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MAXIMUM LIKELIHOOD METHOD: AN ALTERNATIVE TO ORDINARY LEAST SQUARE METHOD IN ESTIMATING THE PARAMETERS OF SIMPLE WEIBULL DISTRIBUTION USING LARGE SAMPLES OF TYPE-I CENSORED OBSERVATION.

Osuji G.A[1], Okoro C.N[1] and Nwosu D.F[2].

1. Department of statistics, Nnamdi Azikiwe University, P.M.B. 5025, Awka, Anambra State, Nigeria.

2. Department of Mathematics / Statistics, Federal Polytechnic Nekede Owerri.

ABSTRACT: *This study is concerned with two-parameter Weibull distribution which is very important in life testing and reliability analysis. Two methods viz: maximum likelihood estimation (MLE) and Ordinary Least Squares (OLS) are good alternatives in estimating the parameters of a simple Weibull distribution as the sample sizes increases. These estimators are derived for Random Type-I censored samples. These methods were compared by looking at their standard errors through simulation study with sample sizes of 100, 300, 500 and 1000. It was observed that MLE stands out when estimating the parameters of the Weibull distribution as the sample size increases compared to the OLSM. We also noted that both OLSM and MLE provides asymptotically normally distributed estimator.*

KEYWORDS: Random Type-I Censoring, Ordinary Least Square Estimation, Maximum Likelihood Estimation, Simulation Study and Weibull Distribution.

INTRODUCTION

As a result of the versatility in fitting time-to-failure of a very extensive variety to complex mechanisms, the Weibull distribution has lately assumed the center stage more especially in the field of life testing and reliability analysis. Censoring is a feature that is recurrent in lifetime and reliability data analysis, it occurs when exact lifetimes or run-outs can only be collected for a portion of the inspection units. According to Horst (2009), "A data sample is said to be censored when, either by accident or deign, the value of the variables under investigation is unobserved for some of the items in the sample". If lifetime are only known to exceed some given time or assumed to have the potential of exceeding but for certain reasons, may be due to removal or withdrawal then it is referred to as right censoring. There are basically two types of right censoring and they are: Type-I and Type-II censoring. Type-I censoring can be classified into two viz: Fixed Type-I censoring and Random Type-I censoring. The main focus of this research is on Random Type-I censoring. This is where the study is designed to end after a specified given time T and the censoring time unlike the Fixed Type-II censoring scheme where at the end of the study every unit that did not have an event observed during the course of study is censored at time T. Therefore in this work, we seek to examine the standard errors

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of the estimated parameters of the Weibull distribution as the sample size increases so as to determine which method is the best for large sample size. It has been observed that the MLE cannot be obtained in a closed form therefore Newton-Raphson method (iterative method) has been proposed to solve the non-linear nature of the equations. For prove on how to obtain MLE of Type-I censoring see, Panam and Saeid (2011), Stefano et al (2007) and Saunders (2007).

Maximum Likelihood Estimation

The probability density function, the cumulative distribution function and the survival function of a two-parameter Weibull distribution with scale parameter $\alpha > 0$ and shape parameter $\beta>0$, are given respectively by

$$f(t_i; \alpha, \beta) = (\frac{\beta}{\alpha})(\frac{t_i}{\alpha})^{\beta-1} \exp[-(\frac{t_i}{\alpha})]^{\beta} \quad (1)$$

The cumulative distribution function is:

$$F(t_i; \alpha, \beta) = 1 - \exp[-(\frac{t_i}{\alpha})]^{\beta} \quad (2)$$

The survival function is given as:

$$S(t_i; \alpha, \beta) = 1 - \{1 - \exp[-(\frac{t_i}{\alpha})]^{\beta}\} \quad (3)$$

Suppose that $t_i, < \ldots < t_t$ is known to have failed during the study and the remaining $t_n - t_T = t_q$

Censored but the censored units do not all have the same censoring time, then the likelihood function of the two-parameter Weibull distribution as stated by Saunders (2007) is

$$L(t_i;\alpha,\beta) = \prod_{i=1}^{r} f(t_i) \prod_{i=1}^{r} F(t) = \prod_{i=1}^{r} h(t_i) \prod_{i=1}^{r} F(t_i) * F(t_q)^{n-r} \quad (4)$$

$$L(t_i;\alpha,\beta) = \prod_{i=1}^{r} [\frac{\beta}{\alpha}\left(\frac{t_i}{\alpha}\right)^{\beta-1} \exp[-\left(\frac{t_i}{\alpha}\right)]^{\beta}][1-(1-\exp[-\left(\frac{t_i}{\alpha}\right)]^{\beta})]^{n-r} \quad (5)$$

