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ISSUES RELATED WITH ARMA (P,Q) PROCESS

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ABSTRACT: *Suppose that* $\{x_t\}$ *be an ARMA (p,q) process with white noise (error) process* $\{a_t\}$ *. Let* $\psi_X(u)$ *and* $\psi_a(u)$ *be the characteristic functions of* $\{x_t\}$ *and* $\{a_t\}$ *respectively. In this paper, a general formula to represent* $\psi_X(u)$ *in terms of* $\psi_a(u)$ *is obtained. By using this formula , we investigate about the distribution of ARMA(p,q) process where it's white noise follow Normal , Cauchy , inverse Gaussian and Gamma distributions . Instead of the hard traditional al method, an easy general formula to determine the coefficients for the causal ARMA (p,q) process will be presented also.*

KEYWORDS: Abid Formula, Characteristic Function, Causal Function, ARMA (P,Q).
Mathematics Subject Classification: 62M10, 37M10, 91B84

INTRODUCTION

Autoregressive schemes with moving average error terms of the form ,

$$(F^p - b_1 F^{p-1} - \ldots - b_p)\, X_t = a_{t+p} + \theta_1 a_{t+p-1} + \ldots + \theta_q a_{t+p-q} \qquad (1.1)$$

Where $F^m X_t = X_{t+m}$, have been considered . The process $\{X_t\}$ is called an autoregressive moving average process of order (p,q) or briefly ARMA(p,q) ,where $\{a_t\}$ is a sequence of white noise (i.i.d.) random variables. Alternatively (1.1) can be written as ,

$$X_t = \frac{\sum_{k=0}^{q} \theta_k \, a_{t+p-k}}{\prod_{j=1}^{p} (F - r_j)} \qquad (1.2)$$

Where $\theta_0 = 1$ and $r_1, r_2, \ldots, r_p$ represent the roots of $F^p - b_1 F^{p-1} - \ldots - b_p = 0$.

Suppose that $\psi_X(u)$ and $\psi_a(u)$ be the characteristic functions of $\{X_t\}$ and $\{a_t\}$ respectively

Andel (1982) represented an AR(1) process in terms of its white noise process as follows ,

$$X_t = \sum_{j=0}^{\infty} r_1^j \, a_{t-j} \qquad (1.3)$$

Then, he wrote the following relationship between the characteristic function of this process $\psi_X(u)$ and the characteristic function of white noise process $\psi_a(u)$ as,

$$\psi_X(u) = \prod_{j=0}^{\infty} \psi_a(u\ r_1^j) \qquad (1.4)$$

Priestley (1984) treated with AR (2) process and represented this process in terms of it's white noise as,

$$X_t = \frac{1}{r_1 - r_2} \sum_{j=0}^{\infty} \left(r_1^{j+1} - r_2^{j+1}\right) a_{t-j} \qquad (1.5)$$

Then one can write the following relationship between $\psi_X(u)$ and $\psi_a(u)$ for AR(2) process ,

$$\psi_X(u) = \prod_{j=0}^{\infty} \psi_a\left[u\left(\frac{r_1^{j+1} - r_2^{j+1}}{r_1 - r_2}\right)\right] \qquad (1.6)$$

Sim in 1986, studied the behavior of AR(1) process where the residuals distributed as Gamma and weibull. In 1990, Sim studied the behavior of AR(1) process where the residuals distributed as Gamma and three parameters exponential. In 1993, Al-osh and Al-zaid studied the properties of ARMA(p,p-1) generalized poisson process, especially for covariance matrix, invertibility and regression behavior. Sim, in 1993, discussed the moments of AR(1) process with residuals distributed as logistic. In 1994, sim studied the order estimation and diagonstic checking for AR(1) process where the residuals distributed as logistic, hyper tangent, Gamma, exponential and Laplace. In 1997, Al-zaid, Al-wasel and Al-nachawati used the simulation to compare between maximum likelihood estimator and Yule-walker estimator for AR(1) generalized poisson parameter. In 1998, the same above researchers, used simulation to compare among Yule-walker estimator, conditional least square estimator, maximum likelihood estimator and moment estimator for AR(1) Binomial parameter.

