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MORE ON SQUARE AND SQUARE ROOT OF A NODE ON T3 TREE

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ABSTRACT: *The article proves several important properties of the square and the square root of a node of T3 tree. The new properties describe how the square and the square root of a node are distributed on the T3 tree and are helpful to locate divisors of a composite integer.*

KEYWORDS: Square, Square Root, Integer, Binary Tree

INTRODUCTION

The square and the square root are undoubtedly very important operations for a number. As a new structure of odd integers, the $\boldsymbol{T}_3$ tree, which was introduced in WANG (2016 & 2018), is of course necessary to make clear these two operations. In fact, for a given node in the tree, the problem where its square and its square root locate is surely a fundamental problem. Some general properties of the square of a node in $\boldsymbol{T}_3$ were mentioned in WANG (2018) and CHEN (2018) disclosed several properties of the square root of a node. However, one can see that, there are still a lot of unknown properties. This paper shows a little more of the properties related with the square and the square root of a node.

PRELIMINARIES

This section lists for later sections the necessary preliminaries, which include definitions, notations and lemmas.

Definitions and Notations

Let S be a set of finite positive integers with s_0 and s_n being the smallest and the biggest nodes respectively; an integer x is said to *be clamped* in S if $s_0 \le x \le s_n$. Symbol $x \triangleq S$ indicates that x is clamped in S. Symbol $\lfloor x \rfloor$ is the floor function, an integer function of real number x that satisfies inequality $x-1<\lfloor x \rfloor \le x$, or equivalently $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.

In this whole paper, symbol $\boldsymbol{T}_3$ is the $\boldsymbol{T}_3$ tree that was introduced in WANG (2016 & 2018) and symbol $N_{(k,j)}$ is by default the node at position j on level k of $\boldsymbol{T}_3$, where $k \ge 0$ and $0 \le j \le 2^k - 1$. By using the asterisk wildcard *, symbol $N_{(k,*)}$ means a node lying on level k. An integer X is said to be clamped on level k of $\boldsymbol{T}_3$ if $2^{k+1} \le X \le 2^{k+2} - 1$ and symbol $X \triangleq k$ indicates X is clamped on level k. If a positive integer X is clamped on level k and there is a node Y of $\boldsymbol{T}_3$ satisfying $X = \lfloor \sqrt{Y} \rfloor$, then X is said to be a *floor square root* of the node Y and Y is called a *square source* of X.

Remark 1. CHEN (2018) put forward the concept that an integer is clamped on a level of $\boldsymbol{T}_3$. In CHEN's paper, a positive integer X was said to be clamped on level k of $\boldsymbol{T}_3$ if $2^{k+1}+1\le X\le 2^{k+2}-1$. Since $2^{k+1}-1$ is the rightmost node on level $k-1$ and $2^{k+1}+1$ is the leftmost node on level k, there is an integer 2^{k+1} between the two. In order to avoid leaving out the number 2^{k+1}, this paper redefines it by $2^{k+1}\le X\le 2^{k+2}-1$.

Lemmas

Lemma 1 (See in WANG (2018)). $\boldsymbol{T}_3$ Tree has the following fundamental properties.

(**P1**). Every node is an odd integer and every odd integer bigger than 1 must be on the $\boldsymbol{T}_3$ tree. Odd integer N with $N>1$ lies on level $\lfloor \log_2 N\rfloor -1$.

(**P2**). On level k with $k=0,1,\dots$, there are 2^k nodes starting by $2^{k+1}+1$ and ending by $2^{k+2}-1$, namely, $N_{(k,j)}\in[2^{k+1}+1,2^{k+2}-1]$ with $j=0,1,\dots,2^k-1$.

(**P3**). $N_{(k,j)}$ is calculated by

$$N_{(k,j)}=2^{k+1}+1+2j,\ j=0,1,\dots,2^k-1$$

(**P4**) Multiplication of arbitrary two nodes of $\boldsymbol{T}_3$, say $N_{(m,\alpha)}$ and $N_{(n,\beta)}$, is a third node of $\boldsymbol{T}_3$. Let $J=2^m(1+2\beta)\ +2^n(1+2\alpha)+2\alpha\beta+\alpha+\beta$; the multiplication $N_{(m,\alpha)}\times N_{(n,\beta)}$ is given by

$$N_{(m,\alpha)}\times N_{(n,\beta)}=2^{m+n+2}+1+2J$$

If $J<2^{m+n+1}$, then $N_{(m,\alpha)}\times N_{(n,\beta)}=N_{(m+n+1,J)}$ lies on level $m+n+1$ of $\boldsymbol{T}_3$; whereas, if $J\ge 2^{m+n+1}$, $N_{(m,\alpha)}\times N_{(n,\beta)}\ =N_{(m+n+2,\chi)}$ with $\chi=J-2^{m+n+1}$ lies on level $m+n+2$ of $\boldsymbol{T}_3$.