The likelihood is

$$\ln L = r\ln(\beta) - r\beta\ln(\alpha) + (\beta-1)\sum_{i=1}^{r}\ln(t_i) - \sum_{i=1}^{r}\left(\frac{t_i}{\alpha}\right)^{\beta} - [n - r\left(\frac{t_i}{n}\right)^{\beta}] \quad (6)$$

Differentiating (7) w.r.t α and β and equating to zero, we have

$$\alpha = [\frac{1}{r}(\sum_{i=1}^{r}(t_i)^{\beta} + (n-r)(t_q)^{\beta})]^{\frac{1}{\beta}} \quad (7)$$

and

$$\frac{r}{\beta} + \sum_{i=1}^{r}\left(\frac{t_i}{\alpha}\right)^{\beta}\ln\left(\frac{t_i}{\alpha}\right) - (n-r)\left(\frac{t_q}{\alpha}\right)^{\beta}\ln\left(\frac{t_q}{\alpha}\right) = 0 \quad (8)$$

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Substituting (8) into (9) we have

$$0 = \frac{r}{\beta} - \sum_{i=1}^{r}\left[\frac{(t_i)^\beta}{[\frac{1}{r}(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta]}\right] \ln\left[\frac{(t_i)}{[[\frac{1}{r}(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta)]^{\frac{1}{\beta}}}\right] - \left[\frac{(n-r)(t_q)^\beta}{[[\frac{1}{r}(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta)]}\right] +$$

$$\ln\left[\frac{(t_q)}{[[\frac{1}{r}(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta)]^{\frac{1}{\beta}}}\right] + \sum_{i=1}^{r}\ln\left[\frac{(t_i)}{[[\frac{1}{r}(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta)]^{\frac{1}{\beta}}}\right] \quad (9)$$

Solving (10) using Newton-Raphson iterative method we have,

$$\text{Let} \quad f'(\beta) = \frac{r}{\beta^2} - \sum_{i=1}^{r}\left[\frac{(t_q)}{[[\frac{1}{r}(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta)]^{\frac{1}{\beta}}}\right] - \left[\frac{(n-r)(t_q)^\beta}{[[\frac{1}{r}(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta)]}\right] \ln^2\left[\frac{(t_q)}{[[\frac{1}{r}(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta)]^{\frac{1}{\beta}}}\right] \quad (10)$$

β is estimated by assuming an initial value and solving equation 12 below repeatedly until it converges ,after which we can determine $\propto$.

$$\beta_{i+1} = \beta_i - \left\{\frac{\frac{r}{\beta} - S_{11} - S_{12} + S_{13}}{-\frac{r}{\beta^2} - S_{14} - S_{15}}\right\} \quad (11)$$

Where

$$S_{11} = \sum_{i=1}^{r}\left[\frac{(t_i)^\beta}{\frac{1}{r}[\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta]}\right] In\left[\frac{t_i}{\left[\frac{1}{r}\left(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta\right)\right]^{1/\beta}}\right]$$

$$S_{12} = \left\{\frac{(n-r)(t_q)^\beta}{\left[\frac{1}{r}\left(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta\right)\right]}\right\} In\left\{\frac{(t_q)}{\left[\frac{1}{r}\left(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta\right)\right]^{1/\beta}}\right\}$$

$$S_{13} = \sum_{i=1}^{r} In\left[\frac{(t_i)}{\left[\frac{1}{r}\left(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta\right)\right]^{1/\beta}}\right]$$

$$S_{14} = \sum_{i=1}^{r}\left[\frac{(t_i)^\beta}{\frac{1}{r}[\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta]}\right] In^2\left[\frac{(t_i)}{\left[\frac{1}{r}\left(\sum_{i=1}^{r}(t_i)^\beta + (n-r)(t_q)^\beta\right)\right]^{1/\beta}}\right]$$

$$S_{15} = \left[\frac{(n-r)(t_q)^{\beta}}{\left[\frac{1}{r}(t_i)^{\beta}+(n-r)(t_q)^{\beta}\right]}\right] In^2 \left[\frac{(t_q)}{\left[\frac{1}{r}\left(\sum_{i=1}^{r}(t_i)^{\beta}+(n-r)(t_q)^{\beta}\right)\right]^{1/\beta}}\right]$$

Ordinary Least Squares Method

For the estimation of Weibull distribution, the least squares method is extremely used in Engineering and Mathematical problems. We can obtain the linear relationship between the two parameters by taking the double logarithm of the CDF of a two parameter Weibull.

