furthermore, since every calculation regarding the natural universe uses a countable set of numbers, a countable number line can satisfy every calculation requirement regarding the natural universe.

A number line thus disciplines mathematical abstractions of numbers that seem infinitely dense by requiring the numeric representation of algebraic expressions and algorithms in order to determine their relative size or magnitude, and placement on the line as an ordered element.

In other words, the number line is like an exclusive club. Applications for membership need to display a full name, which requires that mathematical abstractions and algebraic expressions must display their name as a number, rather than remain in a set of mathematical abstractions and algebraic expressions.

The composition of the number line can now be viewed as uniform, letting a number be drawn out across the entire line as needed. Rather than place every possible number on a number line, making the line seem infinitely dense, a number line places numbers on the line as needed. It is the process of drawing out numbers across the line that is important, rather than placing every conceivable number on a line before it is needed.

In effect, this idea changes the notion of the composition of a number line to availability, applied uniformly across the entire line. This means that a number line can be as simple as the set of natural numbers, adding to it a vast reservoir of fractional numbers obtained by placing a decimal in front of each natural number, to draw out a fractional number as needed.

This concept does not require a direct enumeration of every number conceivable. It reflects a practical viewpoint where a number line retains an orderly arrangement of its elements, which keeps it from becoming a haphazard collection of numbers, algebraic expressions, and algorithms, unable to sort and enumerate sets of objects.

As an aside, the question comes up of whether in an enumeration of numbers, do the rational and irrational numbers describe the fractional numbers obtained by placing a decimal in front of every natural number?

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A number line does not require echelons of calculations to generate new elements. This gives an answer to1/x and similar dilemmas where it is said that all numbers from one to infinity have their reciprocal over the interval from zero to one. Yet adding one to the set of numbers lying from zero to one gives an algebraically equivalent set over the interval of one to two, so that the number line needs to be recalculated for the addition of a new echelon of numbers.

The answer to such issues lies in the idea that a number line supplies numbers as needed rather than enumerating every number conceivable. A number line needs to describe the natural universe rather than answer queries regarding sets of mathematical abstractions, which add echelons of calculations to generate more elements.

Moreover, functions operate over a number line. They do not define a number line. There is no need to redefine a number line after a function has operated over it. A function operates over a domain, which is not recalibrated.

Three

The natural universe itself suggests that the number line is countable since physical measurements of it are associated with a point or element on a line, without requiring that the line be infinitely dense. Indeed, since physical measurements of the natural universe are made using a finite number of points, a countable number line is sufficient to use in describing the natural universe.

In other words, a countable number line lets its elements associate directly with measurements of the natural universe. For example, associating the points of a countable number line with mass lets the total mass in the natural universe remain countable, rather than suggest it has an infinite mass.

In contrast, a number line with infinite density lacks the innate ability to associate its elements with physical measurements of the universe. While it can emulate a countable number line by selecting from it a countable number of points, this begs the question of how only a countable number line was needed in the first place.

Indeed, economics teaches how an infinite density in numbers, such as inflation where too many dollars chase too few goods, creates more inflation and disincentives against investment, ruining a country’s economy, suggesting how a number line with infinite density does not provide a sound foundation for solving problems.

Everyday life tells us that we are bombarded with numbers, which need to be culled or trimmed, to make sense of them. Many numbers are extraneous, unnecessary, or quickly forgotten. Others are erroneous or misleading.

In other words, the mind operates more efficiently when it can focus. This argues for solving problems using a few numbers carefully, letting them represent ideas or physical quantities. An infinite density of numbers makes it difficult to focus on a few key numbers or quantities.

Where it is said that nature abhors a vacuum, the fact is, nature embraces the vacuum. Astronomy teaches that most of the universe is made up of vast spaces with low densities of matter. Moreover, in a solar system, planets orbit a star at relatively long distances similar to how electrons orbit the nucleus of an atom.

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Indeed, the large empty spaces found in nature perform a key function, allowing forces such as gravity to order the universe, and the internal structure of the atom. This suggests that a number line operates naturally with large spaces between its elements, and is used to create order.

Interestingly, within the atom, neutrons, protons, electrons, and subatomic particles exist as discrete units, not fractions. Similarly, the orbit of an electron is described in terms of discrete shells, suggesting how a countable number line is suitable for describing the world of the atom and natural universe.

In other words, the ancient Greek idea of the atom, proposed by Democritus, which said that matter has an elemental composition, still rings true. Indeed, the world of the atom with its electrons, protons, neutrons, and electron shells seems to contain an idea of mathematical smoothness, not based on a number line with infinite density, but consistency in the operation of its components an laws of physics.

