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MEASURE OF DEPARTURE FROM POINT-SYMMETRY FOR TWO-WAY CONTINGENCY TABLES WITH ORDERED CATEGORIES

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ABSTRACT: *For the analysis of two-way contingency tables, the present paper proposes a measure to represent the degree of departure from the point-symmetry for two-way contingency tables with ordered categories. This measure depends on the order of listing the categories. This measure is defined by the cumulative probabilities and expressed by using the Cressie and Read's (1984) power-divergence or Patil and Taillie's (1982) diversity index. It would be useful for comparing the degree of departure from point-symmetry in several tables. An example is given.*

KEYWORDS: Kullback-Leibler Information, Measure, Ordered category, Point-symmetry, Power-divergence.

INTRODUCTION

For an $r \times r$ square contingency table, Wall and Lienert (1976) considered the point-symmetry (PS) model. This model indicates a structure of point-symmetry of the cell probabilities with respect to the center point (when r is even) or the center cell (when r is odd) in square contingency table.

Consider an $r \times c$ rectangular contingency table (including the case of $r = c$). Let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1,...,r; j = 1,...,c$). Tomizawa (1985) extended the PS model for an $r \times c$ contingency table as follows:

$$p_{ij} = \psi_{ij} \quad (i = 1,..,r;\ j = 1,..,c), \tag{1}$$

where $\psi_{ij} = \psi_{i^* j^{**}}$. The symbols "$*$" and "$**$" denote $i^* = r + 1 - i$ and $j^{**} = c + 1 - j$, respectively.

Let

$$E_1 = \left\{ (s,t) \mid (s,t) = (u,v) \text{ for } u = 1,..,\left[\frac{r}{2}\right]; v = 1,..,c; \right.$$

$$\left. \text{and } (s,t) = \left(\left[\frac{r+1}{2}\right], v \right) \text{ for } v = 1,..,\left[\frac{c}{2}\right] \right\},$$

$$E_2 = \left\{ (s,t) \mid (s,t) = (u,v) \text{ for } u = 1,..,r; v = 1,..,\left[\frac{c}{2}\right]; \right.$$

$$\left. \text{and } (s,t) = \left(u, \left[\frac{c+1}{2}\right] \right) \text{ for } u = 1,..,\left[\frac{r}{2}\right] \right\}.$$

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Denote the row and column variables by X and Y, respectively. Then, for a fixed s (s = 1,2), the PS model is also expressed as

$$F_{ij} = G_{ij} \quad \text{for} \quad (i,j) \in E_s, \tag{2}$$

where

$$F_{ij} = \sum_{s=1}^{i}\sum_{t=1}^{j} p_{st}, \quad G_{ij} = \sum_{s=i^*}^{r}\sum_{t=j^{**}}^{c} p_{st}.$$

This indicates that the cumulative probability that an observation will fall in row category i or below and column category j or below is equal to the cumulative probability that the observation falls in row category i^* or above and column category j^{**} or above for $(i,j) \in E_s$.

When the PS model does not hold, we are interested in measuring the degree of departure from point-symmetry. For $r \times c$ contingency tables, Tomizawa, Yamamoto and Tahata (2007) proposed a measure to represent the degree of departure from point-symmetry. This measure, denoted by $\Phi^{(\lambda)}$, is function of the cell probabilities $\{p_{ij}\}$; see Appendix. However, this measure is invariant under symmetric interchange of row and column categories with respect to the center category or point (see Section 4). Thus, this measure would not reflect the order of listing the categories completely. Therefore we are now interested in a measure to represent the degree of departure from point-symmetry which reflects that completely.