Taking the logarithm twice on equation (2), we have;

$$In\left[-In\left(1 - F(t_i)\right)\right] = \beta In(t_i) - \beta In(\alpha) \qquad (12)$$

Equation (12) can be represented by;

$y_i = In\left[-In\left(1 - F(t_i)\right)\right]$ and $x_i = In(t_i)$ (13)

Where $y_i = \beta x_i - \beta In(\alpha)$ (14)

If $A = \sum_{i-1}^{r}(y_i - y)^2$

Then,

$$A = \sum_{i=1}^{r}\left[y_i - \left(\beta x - \beta In(\alpha)\right)\right]^2 \qquad (15)$$

Differentiating (15) with respect to α and β and equating to zero, the estimating equation of LSY as obtained by Zhang et al (2007) is;

$$\hat{\beta}_{xy} = \frac{\sum_{i=1}^{r}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{r}(x_i - \bar{x})^2} \qquad (16)$$

And

$$\hat{\alpha}_{xy} = exp\left[-\left(\frac{\bar{y}}{\hat{\beta}_{xy}} - \bar{x}\right)\right] \qquad (17)$$

Where;

$$\bar{x} = \frac{\sum_{i=1}^{r} x_i}{r} \quad and \ \bar{y} = \frac{\sum_{i=1}^{r} y_i}{r}$$

Fisher Information Matrix

The Fisher information matrix composed of the negative second partial derivatives of log likelihood function can be written as;

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$$F = \begin{bmatrix} \frac{-\partial^2 l}{\partial \alpha^2} & \frac{-\partial^2 l}{\partial \alpha \partial \beta} \\ \frac{-\partial^2 l}{\partial^2 \beta \partial \alpha} & \frac{-\partial^2 l}{\partial \beta^2} \end{bmatrix}$$

$$= \begin{bmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}\hat{\beta}) \\ Cov(\hat{\beta}\hat{\alpha}) & Var(\hat{\alpha}) \end{bmatrix}$$

Where the element of the Fisher information matrix are obtained as;

$$\frac{\partial^2 l}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \sum_{i=1}^{r} \left(\frac{t}{\beta}\right)^{\alpha} In^2 \left(\frac{t}{\beta}\right)$$

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{\alpha}{\beta^2} \left[n - (\alpha - 1) \sum_{i=1}^{r} \left(\frac{t}{\beta}\right)^{\alpha}\right]$$

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = \frac{1}{\beta} \sum_{i=1}^{r} \left(\frac{t}{\beta}\right)^{\alpha} + \frac{\alpha}{\beta} \sum_{i=1}^{r} \left(\frac{t}{\beta}\right)^{\alpha} + In\left(\frac{t}{\beta}\right) - \frac{r}{\beta}$$

The elements of the above matrix otherwise known as Fisher Information Matrix or the variance covariance matrix cannot be obtained algebraically but through the use of statistical software, in this case, Minitab.

Now, the variance covariance matrix of parameters is given by;

$$\Sigma = \begin{bmatrix} \frac{-\partial^2 l}{\partial \alpha} & \frac{-\partial^2 l}{\partial^2 \alpha \partial \beta} \\ \frac{-\partial^2 l}{\partial \beta \partial \alpha} & \frac{-\partial^2 l}{\partial \beta^2} \end{bmatrix}$$

$$= \begin{bmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}\hat{\beta}) \\ Cov(\hat{\beta}\hat{\alpha}) & Var(\hat{\beta}) \end{bmatrix}$$

The 100(1-r) % asymptotic confidence interval for α and β are given respectively by;

$$\left[\hat{\alpha} \pm Z_{1-\frac{r}{2}} \sqrt{var(\hat{\beta})}\right]$$

We now apply the Anderson Darling goodness of fit test to ascertain the goodness of fit to the Weibull distribution which is given by;

AD= - [$\sum_{i=1}^{n} \frac{1-2i}{n}$ { ln($F(t_i)$ + ln(1- F(t_{n+1-i}))} – n]

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Where:

F is the cumulative distribution function of the specified distribution

t_i are the ordered data

At 5% level of significance, the null hypothesis (H_o) is rejected if the P-value is less than the level of significance.

Simulation study

In trying to illustrate and compare the methods as described above, random samples of size 100, 300, 500 and 1000 were used. The scale and shape parameters were chosen to be 1 and 0.5 respectively. The parameters were estimated using the methods above. The comparisons were based on the standard errors of the estimates and the result shown in the table below.

