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MINIMIZATION OF FUNCTIONALS CONTAINING SECOND-ORDER DERIVATIVES USING THE METHOD OF VARIATIONAL CALCULUS

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ABSTRACT: *This paper presents the method of variational calculus in minimizing functionals containing second-order derivatives. The original theory of variational calculus was modified by the introduction of additional variables because of the second order derivatives involved in the functional. The extremum value of the functional was derived by solving the resulting differential equations from the solution Euler's equation of the augmented functional.*

KEYWORDS: Augmented Functional, Variational Calculus, Auxiliary Equation, Lagrange Multiplier, Euler's equation.

INTRODUCTION

Calculus of variation is a field of mathematical analysis that deals with maximizing or minimizing functionals which are mappings from a set of functions to the real numbers. Functionals are often expressed as definite integral involving functions and their derivatives. Calculus of variation developed from a problem stated by Johann Bernoulli which required the form of the curve joining two fixed points in a vertical plane such that a given body sliding down the curve travels from a fixed point to another in a minimum time under constant gravity without any friction, Burghes and Graham (1980). This problem has drawn the attention of a remarkable range of mathematical luminaries like Newton, Euler, Lagrange and Laplace, Clegg (1968). In the nineteenth century, Hamilton, Dirichlet and Hilbert also contributed to this important branch of mathematics, Clegg (1968) and Charles (1987). In recent times, calculus of variations has continued to occupy centre stage, witnessing major theoretical advances, along with wide-ranging applications in physics, engineering and all branches of mathematics, Cassel (2013) and Weinstock (1974).

This method of variational calculus can be used to analyse minimization problems in the equilibrium configurations of almost all continuous physical systems, ranging through elasticity, solid and fluid mechanics, electro-magnetism, gravitation, quantum mechanics and string theory, Arfken, (1985), Jeffreys and Jeffreys (1988). A lot of research work had been done on minimization of functionals. Oke (2014) looked at minimization of quadratic and non-quadratic functionals in its second order method for minimizing unconstrained optimization problems. Oregon (2010) worked on Minimization of functionals using calculus of variation technique while Owonifari (2007) considered the minimization of some selected functionals using variational calculus method, to mention a few. None of these researchers

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worked on functionals containing second-order derivatives. The method considered in this paper gives the extremum values of functionals containing second-order derivatives.

MATERIALS AND METHODS

In this paper, attention was focused on finding the extremum values of functionals of the form $J = \int_a^b F[x, \dot{x}, \ddot{x}, t]\, dt$. Where F is a given function of $x, \dot{x}, \ddot{x}$ and t, $x = x(t)$ gives some path between two fixed points in the plane. $\dot{x}$ and $\ddot{x}$ are first and second order derivatives of $x = x(t)$.

To find the extremum values of this kind of functional, we introduce two additional variables x_1 and x_2 (where $x_1 = x$ and $x_2 = \dot{x}_1$) into the problem because of the second-order derivative contained in the functional. We now write the original functional in form of these two variables with the constraints $x_2 - \dot{x}_1 = 0$.

Two Euler's equations for x_1 and x_2 to be satisfied by the optimum path are thereafter calculated using the augmented functional. The Euler's equations are

$$\frac{\partial F}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial F}{\dot{x}_1}\right) = 0$$

and

$$\frac{\partial F}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial F}{\dot{x}_2}\right) = 0$$

The resulting differential equations from the solution of this Euler's equation will be solved to get the extremum value of the functional.

RESULTS

Example 1

Minimize the functional $J = \int_0^1 [1 + \ddot{x}^2]\, dt$

Where $\dot{x} = \dot{x}(t) = \frac{dx}{dt}$ and $x(t)$ satisfies the boundary condition

$x(0) = 0, \dot{x}(0) = 1, x(1) = 1, \dot{x}(1) = 1$

Solution

Let $x_1 = x$ and $x_2 = \dot{x}_1$.

The functional now becomes

$$J = \int_0^1 [1 + \dot{x}_2{}^2]\, dt$$

with the constraints

$x_2 - \dot{x}_1 = 0.$

The augmented functional is now

$$J^* = \int_0^1 [1 + \dot{x}_2{}^2] + \lambda[x_2 - \dot{x}_1]\, dt$$

where λ is the Lagrange multiplier

The Euler's equations for x_1 and x_2 are

$$0 - \frac{d}{dt}(-\lambda) = 0$$

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$\lambda - \frac{d}{dt}(2\dot{x}_2) = 0$

This implies that

$\dot{\lambda} = 0$

and

$\lambda - 2\ddot{x}_2 = 0$

Solving the two equations above simultaneously, we have

$2\dddot{x}_2 = 0$

Integrating this, we obtain

$x_2(t) = Pt^2 + Qt + R$

where P,Q and R are constants of integration.