Since the PS model can be expressed as equation (1) and the measure $\Phi^{(\lambda)}$ in Appendix is expressed as a function of the cell probabilities $\{p_{ij}\}$, the $\Phi^{(\lambda)}$ would be useful to see how far the cell probabilities $\{p_{ij}\}$ are distant from those with a point-symmetry structure. On the other hand, since the PS model can also be expressed as equation (2), we are also interested in a measure for seeing how far the cumulative probabilities $\{F_{ij}\}$ and $\{G_{ij}\}$ are distant from those with a point-symmetry structure. It seems natural that such a measure should be expressed as a function of $\{F_{ij}\}$ and $\{G_{ij}\}$.

This paper proposes a measure which is expressed as a function of the cumulative probabilities $\{F_{ij}\}$ and $\{G_{ij}\}$. It would be useful for comparing the degree of departure from point-symmetry in several tables.

Measure of departure from point-symmetry

Consider an $r \times c$ contingency table with ordered categories. Assume that $p_{11} + p_{rc} > 0$. For s = 1,2, let

$$\Delta_s = \sum_{(i,j)\in E_s} (F_{ij} + G_{ij}).$$

Also, for $(i,j) \in E_s$ and s = 1,2, let

$$F_{ij(s)} = \frac{F_{ij}}{\Delta_s}, \quad G_{ij(s)} = \frac{G_{ij}}{\Delta_s}, \quad M_{ij(s)} = \frac{F_{ij(s)} + G_{ij(s)}}{2}.$$

We shall consider a measure to represent the degree of departure from point-symmetry, defined by

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$$\Psi^{(\lambda)} = \frac{1}{2}\left(\Psi_1^{(\lambda)} + \Psi_2^{(\lambda)}\right) \quad (\lambda > -1),$$

where

$$\Psi_s^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^{\lambda}-1} I_s^{(\lambda)},$$

with

$$I_s^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{(i,j)\in E_s} \left[F_{ij(s)} \left\{ \left(\frac{F_{ij(s)}}{M_{ij(s)}} \right)^{\lambda} - 1 \right\} + G_{ij(s)} \left\{ \left(\frac{G_{ij(s)}}{M_{ij(s)}} \right)^{\lambda} - 1 \right\} \right],$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. Thus,

$$\Psi_s^{(0)} = \frac{1}{\log 2} I_s^{(0)},$$

where

$$I_s^{(0)} = \sum_{(i,j)\in E_s} \left[F_{ij(s)} \log\left(\frac{F_{ij(s)}}{M_{ij(s)}} \right) + G_{ij(s)} \log\left(\frac{G_{ij(s)}}{M_{ij(s)}} \right) \right].$$

To except difference between row and column variable, this measure $\Psi^{(\lambda)}$ is expressed as the weighted sum of the submeasures $\Psi_1^{(\lambda)}$ and $\Psi_2^{(\lambda)}$. Note that $I_s^{(\lambda)}$ is the power-divergence between $\{F_{ij(s)},G_{ij(s)}\}$ and $\{M_{ij(s)},M_{ij(s)}\}$, and especially $I_s^{(0)}$ is the Kullback-Leibler information between them. For more details of the power-divergence, see Cressie and Read (1984), and Read and Cressie (1988, p. 15). When the PS model holds, we note that $I_s^{(\lambda)} = 0$ ($s = 1,2$). Note that a real value λ is chosen by the user.

Let

$$F_{ij(s)}^{c} = \frac{F_{ij(s)}}{F_{ij(s)} + G_{ij(s)}} \left(= \frac{F_{ij}}{F_{ij} + G_{ij}} \right),$$

and

$$G_{ij(s)}^{c} = \frac{G_{ij(s)}}{F_{ij(s)} + G_{ij(s)}} \left(= \frac{G_{ij}}{F_{ij} + G_{ij}} \right),$$

for $(i,j) \in E_s$ and $s = 1,2$. We note that $\{F_{ij}{}^{c}{}_{(s)} + G_{ij}{}^{c}{}_{(s)} = 1\}$. Also for a fixed s ($s = 1,2$), the PS model can be expressed as as

$$F_{ij(s)}^{c} = G_{ij(s)}^{c} \left(= \frac{1}{2} \right) \quad (i,j) \in E_s.$$

