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MARSHALL-OLKIN EXTENDED POWER FUNCTION DISTRIBUTION

I. E. Okorie[1], A. C. Akpanta[2] and J. Ohakwe[3]

[1]School of Mathematics, University of Manchester, Manchester M13 9PL, UK
[2]Department of Statistics, Abia State University, Uturu, Abia State, Nigeria
[3]Department of Mathematics & Statistics, Faculty of Sciences, Federal University Otuoke, Bayelsa State, P.M.B 126 Yenagoa, Bayelsa, Nigeria

ABSTRACT: *This article introduces the Marshall-Olkin extended power function (MOEP$_F$) distribution as a generalization of the standard power function distribution. The new distribution has a bathtub shaped hazard rate function. The MOEP$_F$ distribution have the beta and power function distribution as special cases. Some statistical and reliability properties of the new distribution were given and the method of maximum likelihood estimates was used to estimate the model parameters. The relevance and exibility of the MOEP$_F$ distribution was demonstrated with two di erent real and uncensored lifetime data sets. The goodness of ts of the distribution was assessed via the p − value criterion. All the three parameters of tted MOEP$_F$ distribution were found statistically signi cant based on their corresponding p − values (p − value = 2.20 × 10^{-16} for each of the three parameters). The MOEP$_F$ distribution is therefore recommended for e ective modelling of lifetime data.*

KEYWORDS: Power Function Distribution,Marshall-Olkin, Reliability; P−value

INTRODUCTION

The two parameter power function distribution is widely used in reliability engineering to model components, systems or device reliability. The intuitive simplicity of the power function distribution makes it most appealing to reliability engineers; for example, Meniconi and Barry[1] proposed the power function distribution for assessing the reliability of electrical components due to its simplicity. But in the ideal context, the goodness of t of a model should not be compromised for its simplicity. The standard probability distributions have been remarked for their lack of ts in modelling data sets that are generated from various complex processes. Over a decade ago, many researchers have proposed various methods of modifying standard distributions as a way of remedying the lack of ts that is akin to them.

Marshall and Olkin [2] introduced a new family of distributions known as the MarshallOlkin extended/generalized distributions. The Marshall-Olkin's approach is well known for its ability of enhancing the exibility of probability distributions through an introduction of an additional parameter to the original distribution. The very robust Marshall-Olkin family of distributions can represent a variety of data sets from a wide range of complex phenomena. The Marshall-Olkin family of distributions can be obtained as follows,

$$\bar{F}(x) = \frac{\gamma \bar{G}(x)}{1-(1-\gamma)\bar{G}(x)}; -\infty < x < \infty; 0 < \gamma < \infty. \quad (1.1)$$

It follows that $F(x) = 1 - \bar{F}(x)$ and

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$$f(x) = \frac{\gamma g(x)}{\left(1-(1-\gamma)\bar{G}(x)\right)^2}; -\infty < x < \infty; 0 < \gamma < \infty \quad , \quad (1.2)$$

where $\bar{G}(x)$ and $g(x)$ are the complementary cumulative density function (survival/reliability function) and density function corresponding to the baseline distribution (original distribution).

So many existing standard distributions have received a fair share of their Marshall-Olkin extended counterparts from various researchers. For instance; Ristić and Kundu [3] introduced the Marshall-Olkin generalized exponential distribution generalizing the exponential distribution. Ghitany et al. [4] introduced the Marshall-Olkin extended Weibull distribution as a generalization of the standard Weibull distribution. Ghitany [5] introduced the MarshallOlkin extended Pareto distribution as a generalization of the standard Pareto distribution. Ristić et al.[6] introduced the Marshall-Olkin extended gamma distribution as a generalization of the standard gamma distribution. Ghitany et al.[7] introduced the Marshall-Olkin extended Lomax distribution as a generalization of the standard Lomax distribution. Jose and Krishna [8] introduced the Marshall-Olkin extended continuous uniform distribution as a generalization of the standard continuous uniform distribution. Al-Saiari et al. [9] introduced the Marshall-Olkin extended Burr type XII distribution as a generalization of the standard Burr type XII distribution. Alizadeh et al. [10] introduced the Marshall-Olkin extended Kumaraswamy distribution as a generalization of the standard Kumaraswamy distribution. Gui [11] introduced the Marshall-Olkin extended log-logistic distribution as a generalization of the standard log-logistic distribution. Pogány et al. [12] introduced the Marshall-Olkin extended exponential Weibull distribution generalizing the exponential Weibull distribution. Jose [13] gave a comprehensive review of the Marshall-Olkin family of distributions and their applications to reliability, time series and stress-strength analysis. For more extensive reviews of the Marshall-Olkin generalized family of distributions see; Nadarajah [14] and Barreto-Souza et al. [15]. Sandhya and Prasanth [16] introduced the Marshall-Olkin extended discrete uniform distribution as a generalizion of the standard discrete uniform distribution; etc. Analogously, this article introduces the three parameter Marshall-Olkin extended power function ($MOEP_F$) distribution as a generalization of the standard two parameter power function distribution. Note; there is a slight di erence in model parameterization between the Marshall-Olkin extended power ($MOEP_o$) distribution that was discussed without any comprehensive account of its distributional properties in Barreto-Souza et al. [15] and the $MOEP_F$ distribution studied here.

The rest of this article is organized as follows: Section 2 introduces the power function distribution and the Marshall-Olkin extended power function ($MOEP_F$) distribution; Section 3 presents some reliability characteristics of the ($MOEP_F$) distribution such as the reliability function, hazard rate function and the mean residual life time; Section 4 presents some statistical properties of the ($MOEP_F$) distribution such as the *kth* crude moment, moment generating function, *pth* quantile function, Rényi entropy measure of the ($MOEP_F$) distribution and the distribution of order statistics of the ($MOEP_F$) distribution; Section 5 proposes parameter estimation of the ($MOEP_F$) distribution by the method of maximum likelihood estimation; Section 6 presents the application of the new ($MOEP_F$) distribution to two real data sets; Section 7 presents the discussion of results and lastly, Section 8 presents the conclusion of the study.