Indeed, mathematical ideas of smoothness seem to be more closely related to the operation of functions rather than requiring a number line or domain with infinite density. For example, while the derivative and integral of calculus have discrete analogs in the difference operator and sum, used when data is bunched into groups, calculus appears to operate over a countable number line rather than require a number line or domain with infinite density.

To derive a derivative or integral, calculus uses a mathematical induction based on Δx, where the limit of Δx is said to approach zero without requiring that the limit be carried out to infinite smallness over an infinitely dense number line. Since this induction is essentially countable, it only requires a number line that is countable.

In other words, calculus uses an idea of continuity that represents smoothness in the operation of a function, rather than the smoothness obtained by using a number line with infinite density.

Four

Related to the idea that the number line should reflect the natural universe in being countable is the idea that mathematical assumptions or axioms should closely conform to the physical systems they seek to represent, similar to how Euclid carefully chose his axioms on geometry to reflect the natural world.

In other words, mathematics carefully chooses axioms to reflect fundamental properties of physical systems, which logic builds upon to give proofs, which become tools used to solve problems.

With this in mind, the questions arises, "Why does a natural universe that is countable need a number line that is infinitely dense or uncountable?" A countable natural universe would suggest that mathematical models of it, such as a number line, should also be countable.

In writing, an excess of words can lead to confusion in both conveying meaning from the viewpoint of a writer, and in understanding the meaning intended from the viewpoint of a reader. Likewise, an excess of numbers can lead to confusion in both conveying meaning from the viewpoint of a writer, and in understanding the meaning intended from the viewpoint of a reader.

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In other words, a number line with infinite density does not seem to provide a sound foundation for describing the natural universe when a simpler, countable number line is sufficient. Sufficiency does not require an excess of information or numbers.

Indeed, just as excess or extraneous information, whether in words or numbers, is often used to mislead or divert attention away from topics of importance, so should the number line be disciplined to exclude an excess of numbers.

Five

An idea often used in management is the quality of resourcefulness, which says that we use the resources we have to solve a problem. Resourcefulness often involves changes in attitude since people sometimes feel they need more resources when the resources they have are adequate, but they need a gentle reminder to try, or be more innovative or effective in using their resources.

Mathematics involves principles of management, such as its management of numbers in the construction of a number line, which provides a tool for the presentation of numbers used in scientific reports and other applications.

Resourcefulness helps solve problems where the data or analysis is incomplete, and time or financial constraints require the use of estimates. It gives solutions that are helpful. It acts intelligently, without expending resources endlessly, while it avoids tasks that are a holdup. It provides explanations without requiring an excess of detail or omitting important steps.

Resourcefulness follows the idea that the natural universe provides natural solutions to problems, although perhaps not the solutions that we expect, or that require that we revise our understanding of a problem.

In mathematics, resourcefulness seeks to use a minimum of assumptions, easily accepted as being true and useful, able to stand the test of time. However, resourcefulness requires skill in choosing assumptions, so useful that they are seen as necessary. Too many assumptions can lower their quality.

Since a countable number line is sufficient to use in solving practical problems, where calculations are made using a finite number of points or numbers, it satisfies a condition of sufficiency, making it sufficient for theory as well as practice.

In other words, as an idea solves problems on a consistent basis, it attains a condition of sufficiency that does not require more detail or explanation, similar to assumptions that have proven themselves true under a variety of conditions.

In contrast, ideas using an excess of words or numbers are likely to result in reasoning that is first excessive, then unsuitable, and finally untenable.

If we do not calculate using a number line with an infinite density of numbers, why do we need a number line that is infinitely dense in the first place?

Six

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As a rule from Euclid, a line connects two points. In other words, a line is a geometric operator that connects two points, which possesses a sense of geometric consistency over its entire length, making it consistent from one end to the other.

A line can be subdivided or added onto. Either procedure adds at least one point to the line or its subdivision. As a corollary, a subdivision or addition to a line leaves the line with the same properties that it had before. In other words, the properties of a line are independent of its length.

While Euclid did not specify the number of points on a line, it may be said that a line contains many points since a subdivision or addition to a line can be repeated, resulting in an accumulation of points.

Lines are commonly used to measure length. A line can be used to measure any finite length. A line with finite length can be used to build a line with infinite length by adding to the line a line of fixed length in a countable process, and letting this process continue indefinitely.

A line does not require an uncountable number of points. There does not appear to be any requirement that a line contain an infinitely dense or uncountable number of points in order for it to operate with geometrical consistency.

Seven

One of the functions of a number line is that it orders or enumerates elements in a set by establishing a one-to-one correspondence between the elements in a set and its elements, using an ordered set of numbers like the natural numbers.