Then the submesure $\Psi_s^{(\lambda)}$ ($s = 1,2$) may be expressed as

$$\Psi_s^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} \sum_{(i,j)\in E_s} \left(F_{ij(s)} + G_{ij(s)}\right) I_{ij(s)}^{(\lambda)},$$

where

$$I_{ij(s)}^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \left[F_{ij(s)}^c \left\{ \left(\frac{F_{ij(s)}^c}{1/2} \right)^\lambda - 1 \right\} + G_{ij(s)}^c \left\{ \left(\frac{G_{ij(s)}^c}{1/2} \right)^\lambda - 1 \right\} \right],$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. Thus,

$$\Psi_s^{(0)} = \frac{1}{\log 2} \sum_{(i,j)\in E_s} \left(F_{ij(s)} + G_{ij(s)}\right) I_{ij(s)}^{(0)},$$

where

$$I_{ij(s)}^{(0)} = F_{ij(s)}^c \log\left(\frac{F_{ij(s)}^c}{1/2} \right) + G_{ij(s)}^c \log\left(\frac{G_{ij(s)}^c}{1/2} \right).$$

Therefore, the $\Psi_s^{(\lambda)}$ would represent essentially the weighted sum of the power-divergence $I_{ij(s)}^{(\lambda)}$.

Furthermore, the submeasure $\Psi_s^{(\lambda)}$ ($s = 1,2$) may be expressed as

$$\Psi_s^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} \sum_{(i,j)\in E_s} \left(F_{ij(s)} + G_{ij(s)}\right) H_{ij(s)}^{(\lambda)},$$

where

$$H_{ij(s)}^{(\lambda)} = \frac{1}{\lambda}\left[1 - \left(F_{ij(s)}^c\right)^{\lambda+1} - \left(G_{ij(s)}^c\right)^{\lambda+1} \right],$$

with

$$\Psi_s^{(0)} = 1 - \frac{1}{\log 2} \sum_{(i,j)\in E_s} \left(F_{ij(s)} + G_{ij(s)}\right) H_{ij(s)}^{(0)},$$

where

$$H_{ij(s)}^{(0)} = -F_{ij(s)}^c \log F_{ij(s)}^c - G_{ij(s)}^c \log G_{ij(s)}^c.$$

Note that $H_{ij(s)}^{(\lambda)}$ is the Patil and Taillie's (1982) diversity index for $\{F_{ij}{}^c{}_{(s)}, G^c{}_{ij(s)}\}$, which includes the Shannon entropy when $\lambda = 0$ in special cases. The $\Psi_s^{(\lambda)}$ would represent essentially the weighted sum of the diversity index $H_{ij(s)}^{(\lambda)}$.

We note that $0 \le H_{ij(s)}^{(\lambda)} \le (2^\lambda - 1)/\lambda 2^\lambda$. Thus, the submeasure $\Psi_s^{(\lambda)}$ ($s = 1,2$) lies between 0 and 1, and the measure $\Psi^{(\lambda)}$ also lies between 0 and 1. We note that for each λ (> -1), (i) $\Psi^{(\lambda)} = 0$

(i.e., $\Psi_1^{(\lambda)} = \Psi_2^{(\lambda)} = 0$) if and only if the PS model holds, and (ii) $\Psi^{(\lambda)} = 1$ (i.e., $\Psi_1^{(\lambda)} = \Psi_2^{(\lambda)} = 1$) if and only if the degree of departure from point-symmetry is the largest in the sense that $F_{ij(s)}^c = 0$ (then $G_{ij(s)}^c = 1$) or $G_{ij(s)}^c = 0$ (then $F_{ij(s)}^c = 1$), namely, $F_{ij} = 0$ (then $G_{ij} > 0$) or $G_{ij} = 0$ (then $F_{ij} > 0$) for all $(i,j) \in E_s$ and $s = 1,2$.