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Marshall-Olkin Extended Power Function Distribution

A random variable *X* is said to follow the power function distribution if its cumulative density function (*cdf*) and probability density function (*pdf*) is given by

$$F(x) = \left(\frac{x}{\psi}\right)^{\xi}; 0 < x < \psi;\ \ \psi, \xi > 0 \quad, \tag{2.1}$$

and

$$f(x) = \gamma\xi\psi^{-\xi}x^{\xi-1}; 0 < x < \psi;\ \psi, \xi > 0, \tag{2.2}$$

respectively.

If *X* is distributed according to Equation 2.2 then, the corresponding Marshall-Olkin generalized form of its *cdf* and *pdf* using Equations 1.1 and 1.2 is given by

$$F(x) = 1 - \frac{\gamma\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)}{\left(\frac{x}{\psi}\right)^{\xi}+\gamma\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)}; 0 < x < \psi;\ \ \psi, \xi, \gamma > 0 \quad, \tag{2.3}$$

and

$$f(x) = \gamma\xi\psi^{-\xi}x^{\xi-1}\left(\left(\frac{x}{\psi}\right)^{\xi}+\gamma\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)\right)^{-2}; 0 < x < \psi;\ \ \psi, \xi, \gamma > 0 \quad, \tag{2.4}$$

respectively, where ψ and γ are the scale parameters and ξ is the shape parameter. The new distribution given by the *pdf* Equation 2.4 is called the Marshall-Olkin extended power function ($MOEP_F$) distribution. Notably, the beta distribution and the power function distribution are special cases of the $MOEP_F$ distribution when $\psi;\gamma = 1$ and $\gamma = 1$, respectively.

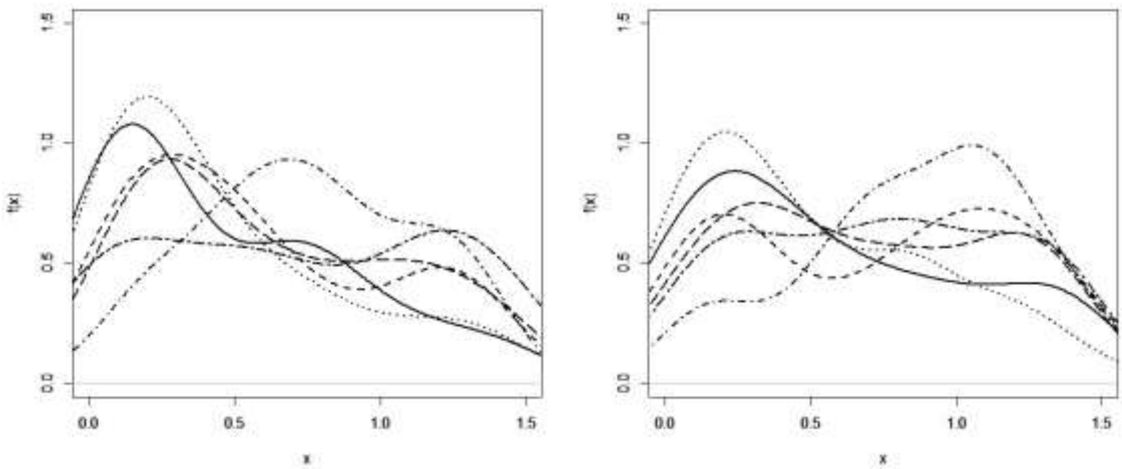

Figure 1: Possible shapes of the density function $f(x)$ of the $MOEP_F$ distribution for xed parameter values of $\psi = 1.50, \xi = 1.00$ and selected values of γ parameter. $\gamma = 0.45$ (solid

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lines), γ = 0.60 (dashed lines), γ = 0.30 (dotted lines), γ = 1.00 (dotdashed lines), γ = 0.50 (long dashed lines) and γ = 0.75 (two dashed lines).

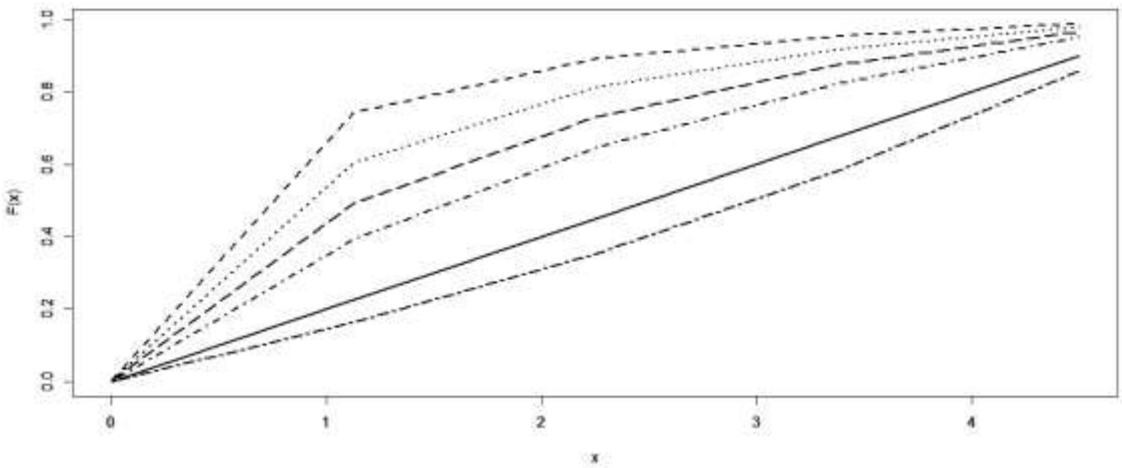

Figure 2: Possible shapes of the cumulative density function $F(x)$ of the $MOEP_F$ distribution for xed parameter values of ψ = 5.00,ξ = 1.00 and selected values of γ parameter. γ = 1.00 (solid line), γ = 0.10 (dashed line), γ = 0.19 (dotted line), γ = 0.45 (dotdashed line), γ = 0.30 (long dashed line) and γ = 1.50 (two dashed line).