Now since $\dot{x}_1 = x_2$, we can integrate again to get

$x_1(t) = \frac{Pt^3}{3} + \frac{Qt^2}{2} + Rt + S$

where S is an additional constant of integration

Applying the end conditions and solving the resulting equations simultaneously, we obtain

$P = 0, Q = 0, R = 1$ and $S = 0$

Putting these values into our equations for $x_1(t)$ and $x_2(t)$, we have

$x_1(t) = t$

$x_2(t) = 1$

$\dot{x}_2(t) = 0$

The optimal path has now been determined. Substituting these in our functional yields

$J = \int_0^1 [1 + 0^2]\, dt = \int_0^1 1\, dt = 1$

Example 2

Minimize the functional $J = \int_0^1 [\ \ddot{x}^2] dt$

Where $\dot{x} = \dot{x}(t) = \frac{dx}{dt}$ and $x(t)$ satisfies the boundary condition

$x(0) = 1, \dot{x}(0) = 1, x(1) = 2, \dot{x}(1) = 0$

Solution

Let $x_1 = x$ and $x_2 = \dot{x}_1$.

The functional now becomes

$J = \int_0^1 [\ \dot{x}_2{}^2] dt$

with the constraints

$x_2 - \dot{x}_1 = 0.$

The augmented functional is now

$J^* = \int_0^1 [\ \dot{x}_2{}^2] + \lambda[x_2 - \dot{x}_1] dt$

where λ is the Lagrange multiplier

The Euler's equations for x_1 and x_2 are

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and

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Solving the two equations above simultaneously, we have

$2\dddot{x}_2 = 0$

Integrating this, we obtain

$x_2(t) = Pt^2 + Qt + R$

where P,Q and R are constants of integration.

Now since $\dot{x}_1 = x_2$, we can integrate again to get

$x_1(t) = \frac{Pt^3}{3} + \frac{Qt^2}{2} + Rt + S$

where S is an additional constant of integration

Applying the end conditions and solving the resulting equations simultaneously, we obtain

$P = -3, Q = 2, R = 1$ and $S = 1$

Putting these values into our equations for $x_1(t)$ and $x_2(t)$, we have

$x_1(t) = -t^3 + t^2 + t + 1$

$x_2(t) = -3t^2 + 2t + 1$

$\dot{x}_2(t) = -6t + 2$

The optimal path has now been determined. Substituting these in our functional yields

$J = \int_0^1 [\,(-6t + 2)^2]\, dt = 4$

Example 3

Minimize the functional $J = \int_0^1 [\,\ddot{x}^2 - x^2 + t^2]\, dt$

Where $\dot{x} = \dot{x}(t) = \frac{dx}{dt}$ and $x(t)$ satisfies the boundary condition

$x(0) = 1, \dot{x}(0) = 1, x\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}, \dot{x}(\frac{\pi}{2}) = e^{\frac{\pi}{2}}.$

Solution

Let $x_1 = x$ and $x_2 = \dot{x}_1$.

The functional now becomes

$J = \int_0^{\frac{\pi}{2}} [\,\dot{x}_2{}^2 - x_1{}^2 + t^2]\, dt$

with the constraints

$x_2 - \dot{x}_1 = 0.$

The augmented functional is now

$J^* = \int_0^{\frac{\pi}{2}} [\,\dot{x}_2{}^2 - x_1{}^2 + t^2] + \lambda[x_2 - \dot{x}_1]\, dt$

where λ is the Lagrange multiplier

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The Euler's equations for x_1 and x_2 are

$-2x_1 - \frac{d}{dt}(-\lambda) = 0$

$\lambda - \frac{d}{dt}(2\dot{x}_2) = 0$

This implies that

$-2x_1 + \dot{\lambda} = 0$

and

$\lambda - 2\ddot{x}_2 = 0$

Solving the two equations above simultaneously, we have

$-2x_1 + 2\frac{d^3x_2}{dt^3} = 0$

Since $x_2 = \dot{x}_1$, we have

$-2x_1 + 2\frac{d^4x_1}{dt^4} = 0$

Putting $x_1 = e^{mx}$ in the fourth order ordinary differential equation above, we have

$2m^4e^{mx} - 2e^{mx} = 0$

$2e^{mx}(m^4 - 1) = 0$

The auxiliary equation is

$m^4 - 1 = 0$

The general solution of the differential equation is therefore found to be

$x_1 = Pe^t + Qe^{-t} + RCost + SSint$

Since $x_2 = \dot{x}_1$, we have

$x_2 = Pe^t - Qe^{-t} - RSint + SCost$

Applying the end conditions and solving the resulting equations simultaneously, we obtain $P = 1, Q = 0, R = 0$ and $S = 0$

Putting these values into our equations for $x_1(t)$ and $x_2(t)$, we have

$x_1(t) = e^t$

$x_2(t) = e^t$

$\dot{x}_2(t) = e^t$

The optimal path has now been determined. Substituting these in our functional yields

$J = \int_0^1 [\, e^{2t} - e^{2t} + t^2]\, dt = 1.2924$

CONCLUSION

The method of variational calculus has been employed to minimize functionals containing second-order derivatives by the introduction of additional variables in the original theory of variational calculus. This method gives the minimum value for any type of functionals containing at most second-order derivatives as can be seen in the examples considered in this paper.

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