According to weighted sum of the power-divergence or weighted sum of the Patil and Taillie's diversity index, the measure $\Psi^{(\lambda)}$ represents the degree of departure from point-symmetry, and the degree increases as the value of $\Psi^{(\lambda)}$ increases.

Approximate confidence interval for measure

Let n_{ij} denote the observed frequency in the ith row and jth column of the table ($i = 1,...,r; j = 1,...,c$). Assuming that a multinomial distribution applies to the $r \times c$ table, we shall consider an approximate standard error and large-sample confidence interval for the measure $\Psi^{(\lambda)}$ using the delta method, descriptions of which are given by Bishop, Fienberg and Holland (1975, Sec 14. 6). The sample version of $\Psi^{(\lambda)}$, i.e., $\hat{\Psi}^{(\lambda)}$, is given by $\Psi^{(\lambda)}$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum\sum n_{ij}$. Using the delta method, $\sqrt{n}(\hat{\Psi}^{(\lambda)} - \Psi^{(\lambda)})$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2[\Psi^{(\lambda)}] = \frac{1}{4}\sum_{k=1}^{r}\sum_{l=1}^{c} p_{ij}\left\{\frac{1}{\Delta_1}\left(d_{kl(1)}^{(\lambda)} - \Psi_1^{(\lambda)} w_{kl(1)}\right) + \frac{1}{\Delta_2}\left(d_{kl(2)}^{(\lambda)} - \Psi_2^{(\lambda)} w_{kl(2)}\right)\right\}^2,$$

where, for $\lambda > -1; \lambda \neq 0$,

$$d_{kl(s)}^{(\lambda)} = \begin{cases} \dfrac{1}{2^\lambda - 1}\displaystyle\sum_{(i,j)\in W_{kl(s)}} \left[(2F_{ij(s)}^c)^\lambda - 1 + \lambda G_{ij(s)}^c\left\{(2F_{ij(s)}^c)^\lambda - (2G_{ij(s)}^c)^\lambda\right\}\right] & \text{for } (k,l) \in E_s, \\ \dfrac{1}{2^\lambda - 1}\displaystyle\sum_{(i,j)\in W_{k^*l^{**}(s)}} \left[(2G_{ij(s)}^c)^\lambda - 1 + \lambda F_{ij(s)}^c\left\{(2G_{ij(s)}^c)^\lambda - (2F_{ij(s)}^c)^\lambda\right\}\right] & \text{for } (k^*,l^{**}) \in E_s, \\ 0 \text{ for otherwise,} & \end{cases}$$

and for $\lambda = 0$,

$$d_{kl(s)}^{(0)} = \begin{cases} \dfrac{1}{\log 2}\displaystyle\sum_{(i,j)\in W_{kl(s)}} \log\left(2F_{ij(s)}^c\right) & \text{for } (k,l) \in E_s, \\ \dfrac{1}{\log 2}\displaystyle\sum_{(i,j)\in W_{k^*l^{**}(s)}} \log\left(2G_{ij(s)}^c\right) & \text{for } (k^*,l^{**}) \in E_s, \\ 0 & \text{for otherwise,} \end{cases}$$

with

$$w_{kl(s)} = \begin{cases} \#(W_{kl(s)}) & \text{for } (k,l) \in E_s, \\ \#(W_{k^*l^{**}(s)}) & \text{for } (k^*,l^{**}) \in E_s, \\ 0 & \text{for otherwize}, \end{cases}$$

and where,

$$W_{kl(1)} = \left\{ (s,t) \mid (s,t) = (u,v) \text{ for } u = k,..,\left[\frac{r}{2}\right]; v = l,..,c; \right.$$
$$\left. \text{and } (s,t) = \left(\left[\frac{r+1}{2}\right], v\right) \text{ for } v = l,..,\left[\frac{c}{2}\right] \right\},$$
$$W_{kl(2)} = \left\{ (s,t) \mid (s,t) = (u,v) \text{ for } u = k,..,r; v = l,..,\left[\frac{c}{2}\right]; \right.$$
$$\left. \text{and } (s,t) = \left(u, \left[\frac{c+1}{2}\right]\right) \text{ for } u = k,..,\left[\frac{r}{2}\right] \right\},$$

and $\#(A)$ is the number of elements in set A.