Some Reliability Properties of the $MOEP_F$ Distribution

The Reliability Function

The reliability function gives the probability that a system will not fail until some speci ed time t under certain prede ned conditions. It could be expressed mathematically as $\bar{F}(x) = P(X > x) = 1 - F(x)$, using Equation 1.1 the reliability function of a $MOEP_F$ random

variable is given by

$$\bar{F}(x) = \frac{\gamma\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)}{\left(\frac{x}{\psi}\right)^{\xi}+\gamma\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)}; 0<x<\psi;\ \ \psi,\xi,\gamma>0 \quad . \qquad (3.1)$$

The Hazard Rate Function

The hazard rate function of a system is the probability that the system fails given that it has not failed up to time t. it is given by

$$h(x) = \frac{f(x)}{F(x)}. \qquad (3.2)$$

Thus, the hazard rate function of the ($MOEP_F$) distribution is given by

$$h(x) = \frac{\xi\psi^{-\xi}x^{\xi-1}}{\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)\left(\left(\frac{x}{\psi}\right)^{\xi}+\gamma\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)\right)}; 0<x<\psi;\ \ \psi,\xi,\gamma>0 \quad . \qquad (3.3)$$

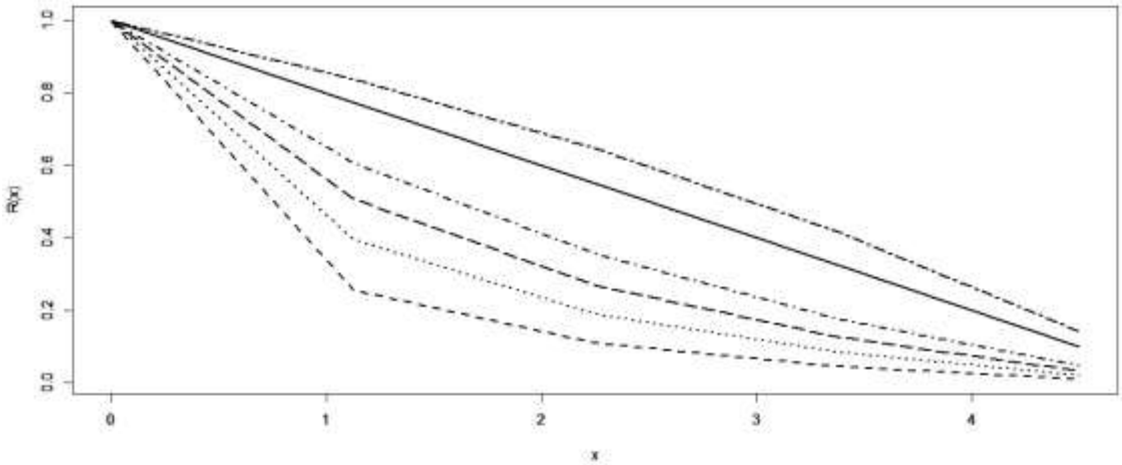

Figure 3: Possible shapes of the reliability function $R(x)$ of the $MOEP_F$ distribution for xed parameter values of $\psi = 5.00, \xi = 1.00$ and selected values of γ parameter. $\gamma = 1.00$ (solid line), $\gamma = 0.10$ (dashed line), $\gamma = 0.19$ (dotted line), $\gamma = 0.45$ (dotdashed line), $\gamma = 0.30$ (long dashed line) and $\gamma = 1.50$ (two dashed line).

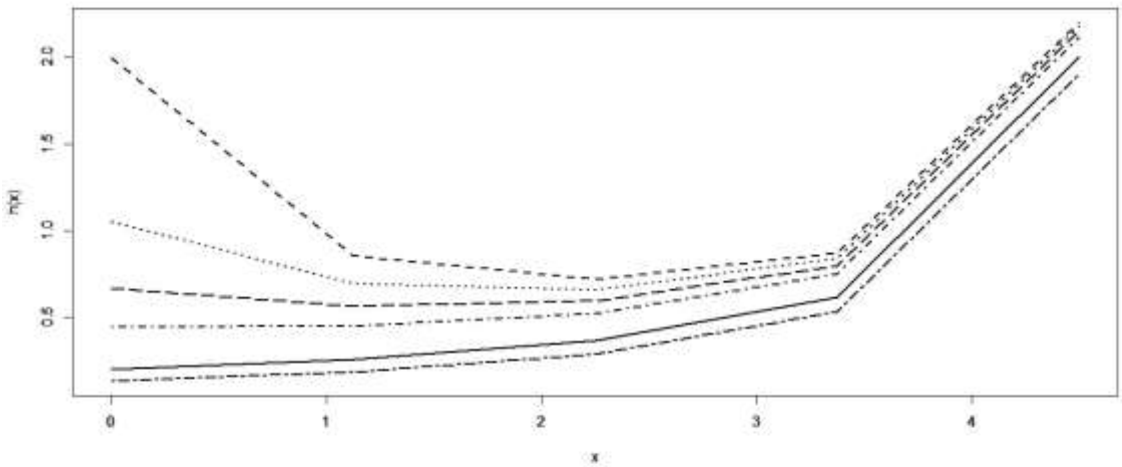

Figure 4: Possible shapes of the hazard rate function $h(x)$ of the $MOEP_F$ distribution for xed parameter values of $\psi = 5.00, \xi = 1.00$ and selected values of γ parameter. $\gamma = 1.00$ (solid line), $\gamma = 0.10$ (dashed line), $\gamma = 0.19$ (dotted line), $\gamma = 0.45$ (dotdashed line), $\gamma = 0.30$ (long dashed line) and $\gamma = 1.50$ (two dashed line).