Since Euclid's definition of a line implies a sense of geometric consistency over a line, a process of enumeration or enumerator should be able to enumerate a line from one end to the other. It should be able to enumerate the elements that make up a line, which consists of a set of points, and does not include an intermediate medium, except perhaps line segments, which consist of smaller sets of points.

Operating over a line, an enumerator enumerates the entire line. But if an enumerator operates over a line with infinite density, it cannot traverse the line, no matter how fast or slowly it enumerates the elements on the line.

In other words, a number line needs to be able to establish a one-to-one correspondence between an enumerator and its own elements. Since this correspondence is established in a step-by-step process, it is countable. Enumeration requires that a number line be countable.

Eight

Where Euclid's rule says that a line connects two points, his rule may be reworded to say that a line can connect any two points in a space, which is assumed to possess continuity without barriers or singularities.

Euclid's rule implies a sense of geometrical consistency in its operation. In other words, a line can connect any two points in a space since Euclid did not place any restrictions on the location of the points, except the obvious requirement that the points are distinct and geometrically separate, requiring a line to connect them.

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In other words, as a geometrical operator, a line is free to operate over the entire space, able to connect any two points just as Euclid imposed no barriers to a line connecting two points.

Just as Euclid used two points to define a line, the analytic geometry of Descartes requires two points to define a linear equation of the form $y = mx + b$ in the xy plane. Both Euclid and Descartes recognized how two points can define a line.

Adding points to the determination of a line involves the repetition of information, a line with multiple variables, or the determination of a curve. Euclid's rule may also be said to connect two points using a line with minimum length, which results in a straight line in a linear space.

Nine

The number line should mesh with Euclidian geometry, including the tools used in the construction of a line and physical representations of points, which give them their sense of precision and displacement.

A physical representation of a point represents a practical application of the point as an abstract concept. For example, in a land survey, a point is defined on the ground with a stake, and then transposed onto a map with some small displacement, all the while retaining its character as an abstract mathematical concept.

The physical representation of a point lets a point be pointed to, exactly, but does not require that every point on a line have the same physical representation or displacement. A line can contain many points without specifying their exact number or physical displacement.

In practical geometry, it may be observed that the question is almost never asked of how many points there are on a line since, in most applications, only a small number of points are of interest and physically determined.

A line only needs a finite number of points to be operational, just as Euclid required only two points to define a line.

Ten

As a tool of science, the number line appears in the x and y axis, which are commonly used for scientific analysis where functions are plotted, representing physical quantities such as time, distance, and mass.

As a practical tool, the points on a number line need to be visible whether on a computer screen, chalkboard, or piece of paper, whose physical displacement is a flexible property of the point as a mathematical abstraction, determined by the tools used in their construction.

In contrast, the idea that the number line has an infinite density seems to force an infinite density upon a countable universe, and disassociates the physical displacement of a point from its property as a mathematical abstraction.

Zeno's Paradox

Where the normal induction process extends the operation of a function from n to n+1, and shows since the function is true at $n = 1$ it is true over the natural numbers by moving through the natural numbers in an orderly, sequential manner, an anti-induction process such as

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Zeno's Paradox confines the operation of a function to a shortened, selective interval, using a number line of practically infinite density to create a series of increasingly small intervals.

In other words, where a normal induction process extends the operation of a function from n to n+1, an anti-induction process makes it impossible to extend the operation of a function from n to n+1 or the endpoint of an interval, by arguing it must operate over a series of increasingly small intervals.

A normal induction process is similar to the derivative or integral of calculus that relates the rate of change of $f(x+\Delta x)$ to $f(x)$ by taking the limit as $\Delta x \rightarrow 0$. An induction proves a function over the natural numbers. Calculus proves a derivative or integral, using the limit, letting it approach zero in an orderly, sequential manner similar to the orderly, sequential process of a mathematical induction.

An anti-induction process such as Zeno's Paradox uses a sophistry to limit the operation of a function to a selective part of its domain in order to justify a false conclusion when it would normally present a true conclusion if allowed to operate over its full domain.

Zeno's Paradox uses a process of interrogation applied only under the condition where the tortoise retains his head start against Achilles. To win his case, Zeno repeats his same argument over increasingly small intervals of time, thus making it impossible for Achilles to catch up to the tortoise, and he never presents his argument after Achilles has passed the tortoise.

Zeno's Paradox illustrates the hazard of using a line with infinite density for practical applications such as calculating the distance covered by a moving object. Using such a line, the distance covered by a moving object can be arbitrarily truncated by arguing it must be calculated over increasingly small intervals, which by design do not reach the endpoint of an interval.

In contrast, logical and geometrical consistency requires that functions such as the calculation of distance be applied consistently over a line, and reach an endpoint. It may be observed that if Zeno's argument is applied from the tortoise to the starting line, the tortoise never started the race, proving its absurdity.

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