Let $\hat{\sigma}^2[\Psi^{(\lambda)}]$ denote $\sigma^2[\Psi^{(\lambda)}]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then, $\hat{\sigma}[\Psi^{(\lambda)}]/\sqrt{n}$ is an approximate estimated standard error for $\hat{\Psi}^{(\lambda)}$, and $\hat{\Psi}^{(\lambda)} \pm z_{\alpha/2}\hat{\sigma}[\Psi^{(\lambda)}]/\sqrt{n}$ is an approximate $100(1 - \alpha)\%$ confidence interval for $\Psi^{(\lambda)}$, where $z_{\alpha/2}$ is the percentage point from the standard normal distribution that corresponds to two-tail probability equal to α.

Comparison between measures

We shall compare the measures $\Psi^{(\lambda)}$ with $\Phi^{(\lambda)}$ defined by Tomizawa et al. (2007). Consider the artificial data in Table 1a, and the modified data in Table 1b which are obtained by interchanging categories 2 and 3 in Table 1a. Then from Tables 2a and 2b, we can see that for each λ, (i) the value of $\hat{\Phi}^{(\lambda)}$ for Table 1a is equal to that for Table 1b, and (ii) the value of $\hat{\Psi}^{(\lambda)}$ for Table 1a is greater than that for Table 1b. Generally, the measure $\Phi^{(\lambda)}$ is invariant under symmetric interchange with respect to the center category or point, however the new measure $\Psi^{(\lambda)}$ is not invariant under arbitrary permutations of row and column categories (including symmetric interchange with respect to the center category or point).

Analysis of data

Consider two sets of data on unaided distance vision of 3168 pupils comprising nearly equal number of boys and girls aged 6-12 at elementary schools in Tokyo, Japan, examined in June 1984 (Table 3) and 4746 students aged 18 to about 25 including about 10% women in Faculty of Science and Technology, Tokyo University of Science in Japan, examined in April 1982 (Table 4). In these data, the row variable is the right eye grade and the column variable is the left eye grade with the categories ordered from the Best (1) to the Worst (4). The data in Table 3 have been analyzed by Miyamoto et al. (2004). The data in Table 4 have also been analyzed by Tomizawa (1984, 1985).

Since the approximate confidence intervals for $\Psi^{(\lambda)}$ applied to the data in Tables 3 and 4 do not include 0 for each λ (See Table 5), these would indicate that there is not a structure of point-

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symmetry in each table. When the degree of departure from point-symmetry in Tables 3 and 4 are compared using the approximate confidence interval for $\Psi^{(\lambda)}$, the degree of departure is greater in Table 3 than in Table 4.

CONCLUDING REMARKS

The measure $\Psi^{(\lambda)}$ always ranges between 0 and 1 with independent of the sample size n. Therefore, $\Psi^{(\lambda)}$ may be useful for comparing the degree of departure from point-symmetry in several tables. Also the measure $\Psi^{(\lambda)}$ would be useful when we want to see how degree the departure from point-symmetry is toward the maximum departure from point-symmetry.