Table 1

Censoring Information	Count
Uncensored value	100

Estimation Method: Least Squares (failure time(X) on rank(Y))

Distribution: Weibull

Parameter Estimates

		Standard	95.0% Normal CI	
Parameter	Estimate	Error	Lower	Upper
Shape	0.917671	0.0711279	0.788336	1.06823
Scale	0.568663	0.0652761	0.454095	0.712137

Log-Likelihood = -45.889

Goodness-of-Fit
Anderson-Darling (adjusted) = 0.845

Censoring Information	Count
Uncensored value	100

Estimation Method: Maximum Likelihood

Distribution: Weibull

Parameter Estimates

		Standard	95.0% Normal CI	
Parameter	Estimate	Error	Lower	Upper
Shape	0.926735	0.0707142	0.798004	1.07623

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Scale	0.564052	0.0641156	0.451402	0.704815

Log-Likelihood = -45.874

Goodness-of-Fit
Anderson-Darling (adjusted) = 0.829

Table 2

Censoring Information	Count
Uncensored value	300

Estimation Method: Least Squares (failure time(X) on rank(Y))

Distribution: Weibull

Parameter Estimates

		Standard	95.0% Normal CI	
Parameter	Estimate	Error	Lower	Upper
Shape	1.02830	0.0479686	0.938450	1.12675
Scale	0.501472	0.0296307	0.446634	0.563044

Log-Likelihood = -83.324

Goodness-of-Fit
Anderson-Darling (adjusted) = 0.275

Censoring Information	Count
Uncensored value	300

Estimation Method: Maximum Likelihood

Distribution: Weibull

Parameter Estimates

		Standard	95.0% Normal CI	
Parameter	Estimate	Error	Lower	Upper
Shape	1.05455	0.0474185	0.965591	1.15171
Scale	0.496706	0.0286247	0.443655	0.556100

Log-Likelihood = -83.108

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Table 3

Censoring Information	Count
Uncensored value	500

Estimation Method: Least Squares (failure time(X) on rank(Y))

Distribution: Weibull

Parameter Estimates

		Standard	95.0% Normal CI	
Parameter	Estimate	Error	Lower	Upper
Shape	0.887444	0.0343636	0.822584	0.957417
Scale	0.480563	0.0254492	0.433185	0.533123

Log-Likelihood = -135.947

Goodness-of-Fit
Anderson-Darling (adjusted) = 1.102
Correlation Coefficient = 0.997

Censoring Information	Count
Uncensored value	500

Estimation Method: Maximum Likelihood

Distribution: Weibull

Parameter Estimates

		Standard	95.0% Normal CI	
Parameter	Estimate	Error	Lower	Upper
Shape	0.945619	0.0333937	0.882383	1.01339
Scale	0.470489	0.0233749	0.426835	0.518608

Log-Likelihood = -133.932

Goodness-of-Fit
Anderson-Darling (adjusted) = 0.645

Table 4

Censoring Information	**Count**
Uncensored value	**1000**

Estimation Method: Least Squares (failure time(X) on rank(Y))

Distribution: Weibull

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Parameter Estimates

Parameter	Estimate	Standard Error	95.0% Normal CI Lower	Upper
Shape	0.978834	0.0242224	0.932492	1.02748
Scale	0.499534	0.0169948	0.467311	0.533979

Log-Likelihood = -305.911

Goodness-of-Fit
Anderson-Darling (adjusted) = 0.973

Censoring Information	Count
Uncensored value	1000
Censoring Information	Count
Uncensored value	1000

Estimation Method: Maximum Likelihood

Distribution: Weibull

Parameter Estimates

Parameter	Estimate	Standard Error	95.0% Normal CI Lower	Upper
Shape	0.985418	0.0241313	0.939238	1.03387
Scale	0.496402	0.0167761	0.464587	0.530396

Log-Likelihood = -305.833

Goodness-of-Fit
Anderson-Darling (adjusted) = 0.891

Distribution: Weibull

Parameter Estimates

Parameter	Estimate	Standard Error	95.0% Normal CI Lower	Upper
Shape	0.985418	0.0241313	0.939238	1.03387
Scale	0.496402	0.0167761	0.464587	0.530396

Log-Likelihood = -305.833

Goodness-of-Fit
Anderson-Darling (adjusted) = 0.891

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Table 5: Summary result of simulation study with α= 1, β= 0.5

	OLS		MLE	
n	A B	SE	α β	SE
100	0.917671 0.568663	0.0711279 0.0652761	0.926735 0.564052	0.0707142 0.0641156
300	1.02830 0.501472	0.0479686 0.0296307	1.05455 0.496706	0.0474185 0.0286247
500	0.887444 0.480563	0.0343636 0.0254492	0.945619 0.470489	0.0333937 0.0233749
1000	0.978834 0.499534	0.0242224 0.0169948	0.9851418 0.496402	0.0211313 0.0167761

DISCUSSION AND CONCLUSIONS

From the results in Table 1, it is easy to find that the estimates of the parameter perform well. For fixed $\propto and\ \beta$, the S.E of $\propto$ and β decreases as n increases. This indicates that both OLS and MLE provides asymptomatically, normally distributed and consistent estimator for the parameters.

From the above result, we could find out that the MLE estimates has a minimum S.E for all sample sizes compared to the estimates of OLS, therefore making MLE a better option to OLS for large sample sizes.

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