The Mean Residual Life Time

The remaining life time of a system that has not failed up to time t is random because the failure time is not known. The expected value of this random failure times is known as the mean residual life time denoted by $M(t)$. $M(t)$ only exist for $\bar{F}(t) > 0$ and its mathematical representation is given by $M(t) = E(X - t|X > t)$. Hence,

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$$M(t) = \int_0^\infty \frac{\bar{F}(y+t)}{\bar{F}(t)} dy = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x) dx \quad (3.4)$$

The mean residual lifetime of the $MOEP_F$ distribution could be obtained as follows

$$M(t) = \frac{1}{\bar{F}(t)} \int_t^\psi \bar{F}(x) dx$$

$$M(t) = \frac{\gamma\left(\left(\frac{t}{\psi}\right)^\xi + \gamma\left(1-\left(\frac{t}{\psi}\right)^\xi\right)\right)}{\gamma\left(1-\left(\frac{t}{\psi}\right)^\xi\right)} \int_t^\psi \frac{1-\left(\frac{x}{\psi}\right)^\xi}{\left(\frac{x}{\psi}\right)^\xi + \gamma\left(1-\left(\frac{x}{\psi}\right)^\xi\right)} dx \quad (3.5)$$

(3.6)

Setting $y = \left(\frac{x}{\psi}\right)^\xi; x = \psi y^{\frac{1}{\xi}}; dx = \frac{\psi}{\xi} y^{\frac{1}{\xi}-1} dy$; and $I_t = \frac{\left(\frac{t}{\psi}\right)^\xi + \gamma\left(1-\left(\frac{t}{\psi}\right)^\xi\right)}{1-\left(\frac{t}{\psi}\right)^\xi}$, gives

$$M(t) = I_t \int_{\left(\frac{t}{\psi}\right)^\xi}^1 (1-y)(y+\gamma(1-y))^{-1} \frac{\psi}{\xi} y^{\frac{1}{\xi}-1} dy \quad (3.7)$$

$$M(t) = \frac{\psi I_t}{\xi} \int_{\left(\frac{t}{\psi}\right)^\xi}^1 (1-y) \sum_{i=0}^\infty (-1)^i \binom{i+1-1}{i} y^i \gamma^{-1-i} (1-y)^{-1-i} y^{\frac{1}{\xi}-1} dy \quad (3.8)$$

$$M(t) = \frac{\psi I_t}{\xi} \sum_{i=0}^\infty (-1)^i \gamma^{-1-i} \int_{\left(\frac{t}{\psi}\right)^\xi}^1 y^{\frac{1}{\xi}+i-1} (1-y)^{-i} dy \quad (3.9)$$

$$M(t) = \frac{\psi I_t}{\xi} \sum_{i=0}^\infty \sum_{j=0}^\infty \gamma^{-1-i} \binom{j+i-1}{j} \int_{\left(\frac{t}{\psi}\right)^\xi}^1 y^{\frac{1}{\xi}-j-1} dy$$

$$M(t) = \frac{\psi I_t}{\xi} \sum_{i=0}^\infty \sum_{j=0}^\infty \gamma^{-1-i} \binom{j+i-1}{j} \frac{1-\left(\frac{t}{\xi}\right)^{1-\xi j}}{\frac{1}{\xi}-j}$$

$$M(t) = \frac{\psi I_t}{\xi} \sum_{i=0}^\infty (-1)^i \gamma^{-1-i} \int_{\left(\frac{t}{\psi}\right)^\xi}^1 y^{\frac{1}{\xi}+i-1} \sum_{j=0}^\infty (-1)^j \binom{j+i-1}{j} (-1)^{-i-j} y^{-i-j} dy \quad ($$

(3.11)

(3.12)

Some Statistical Properties of the MOEP$_F$ Distribution

The kth Crude Moment of the MOEP$_F$ Distribution

The crude moment of a random variable plays a very vital role in statistics because so many other essential properties of the distribution can be derived from it, more importantly some descriptive statistics such as the mean, variance, coe cient of variation, skewness and kurtosis statistics. The *kth* crude moment $y = \left(\frac{x}{\psi}\right)^\xi; x = \psi y^{\frac{1}{\xi}}$ of any continuous random variable

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X is generally given by $E(x^k) = ^{R}{}_{all\,x}\, x^k f(x)dx$. Hence, the kth crude moment of the $MOEP_F$ distribution could be obtained as follows,

$$E(x^k) = \int_0^{\psi} \gamma\xi\psi^{-\xi}x^{k+\xi-1}\left(\left(\frac{x}{\psi}\right)^{\xi} + \gamma\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)\right)^{-2} dx \quad (4.1)$$