(P5) Product $N_{(m,\alpha)}\times N_{(m,\alpha)}=N^2_{(m,\alpha)}$ is a left node of $\boldsymbol{T}_3$, and it lies on level $2m+1$ or $2m+2$

Lemma 2 (See in WANG (2017)). For real numbers x, y and positive integer i, it holds

(P13) $x\le y\Rightarrow\lfloor x\rfloor\le\lfloor y\rfloor$; $x<n\Rightarrow\lfloor x\rfloor<n$, where n is an integer.

(P31) $i-1\le 2\left\lfloor \frac{i}{2}\right\rfloor\le i$.

MAIN RESULTS AND PROOFS

Theorem 1. Let k be a positive integer; then there are $2k+1$ consecutive integers $n_1,n_2,\dots,n_{2k+1}$ that satisfy $\lfloor\sqrt{n_i}\rfloor_{(i=1,2,\dots,2k+1)}=k$.

Proof. Consider an arbitrary integer n such that $\lfloor\sqrt{n}\rfloor=k$; then by definition of the floor function it holds $k\le\sqrt{n}<k+1$. That is

$$k^2\le n<k^2+2k+1$$

Hence the $2k+1$ integers, $n_1 = k^2 + 0$, $n_2 = k^2 + 1, n_3 = k^2 + 2, \ldots$, and $n_{2k+1} = k^2 + 2k$, are the integers satisfying $\lfloor \sqrt{n_i} \rfloor_{(i=1,2,\ldots,2k+1)} = k$.

□

Proposition 1. Let $N_{(m,\alpha)}$ be a node of $\boldsymbol{T}_3$ with $m > 0$; then when $0 \le \alpha \le \left\lfloor \frac{\sqrt{2^{2m+3}+1}-1}{2} \right\rfloor - 2^m$, $N^2_{(m,\alpha)}$ lies on level $2m+1$; otherwise it lies on level $2m+2$. Particularly, $N^2_{(m,0)} = N_{(2m+1,2^{m+1})}$ and $N^2_{(m,2^m-1)} = N_{(2m+2,2^{2m+2}-2^{m+2})}$.

Proof. Direct calculation shows

$$N^2_{(m,\alpha)} = (2^{m+1} + 2\alpha + 1)^2 = 2^{2m+2} + 2(2\alpha^2 + 2^{m+2}\alpha + 2^{m+1} + 2\alpha) + 1$$

By Lemma 1 (**P4** & **P5**), it knows that $N^2_{(m,\alpha)}$ lies on level $2m+1$ if and only if $J = 2\alpha^2 + 2^{m+2}\alpha + 2^{m+1} + 2\alpha < 2^{2m+1}$. Consequently,

$$\begin{aligned}
&\alpha^2 + 2^{m+1}\alpha + 2^m + \alpha < 2^{2m} \\
&\Rightarrow \alpha^2 + (2^{m+1}+1)\alpha + 2^m - 2^{2m} < 0 \\
&\Rightarrow 0 \le \alpha < \frac{-(2^{m+1}+1) + \sqrt{(2^{m+1}+1)^2 - 4\times(2^m - 2^{2m})}}{2} \\
&\Rightarrow 0 \le \alpha < \frac{\sqrt{2^{2m+3}+1} - (2^{m+1}+1)}{2} \\
&\Rightarrow 0 \le \alpha < \frac{\sqrt{2^{2m+3}+1} - 1}{2} - 2^m \\
&\Rightarrow 0 \le \alpha \le \left\lfloor \frac{\sqrt{2^{2m+3}+1} - 1}{2} \right\rfloor - 2^m
\end{aligned}$$

which validates the first part of the proposition.