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APPENDIX

For an $r \times c$ table, the measure to represent the degree of departure from point-symmetry considered by Tomizawa et al. (2007) is given as follows: Assume that $\{p_{ij} + p_{i*j**}\} > 0$,

$$\Phi^{(\lambda)} = \frac{1}{2^{\lambda} - 1} \sum_{(i,j)\in D} q_{ij} \left[\left(\frac{q_{ij}}{q_{ij}^{PS}} \right)^{\lambda} - 1 \right] \quad (\lambda > -1),$$

where

$$\delta = \sum_{(i,j)\in D} p_{ij}, \quad q_{ij} = \frac{p_{ij}}{\delta}, \quad q_{ij}^{PS} = \frac{q_{ij} + q_{i*j**}}{2},$$

and (i) when r is odd and c is odd,

$$D = \left\{ (i,j) \mid i = 1,..,r;\ j = 1,..,c;\ (i,j) \neq \left(\frac{r+1}{2}, \frac{c+1}{2} \right) \right\},$$

and (ii) otherwise,

$$D = \{ (i,j) \mid i = 1,..,r;\ j = 1,..,c \}.$$

The value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$.

Table 1: (a) Artificial data and (b) modified data obtained by interchanging the row and column categories 2 and 3.

(a) $n = 300$ (sample size)

	(1)	(2)	(3)	(4)
(1)	80	40	30	10
(2)	8	60	20	15
(3)	4	2	9	3
(4)	1	6	7	5

(b) $n = 300$ (sample size)

	(1)	(3)	(2)	(4)
(1)	80	30	40	10
(3)	4	9	2	3
(2)	8	20	60	15
(4)	1	7	6	5

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Table 2: The values of $\hat{\Psi}^{(\lambda)}$ and $\hat{\Phi}^{(\lambda)}$ applied to Tables 1a and 1b.

(a) For Table 1a

Values of λ	$\hat{\Psi}^{(\lambda)}$	$\hat{\Phi}^{(\lambda)}$
-0.4	0.316	0.368
0	0.416	0.481
0.6	0.489	0.559
1.0	0.508	0.579
1.8	0.511	0.583

(b) For Table 1b

Values of λ	$\hat{\Psi}^{(\lambda)}$	$\hat{\Phi}^{(\lambda)}$
-0.4	0.158	0.368
0	0.209	0.481
0.6	0.245	0.559
1.0	0.255	0.579
1.8	0.257	0.583

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Table 3: Unaided distance vision of 3168 pupils comprising nearly equal number of boys and girls aged 6-12 at elementary schools in Tokyo, Japan, examined in June 1984; from Tomizawa (1985).

Right eye	Left eye grade				
grade	Best (1)	Second (2)	Third (3)	Worst (4)	Total
Best (1)	2470	126	21	10	2627
Second (2)	96	138	33	5	272
Third (3)	10	42	75	15	142
Worst (4)	12	7	16	92	127
Total	2588	313	145	122	3168

Table 4: Unaided distance vision of 4746 students aged 18 to about 25 including about 10% women in Faculty of Science and Technology, Tokyo University of Science, Japan, examined in April 1982; from Tomizawa (1984).

Right eye	Left eye grade				
grade	Best (1)	Second (2)	Third (3)	Worst (4)	Total
Best (1)	1291	130	40	22	1483
Second (2)	149	221	114	23	507
Third (3)	64	124	660	185	1033
Worst (4)	20	25	249	1429	1723
Total	1524	500	1063	1659	4746

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Table 5: Estimate of measure $\Psi^{(\lambda)}$, approximate standard error for $\hat{\Psi}^{(\lambda)}$ and approximate 95% confidence interval for $\Psi^{(\lambda)}$, applied to Tables 3 and 4.

(a) For Table 3

Values of λ	Estimated measure	Standard error	Confidence interval
-0.4	0.562	0.016	(0.531, 0.592)
0	0.696	0.015	(0.666, 0.727)
0.6	0.775	0.014	(0.748, 0.802)
1.0	0.793	0.013	(0.767, 0.819)
1.8	0.796	0.013	(0.770, 0.821)

(b) For Table 4

Values of λ	Estimated measure	Standard error	Confidence interval
-0.4	0.007	0.002	(0.004, 0.010)
0	0.010	0.002	(0.006, 0.015)
0.6	0.013	0.003	(0.007, 0.019)
1.0	0.014	0.003	(0.008, 0.020)
1.8	0.014	0.003	(0.008, 0.021)

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