ψ 1ξ−1 making the following substitutions,; and $dx = {}_{\xi}y\; dy$, gives

$$E(x^k) = \gamma\xi\psi^{-\xi}\int_0^1 \left(\psi y^{\frac{1}{\xi}}\right)^{k+\xi-1}(y+\gamma(1-y))^{-2}\frac{\psi}{\xi}y^{\frac{1}{\xi}-1}dy \quad (4.2)$$

$$E(x^k) = \frac{\gamma\xi\psi^{-\xi}\psi^{k+\xi-1}\psi}{\xi}\int_0^1 y^{\frac{k}{\xi}-\frac{1}{\xi}+1}y^{\frac{1}{\xi}-1}(y+\gamma(1-y))^{-2}dy \quad (4.3)$$

$$E(x^k) = \gamma\psi^k\int_0^1 y^{\frac{k}{\xi}}\sum_{i=0}^{\infty}(-1)^i\binom{i+2-1}{i}y^i\gamma^{-2-i}(1-y)^{-2-i}dy$$

$$E(x^k) = \psi^k\sum_{i=0}^{\infty}(-1)^i\binom{i+1}{i}\gamma^{-1-i}\int_0^1 y^{\frac{k}{\xi}+i}(1-y)^{-2}(1-y)^{-i}dy$$

$$E(x^k) = \gamma\psi^k\int_0^1 y^{\frac{k}{\xi}}(y+\gamma(1-y))^{-2}dy \quad (4.4)$$

(4.5)

(4.6)

$$E(x^k) = \psi^k\sum_{i=0}^{\infty}(-1)^i\binom{i+1}{i}\gamma^{-1-i}\int_0^1 y^{\frac{k}{\xi}+i}\sum_{j=0}^{\infty}(-1)^j\binom{j+2-1}{j}(-1)^{-2-j}y^{-2-j}$$
$$\times\sum_{l=0}^{\infty}(-1)^l\binom{l+i-1}{l}(-1)^{-i-l}y^{-i-l}dy \quad (4$$

$$E(x^k) = \psi^k\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{-2}\gamma^{-1-i}\binom{i+1}{i}\binom{j+1}{j}\binom{l+i-1}{l}\int_0^1 y^{\frac{k}{\xi}-l-j-2}dy \quad ($$

$$E(x^k) = \psi^k\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\gamma^{-1-i}(i+1)(j+1)\binom{l+i-1}{l}\frac{1}{\frac{k}{\xi}-l-j-1} \quad (4.9)$$

In reliability theory the mean (often referred to as the mean time to failure ($MTTF$)) is a very important characteristics of a lifetime distribution. Under certain prede ned conditions $MTTF$ could be interpreted as the expected length of time a non-repairable system can last in operation before it fails. The mean of the $MOEP_F$ distribution could be obtained by evaluating Equation 4.9 at k = 1 as presented below,

$$E(x) = \psi\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\gamma^{-1-i}(i+1)(j+1)\binom{l+i-1}{l}\frac{1}{\frac{1}{\xi}-l-j-1}. \quad (4.10)$$

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Evaluating Equation 4.9 at $k = 2$ gives the second order moment of the $MOEP_F$ distribution

as

$$E(x^2) = \psi^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \gamma^{-1-i}(i+1)(j+1)\binom{l+i-1}{l}\frac{1}{\frac{2}{\xi}-l-j-1}. \quad (4.11)$$

The variance $V(x)$ could be obtained by substituting Equations 4.10 and 4.11 in the following expression $V(x) = E(x^2) - (E(x))^2$. Other higher order moments like $E(x^3)$ and $E(x^4)$ are required for the computation of the skewness and kurtosis statistics of the $MOEP_F$ distribution.

The Moment Generating Function of the $MOEP_F$ Distribution

The moment generating function (*mgf*) of a random variable X is generally de ned by

$$M_x(t) = E(e^{tx}) = E\left(\sum_{k=0}^{\infty}\frac{(tx)^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{t^k}{k!}E(x^k). \quad (4.12)$$

It follows from Equation 4.13 that the *mgf* of the $MOEP_F$ distribution is given by

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\psi t)^k}{k!}.\gamma^{-1-i}(i+1)(j+1)\binom{l+i-1}{l}\frac{1}{\frac{k}{\xi}-l-j-1} \quad (4.13)$$

The *pth* Quantile Function of the $MOEP_F$ Distribution

The *pth* quantile function of the $MOEP_F$ distribution is given by

$$x_p = \psi\left(\frac{\gamma p}{\gamma p - p + 1}\right)^{\frac{1}{\xi}}. \quad (4.14)$$

We can simulate random variables from the $MOEP_F$ distribution through the inversion of the *cdf* method by simply substituting p in Equation 4.14 with a $U(0,1)$ variates. Also, we can obtain the median of the $MOEP_F$ distribution by evaluating Equation 4.14 at $p = 1/2$.

The Re´nyi Entropy Measure of the $MOEP_F$ Distribution

The Re´nyi entropy is used to quantify the uncertainty or variation in a random variable X. The Re´nyi's entropy measure has been noted as a powerful tool for comparing the tails and shapes behaviour of many standard probability distributions, Song [17] . The Re´nyi entropy measure is generally given by

$$H_\delta(x) = \lim_{n\to\infty}(I_\delta(p_n) - \ln(n)) = \frac{1}{1-\delta}\ln\int p^\delta(x)dx \quad (4.15)$$

$$H_\delta(x) = \frac{1}{1-\delta}\ln(I_\delta). \quad (4.16)$$

where I_δ for the $MOEP_F$ distribution could be obtained as follows

$$y = \left(\frac{x}{\psi}\right)^{\xi}; x = \psi y^{\frac{1}{\xi}}$$

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$$I_\delta = \int p^\delta(x)dx = \int_0^\psi \left(\gamma\xi\psi^{-\xi}x^{\xi-1}\left(\left(\frac{x}{\psi}\right)^\xi + \gamma\left(1-\left(\frac{x}{\psi}\right)^\xi\right)\right)^{-2}\right)^\delta dx \quad (4.17)$$