The second part is easily obtained by the following calculations.

$$N^2_{(m,0)} = (2^{m+1} + 2\times 0 + 1)^2 = 2^{2m+2} + 2\times 2^{m+1} + 1 = N_{(2m+1,2^{m+1})}$$

$$\begin{aligned}
N^2_{(m,2^m-1)} &= (2^{m+1} + 2\times(2^m - 1) + 1)^2 = (2^{m+1} + 2^{m+1} - 1)^2 \\
&= 2^{2m+2} + 2^{2m+2} + 1 + 2^{2m+3} - 2^{m+2} - 2^{m+2} \\
&= 2^{2m+3} + 2^{2m+3} - 2^{m+3} + 1 \\
&= 2^{2m+3} + 2\times(2^{2m+2} - 2^{m+2}) + 1 \\
&= N_{(2m+2,2^{2m+2}-2^{m+2})}
\end{aligned}$$

□

Remark 1. The condition $m > 0$ in Proposition 1 is proposed because it can get rid of the case $N^2_{(0,0)} = 3^2 = 9 = N_{(2,0)}$, which is the unique example that violates $N^2_{(m,0)} = N_{(2m+1,2^{m+1})}$.

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Proposition 2. Let k be a positive integer, $N_{(2k+1,*)}$ and $N_{(2k+2,*)}$ be nodes of $\boldsymbol{T}_3$; then $(\lfloor\sqrt{N_{(2k+1,*)}}\rfloor\le\lfloor 2^{k+1}\sqrt{2}\rfloor)\triangleq k$ and $(\lfloor 2^{k+1}\sqrt{2}\rfloor\le\lfloor\sqrt{N_{(2k+2,*)}}\rfloor)\triangleq k$. On level $2k$ there is not a node $N_{(2k,*)}$ satisfying $N_{(2k,*)}\triangleq k$ and there is neither a node $N_{(2k+3,*)}$ satisfying $N_{(2k+3,*)}\triangleq k$ on level $2k+3$.

Proof. $N_{(2k+1,*)}$ and $N_{(2k+2,*)}$ being the nodes on levels $2k+1$ and $2k+2$ respectively yields

$$2^{2k+2}<2^{2k+2}+1\le N_{(2k+1,*)}\le 2^{2k+3}-1<2^{2k+3}$$
$$\Rightarrow 2^{k+1}=\lfloor\sqrt{2^{2k+2}}\rfloor\le N_{(2k+1,*)}\le\lfloor 2^{k+1}\sqrt{2}\rfloor$$

and

$$2^{2k+3}<2^{2k+3}+1\le N_{(2k+2,*)}\le 2^{2k+4}-1<2^{2k+4}$$
$$\Rightarrow\lfloor 2^{k+1}\sqrt{2}\rfloor\le\lfloor\sqrt{N_{(2k+2,*)}}\rfloor<2^{k+2}$$

Note that $1+\frac{1}{2}-\frac{1}{4}<\sqrt{2}<1+\frac{1}{2}$, it yields

$$2^{k+1}+2^{k}-2^{k-1}\le\lfloor 2^{k+1}\sqrt{2}\rfloor\le(\lfloor 2^{k+1}(1+\frac{1}{2})\rfloor=2^{k+1}+2^{k}$$

Hence it holds

$$2^{k+1}\le N_{(2k+1,*)}\le\lfloor 2^{k+1}\sqrt{2}\rfloor\le 2^{k+1}+2^{k}$$

and

$$2^{k+1}+2^{k}-2^{k-1}\le\lfloor 2^{k+1}\sqrt{2}\rfloor\le\lfloor\sqrt{N_{(2k+2,*)}}\rfloor<2^{k+2}$$

That is $(\lfloor\sqrt{N_{(2k+1,*)}}\rfloor\le\lfloor 2^{k+1}\sqrt{2}\rfloor)\triangleq k$ and $(\lfloor 2^{k+1}\sqrt{2}\rfloor\le\lfloor\sqrt{N_{(2k+2,*)}}\rfloor)\triangleq k$.

Considering the biggest node $N_{(2k,2^{2k}-1)}$ on level $2k$, it yields

$$N_{(2k,*)}\le N_{(2k,2^{2k}-1)}=2^{2k+2}-1<2^{2k+2}\Rightarrow\lfloor\sqrt{N_{(2k,*)}}\rfloor\le\lfloor\sqrt{N_{(2k,2^{2k}-1)}}\rfloor<2^{k+1}$$

Likewise, considering the smallest node $N_{(2k+3,0)}$ on level $2k+3$, it yields

$$N_{(2k+3,0)}=2^{2k+4}+1>2^{2k+4}\Rightarrow\lfloor\sqrt{N_{(2k+3,*)}}\rfloor\ge\lfloor\sqrt{N_{(2k+3,0)}}\rfloor\ge 2^{k+2}>2^{k+2}-1$$

Hence there is not a node $N_{(2k,*)}$ satisfying $N_{(2k,*)}\triangleq k$, and there is neither a node $N_{(2k+3,*)}$ satisfying $N_{(2k+3,*)}\triangleq k$.