(4.18)

$$I_\delta = \left(\gamma\xi\psi^{-\xi}\right)^\delta \int_0^\psi x^{\delta(\xi-1)}\left(\left(\frac{x}{\psi}\right)^\xi + \gamma\left(1-\left(\frac{x}{\psi}\right)^\xi\right)\right)^{-2\delta} dx$$

also, setting; and $dx = \frac{\psi}{\xi} y^{\frac{1}{\xi}-1} dy$, we have

$$I_\delta = \left(\gamma b\psi^{-\xi}\right)^\delta \int_0^1 \left(\psi y^{\frac{1}{\xi}}\right)^{\delta(\xi-1)} (y+\gamma(1-y))^{-2\delta}\frac{\psi}{\xi}y^{\frac{1}{\xi}-1}dy$$

$$I_\delta = \frac{\psi(\gamma\xi\psi^{-\xi})^\delta\psi^{\xi\delta-\delta}}{\xi}\int_0^1 y^{\frac{\delta}{\xi}(\xi-1)}y^{\frac{1}{\xi}-1}(y+\gamma(1-y))^{-2\delta}dy \quad (4.19)$$

(4.20)

$$I_\delta = \frac{\psi(\gamma\xi)^\delta\psi^{\xi\delta-\delta}\psi^{-\xi\delta}}{\xi}\int_0^1 y^{\delta-\frac{\delta}{\xi}+\frac{1}{\xi}-1}(y+\gamma(1-y))^{-2\delta}dy \quad (4.21)$$

$$I_\delta = \frac{\psi(\gamma\xi)^\delta}{\xi\psi^\delta}\int_0^1 y^{\delta-\frac{\delta}{\xi}+\frac{1}{\xi}-1}\sum_{i=0}^\infty(-1)^i\binom{i+2\delta-1}{i}y^i\gamma^{-2\delta-i}(1-y)^{-2\delta-i}dy \quad (4.22)$$

$$I_\delta = \frac{\psi(\gamma\xi)^\delta}{\xi\psi^\delta}\sum_{i=0}^\infty(-1)^i\binom{i+2\delta-1}{i}\gamma^{-2\delta-i}\int_0^1 y^{\delta-\frac{\delta}{\xi}+\frac{1}{\xi}+i-1}(1-y)^{-2\delta-i}dy \quad (4.23)$$

$$I_\delta = \frac{\psi(\gamma\xi)^\delta}{\xi\psi^\delta}\sum_{i=0}^\infty(-1)^i\binom{i+2\delta-1}{i}\gamma^{-2\delta-i}\int_0^1 y^{\delta-\frac{\delta}{\xi}+\frac{1}{\xi}+i-1}(1-y)^{-2\delta}(1-y)^{-i}dy \quad ($$

$$I_\delta = \frac{\psi(\gamma\xi)^\delta}{\xi\psi^\delta}\sum_{i=0}^\infty(-1)^i\binom{i+2\delta-1}{i}\gamma^{-2\delta-i}\int_0^1 y^{\delta-\frac{\delta}{\xi}+\frac{1}{\xi}+i-1}\sum_{j=0}^\infty(-1)^j\binom{j+2\delta-1}{j}$$
$$\times(-1)^{-2\delta-j}y^{-2\delta-j}\sum_{k=0}^\infty(-1)^k\binom{k+i-1}{k}(-1)^{-i-k}y^{-i-k}dy \quad (4$$

$$I_\delta = \frac{\psi(\gamma\xi)^\delta}{\xi\psi^\delta}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty(-1)^{-2\delta}\gamma^{-2\delta-i}\binom{i+2\delta-1}{i}\binom{j+2\delta-1}{j}$$
$$\times\binom{k+i-1}{k}\int_0^1 y^{-\delta-\frac{\delta}{\xi}+\frac{1}{\xi}-k-j-1}dy \quad (4$$

$$I_\delta = \frac{\psi(\gamma\xi)^\delta}{\xi\psi^\delta}\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty(-1)^{-2\delta}\gamma^{-2\delta-i}\binom{i+2\delta-1}{i}\binom{j+2\delta-1}{j}$$
$$\times\binom{k+i-1}{k}\frac{1}{\frac{1}{\xi}-\frac{\delta}{\xi}-\delta-k-j}. \quad (4$$

Order Statistics of the $MOEP_F$ Random Variable

The distribution of the *rth* order statistics denoted by $f_{x(r)}(x)$ of an n sized random sample $X_1, X_2, X_3, \ldots, X_n$ is generally given by

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$$f_{x_{(r)}}(x)=\frac{n!}{(r-1)!(n-r)!}(F_x(x))^{r-1}(1-F_x(x))^{n-r}f_x(x) \quad (4.28)$$

Then the density of the *rth* order statistics of the $MOEP_F$ distribution is obtained as

$$f_{x_{(r)}}(x)=\frac{n!\gamma\xi\psi^{-\xi}x^{\xi-1}}{(r-1)!(n-r)!}\left(\frac{x}{\psi}\right)^{\xi(r-1)}\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)^{n-r} \quad (4.29)$$

The density of the *rth* smallest order statistics of the $MOEP_F$ distribution could be obtained as

$$f_{x_{(1)}}(x)=n\gamma\xi\psi^{-\xi}x^{\xi-1}\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)^{n-1} \quad (4.30)$$

$$f_{x_{(1)}}(x)=n\gamma\xi\psi^{-\xi}x^{\xi-1}\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)^{n}\left(1-\left(\frac{x}{\psi}\right)^{\xi}\right)^{-1} \quad (4.31)$$

$$f_{x_{(1)}}(x)=n\gamma\xi\psi^{-\xi}x^{\xi-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j-1}\binom{n}{j}\binom{i+1-1}{i}\left(\frac{x}{\psi}\right)^{j}\left(\frac{x}{\psi}\right)^{-\xi-\xi i} \quad (4.32)$$

$$f_{x_{(1)}}(x)=n\gamma\xi\psi^{-\xi}x^{\xi-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j-1}\binom{n}{j}\left(\frac{x}{\psi}\right)^{j-\xi-\xi i} \quad (4.33)$$

The density of the *rth* largest order statistics of the $MOEP_F$ distribution is given by

$$f_{x_{(n)}}(x)=n\gamma\xi\psi^{-\xi}x^{\xi-1}\left(\frac{x}{\psi}\right)^{\xi(n-1)} \quad (4.34)$$

Estimation of the Parameters of the $MOEP_F$ Distribution

Here, we propose to estimate the parameters of the $MOEP_F$ distribution through the method of maximum likelihood estimates (*mle*). Suppose the following sample $x_1,x_2,x_3,\ldots,x_n$ of size n is drawn from the $MOEP_F$ distribution then, the *mle* of its parameters could be obtained as follows

$$L(x;\psi,\xi,\gamma)=\prod_{i=1}^{n}\gamma\xi\psi^{-\xi}x_i^{\xi-1}\left(\left(\frac{x_i}{\psi}\right)^{\xi}+\gamma\left(1-\left(\frac{x_i}{\psi}\right)^{\xi}\right)\right)^{-2} \quad (5.1)$$

$$L(x;\psi,\xi,\gamma)=\left(\gamma\xi\psi^{-\xi}\right)^{n}\prod_{i=1}^{n}x_i^{\xi-1}\left(\left(\frac{x_i}{\psi}\right)^{\xi}+\gamma\left(1-\left(\frac{x_i}{\psi}\right)^{\xi}\right)\right)^{-2} \quad (5.2)$$

$$\ln(L(x;\psi,\xi,\gamma))=n\ln(\gamma\xi\psi^{-\xi})+(\xi-1)\sum_{i=1}^{n}\ln(x_i)-2\sum_{i=1}^{n}\ln\left\{\left(\frac{x_i}{\psi}\right)^{\xi}+\gamma\left(1-\left(\frac{x_i}{\psi}\right)^{\xi}\right)\right\} \quad (5.3)$$

$$\frac{\partial\ln(L(x;\psi,\xi,\gamma))}{\partial\psi}=-\frac{n}{\psi}-2\sum_{i=1}^{n}\left\{\frac{-\frac{\xi x_i}{\psi^2}\left(\frac{x_i}{\psi}\right)^{\xi-1}+\frac{\gamma\xi x_i}{\psi^2}\left(\frac{x_i}{\psi}\right)^{\xi-1}}{\left(\frac{x_i}{\psi}\right)^{\xi}+\gamma\left(1-\left(\frac{x_i}{\psi}\right)^{\xi}\right)}\right\}=0 \quad (5.4)$$

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$$\frac{\partial \ln(L(x;\psi,\xi,\gamma))}{\partial \xi} = \frac{n}{\xi} + \sum_{i=1}^{n} \ln(x_i) - 2\sum_{i=1}^{n} \left\{ \frac{\left(\frac{x_i}{\psi}\right)^{\xi} \ln\left(\frac{x_i}{\psi}\right) + \gamma\left(\frac{x_i}{\psi}\right)^{\xi} \ln\left(\frac{x_i}{\psi}\right)}{\left(\frac{x_i}{\psi}\right)^{\xi} + \gamma\left(1-\left(\frac{x_i}{\psi}\right)^{\xi}\right)} \right\} = 0 \quad (5.$$

$$\frac{\partial \ln(L(x;\psi,\xi,\gamma))}{\partial \gamma} = \frac{n}{\gamma} - 2\sum_{i=1}^{n} \left\{ \frac{1-\left(\frac{x_i}{\psi}\right)^{\xi}}{\left(\frac{x_i}{\psi}\right)^{\xi} + \gamma\left(1-\left(\frac{x_i}{\psi}\right)^{\xi}\right)} \right\} = 0 \quad (5.$$

There is no known closed-form analytical solution for Equations 5.4, 5.5 and 5.6 thus, we recommend the use of some nonlinear numerical optimization technique such as the Newton Raphson algorithm as an e ective way of circumventing this intractable analytical issue.

Application

In this section we would t the $MOEP_F$ distribution to two real data sets to illustrate its applicability and exibility. The goodness of t of the new distribution would be assessed through the signi cance of the model parameters using the $p - value$ criterion. The rst uncensored data set in Table 1 represents the failure times in weeks of 50 items that were subjected to use at time 0. The data set was reported in Murthy et al. [18]. The second uncensored data set in Table 2 shows the 45 yearly survival times data of a group of patients who received only chemotherapy treatment. The data set was reported in Bekker et al. [19]. Results from the model ttings for each of the reported data set are presented in Tables 3 and 4.

Table 1: Data 1

0.013	0.065	0.111	0.111	0.163	0.309	0.426	0.535	0.684	0.747	0.997	1.284
		1.304	1.647	1.829	2.336	2.838	3.269	3.977	3.981		
4.520	4.789	4.849	5.202	5.291	5.349	5.911	6.018	6.427	6.456		
6.572	7.023	7.087	7.291	7.787	8.596	9.388	10.261	10.713	11.658		
13.006	13.388	13.842	17.152	17.283	19.418	23.471	24.777	32.795	48.105		

Table 2: Data 2

0.047	0.115	0.121	0.132	0.164	0.197	0.203	0.260	0.282	0.296
0.334	0.395	0.458	0.466	0.501	0.507	0.529	0.534	0.540	0.641
0.644	0.696	0.841	0.863	1.099	1.219	1.271	1.326	1.447	1.485
1.553	1.581	1.589	2.178	2.343	2.416	2.444	2.825	2.830	3.578
3.658	3.743	3.978	4.003	4.033					

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Table 3: Results from Data 1

Model	Parameters	Estimates	STD Errors	p-values	logLik	AIC
	γ	14.879	0.00049513	2.20×10^{-16}		
MOEP$_F$	ψ	47.880	0.00159260	2.20×10^{-16}	28030024	-56060041
	ξ	14.878	0.00049510	2.20×10^{-16}		