□

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Proposition 3. Node $N_{(k,j)}$ ($k>1$) of T_3 satisfies

$$2^{\left\lfloor \frac{k+1}{2} \right\rfloor}-1<2^{\left\lfloor \frac{k+1}{2} \right\rfloor}\le\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor\le 2^{\left\lfloor \frac{k+1}{2} \right\rfloor+1}-1<2^{\left\lfloor \frac{k+1}{2} \right\rfloor+1}$$

Or equivalently $\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor \triangleq \left\lfloor \frac{k-1}{2} \right\rfloor$.

Proof. Since $2^{k+1}+1\le N_{(k,j)}\le 2^{k+2}-1$, it yields $2^{k+1}<N_{(k,j)}<2^{k+2}$; hence it holds

$$2^{\frac{k+1}{2}}<\sqrt{N_{(k,j)}}<2^{\frac{k}{2}+1}$$

By Lemma 2(**P31**), the inequality $k\le 2\left\lfloor \frac{k+1}{2} \right\rfloor\le k+1$ yields $2^{\left\lfloor \frac{k+1}{2} \right\rfloor}\le 2^{\frac{k+1}{2}}$ and $2^{\frac{k}{2}+1}\le 2^{\left\lfloor \frac{k+1}{2} \right\rfloor+1}$, hence it holds

$$2^{\left\lfloor \frac{k+1}{2} \right\rfloor}<\sqrt{N_{(k,j)}}<2^{\left\lfloor \frac{k+1}{2} \right\rfloor+1}$$

By Lemma 2 (**P13**) it immediately leads to

$$2^{\left\lfloor \frac{k+1}{2} \right\rfloor}\le\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor<2^{\left\lfloor \frac{k+1}{2} \right\rfloor+1}$$

which is $\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor \triangleq (\left\lfloor \frac{k+1}{2} \right\rfloor-1=\left\lfloor \frac{k-1}{2} \right\rfloor)$.

□

Remark 2. This proposition is a modification of the Corollary 2 in CHEN (2018). That paper claimed that $\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor \triangleq \left\lfloor \frac{k+1}{2} \right\rfloor-1$ or $\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor \triangleq \left\lfloor \frac{k}{2} \right\rfloor$. However, seeing from Propositions 1, 2 and 3, one can see that $\left\lfloor \sqrt{N_{(k,j)}} \right\rfloor \triangleq \left\lfloor \frac{k+1}{2} \right\rfloor$ never occurs.

Theorem 2. Given $N>3$ be an odd integer; then $\left\lfloor \sqrt{N} \right\rfloor \triangleq \left\lfloor \frac{\lfloor \log_2 N \rfloor}{2} \right\rfloor-1$.

Proof. Let $k=\lfloor \log_2 N \rfloor-1$. By Lemma 1(**P1**), N is a node on level k of T_3. By Proposition 3 it knows $\left\lfloor \sqrt{N} \right\rfloor \triangleq \left\lfloor \frac{k-1}{2} \right\rfloor$, that is $\left\lfloor \sqrt{N} \right\rfloor \triangleq \left\lfloor \frac{\lfloor \log_2 N \rfloor}{2} \right\rfloor-1$.

□

Example 1. Table 1 lists several odd integers that are randomly picked, and their positions in T_3 as well as their square roots in T_3. It can see that, $\left\lfloor \sqrt{N} \right\rfloor \triangleq \left\lfloor \frac{\lfloor \log_2 N \rfloor}{2} \right\rfloor-1$ holds for each number. Readers can check it manually or with Mathematica.