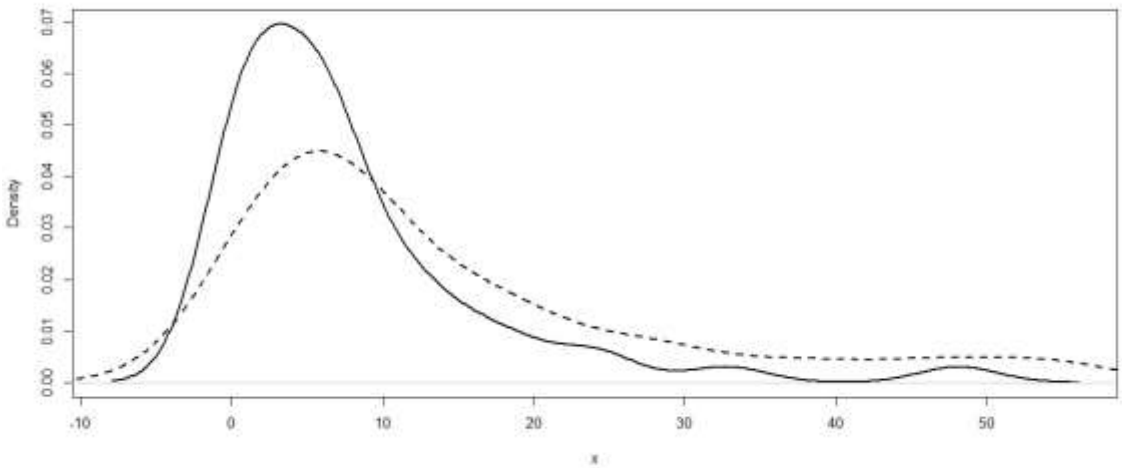

Figure 5: Density plots of the simulated *MOEP$_F$* random variables (dashed lines) superimposed on the empirical density of data 1 (solid lines).

Table 4: Results from Data 2

Model	Parameters	Estimates	STD Errors	p-values	logLik	AIC
	γ	4.86732097	0.00014760	2.20×10^{-16}		
MOEP$_F$	ψ	3.81636038	0.00011537	2.20×10^{-16}	4369642	-8739277
	ξ	4.81109760	0.00014600	2.20×10^{-16}		

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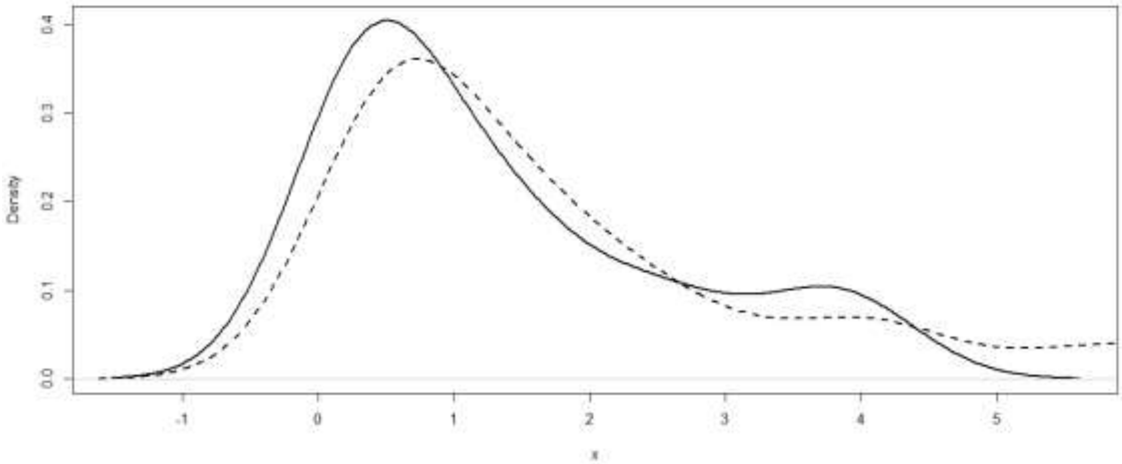

Figure 6: Density plots of the simulated $MOEP_F$ random variables (dashed lines) superimposed on the empirical density of data 2 (solid lines).

DISCUSSION OF RESULTS

The density plots in Figure 1 depicts some funny unpredictable shapes of the $MOEP_F$ distribution. The reliability function is a decreasing function of x and the hazard rate function could either be increasing or bathtub shaped a unique feature which makes it more suitable for analyzing lifetime data sets. The results in Tables 3 and 4 shows that the parameters of the tted $MOEP_F$ distribution are highly signi cant suggesting that the model is adequate for the two lifetime data sets under consideration. Given the complexities of the two data sets and without looking too closely to the density plots in Figures 5 and 6 we can see a good t of the $MOEP_F$ distribution to the data sets.

CONCLUSIONS

This article introduces a new lifetime distribution - the Marshall-Olkin extended power function ($MOEP_F$) distribution. The new distribution generalizes the power function distribution and have beta and power function distributions as sub-models. We have given explicit mathematical expressions for some of its basic statistical properties such as the probability density function, cumulative density function, *kth* crude moment, variance, moment generating function, *pth* quantile function, the *rth* order statistics, and the R*e*´nyi's entropy measure. Also, some of its reliability characteristics like the reliability function, hazard rate function and the mean residual life time was given. Estimation of the model parameters was approached through the method of maximum likelihood estimation. The exibility, applicability and robustness of the new lifetime distribution was demonstrated with two real data sets and the results obtained shows that the $MOEP_F$ distribution provides good ts to the two lifetime data sets. We propose the $MOEP_F$ distribution for modelling complex lifetime data sets in particular because it would receive reasonably high rate of application in this direction as a result of its bathtub shaped hazard rate characteristics.

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