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Table 1. Odd Integers and their square roots in T_3

Odd Integer N	N's in T_3	$\left\lfloor \frac{\lfloor \log_2 N \rfloor}{2} \right\rfloor - 1$	$\lfloor \sqrt{N} \rfloor$ & its level
1517	$N_{(9,246)}$	4	$\lfloor \sqrt{1517} \rfloor = 38 \triangleq 4$
20491	$N_{(13,2053)}$	6	$\lfloor \sqrt{20491} \rfloor = 143 \triangleq 6$
386757	$N_{(17,62306)}$	8	$\lfloor \sqrt{386757} \rfloor = 621 \triangleq 8$
6947533	$N_{(21,1376614)}$	10	$\lfloor \sqrt{6947533} \rfloor = 2635 \triangleq 10$
104678919	$N_{(25,18785027)}$	12	$\lfloor \sqrt{104678919} \rfloor = 10231 \triangleq 12$

Remark 3. Table 1 can be easily checked manually or with Mathematica. When programmed as follows,

```
f[x_]:=Floor[Floor[Log[x]/Log[2]]/2]-1;
g[x_]:=Floor[Log[Sqrt[x]]/Log[2]]-1;
r[x_]:=Floor[Sqrt[x]];
inData={1517,20491,386757,6947533,104678919};
r1=Table[f[inData[[i]]],{i,5}];
r2=Table[g[inData[[i]]],{i,5}];
r3=Table[r[inData[[i]]],{i,5}];
t={r1,r2,r3}//MatrixForm
```

the screenshot in Mathematica 7.0 is as figure 1

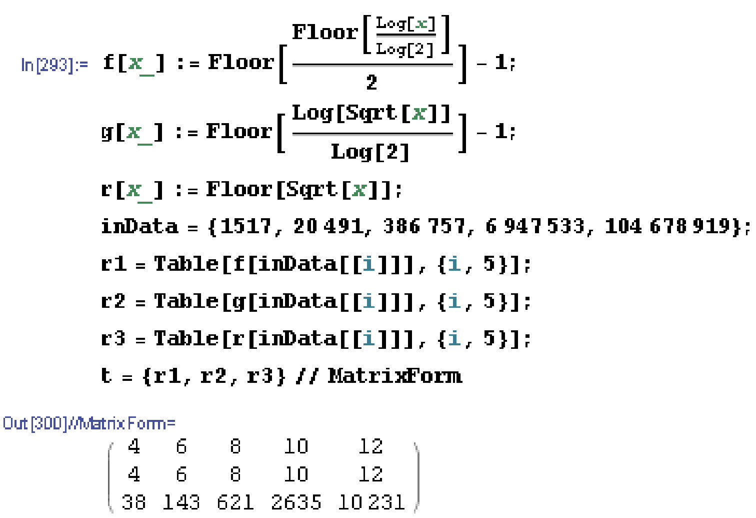

Figure 1. Screenshot of the program and outputs

Corollary 1. Let $N > 3$ be an odd integer and $k = \lfloor \log_2 N \rfloor - 1$; then $(\lfloor \sqrt{N} \rfloor \le \lfloor 2^{\lfloor \frac{k+1}{2} \rfloor} \sqrt{2} \rfloor) \triangleq \lfloor \frac{k-1}{2} \rfloor$ when k is odd whereas $(\lfloor 2^{\lfloor \frac{k+1}{2} \rfloor} \sqrt{2} \rfloor \le \lfloor \sqrt{N} \rfloor) \triangleq \lfloor \frac{k-1}{2} \rfloor$ when k is even.

Proof. By Theorem 2, $\lfloor \sqrt{N} \rfloor \triangleq \lfloor \frac{k-1}{2} \rfloor$ is sure. Now consider the fact that, whether $k = 2l+1$ or $k = 2l+2$, $\lfloor \frac{k+1}{2} \rfloor = l+1$ always holds. By Proposition 2, it knows that, $(\lfloor \sqrt{N} \rfloor \le \lfloor 2^{\lfloor \frac{k+1}{2} \rfloor} \sqrt{2} \rfloor) \triangleq \lfloor \frac{k-1}{2} \rfloor$ when k is odd whereas $(\lfloor 2^{\lfloor \frac{k+1}{2} \rfloor} \sqrt{2} \rfloor \le \lfloor \sqrt{N} \rfloor) \triangleq \lfloor \frac{k-1}{2} \rfloor$ when k is even.

CONCLUSION

The square and the square root are important numbers. For an integer, the square root is essentially important because it is the cut-off point of divisors of a composite integer. Study of these numbers is helpful to understand distribution of the divisors on $\boldsymbol{T}_3$. The Theorem 2 proved in this paper is of course a foundation for us to know where the square root of a node lies. Hope it is helpful in the future.

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