Abid (2001) wrote the relationship between $\psi_X(u)$ and $\psi_a(u)$ for AR(p) process as follows,

$$\psi_X(u) = \begin{cases} \prod_{j=0}^{\infty} \psi_a\left(u\,\eta_p\left(j+p-1, r_1, r_2, r_p\right)\right) & , p = 2, 3, \ldots \\ \prod_{j=0}^{\infty} \psi_a\left(t\, r_1^j\right) & , p = 1 \end{cases} \qquad (1.7)$$

$$\eta_k(j+k-1, r_1, r_2, \ldots, r_k) = \frac{1}{r_{k-1} - r_k}\left\{\eta_{k-1}(j+k-1, r_1, r_2, \ldots, r_{k-1}) - \eta_{k-1}(j+k-1, r_1, r_2, \ldots, r_{k-2}, r_k)\right\}$$

Where

$$= \frac{1}{r_{k-1} - r_k}\left\{\frac{1}{r_{k-2} - r_{k-1}}\left(\eta_{k-2}(j+k-1, r_1, \ldots, r_{k-2}) - \eta_{k-2}(j+k-1, r_1, \ldots, r_{k-3}, r_{k-1})\right) - \frac{1}{r_{k-2} - r_k}\left(\eta_{k-2}(j+k-1, r_1, \ldots, r_{k-2}) - \eta_{k-2}(j+k-1, r_1, \ldots, r_{k-3}, r_k)\right)\right\} \qquad (1.8)$$

and continue until substitution of η_2 as $\eta_2(l, a, b) = \dfrac{a^l - b^l}{a - b}$.

Depending on the following formula he derived for representation of AR(p) process in terms of it's white noise process ,

$$X_t = \begin{cases} \sum_{j=0}^{\infty} \eta_p\left(j+p-1, r_1, r_2, \ldots, r_p\right) a_{t-j} & , p = 2,3,\ldots \\ \sum_{j=0}^{\infty} r_1^{\,j}\, a_{t-j} & , p = 1 \end{cases} \qquad (1.9)$$

The <u>first goal</u> of this paper is to **derive** a general formula to represent $\psi_X(u)$ in terms of $\psi_a(u)$ for ARMA(p,q) process .

An ARMA(p,q) process $\{X_t\}$ is a causal function of $\{a_t\}$ if it can be written as the MA(∞) process ,

$$X_t = \sum_{h=0}^{\infty} g_h\, a_{t-h} \qquad (1.10)$$

Where the coefficients $\{g_j\}$ satisfy $\sum_{h=0}^{\infty} |g_h| < \infty$.

So, the <u>second goal</u> of this work is to **find** a general formula to determine $g_h \ (h = 0,1,2,\ldots)$, in terms of $\theta's$ and $r's$.

REPRESENTATION OF ARMA(p,q) PROCESS IN TERMS OF IT'S WHITE NOISE

Firstly , the formula of ARMA(1,q) can be written as,

$$X_t = \frac{\sum_{k=0}^{q} \theta_k a_{t+1-k}}{F - r_1} \qquad (2.1)$$

$$= \frac{1}{F}\left(1 + \frac{r_1}{F} + \frac{r_1^2}{F^2} + \ldots\right) \sum_{k=0}^{q} \theta_k a_{t+1-k}$$

$$= \sum_{k=0}^{q} \theta_k (B + r_1 B^2 + r_1^2 B^3 + \ldots)\, a_{t+1-k}$$

$$= \sum_{k=0}^{q} \theta_k \left(a_{t-k} + r_1 a_{t-k-1} + r_1^2 a_{t-k-2} + \ldots\right)$$

$$= \sum_{k=0}^{q} \theta_k \sum_{j=0}^{\infty} r_1^{\,j} a_{t-k-j}$$

$$= \sum_{k=0}^{q} \sum_{j=0}^{\infty} \theta_k r_1^{\,j} a_{t-k-j} \quad , \ \theta_0 = 1 \quad , \qquad (2.2)$$

Also, we can write ARMA(2,q) process as,

$$X_t = \frac{\sum_{k=0}^{q} \theta_k a_{t+2-k}}{(F - r_1)(F - r_2)} \qquad (2.3)$$

$$= \sum_{k=0}^{q} \theta_k \left(\frac{1}{(F-r_1)(F-r_2)} \right) a_{t+2-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{r_1 - r_2} \left(\frac{1}{F\left(1-\frac{r_1}{F}\right)} - \frac{1}{F\left(1-\frac{r_2}{F}\right)} \right) a_{t+2-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{r_1 - r_2} . \frac{1}{F} \left(1 + \frac{r_1}{F} + \frac{r_1^2}{F^2} + \ldots - \left(1 + \frac{r_2}{F} + \frac{r_2^2}{F^2} + \ldots \right) \right) a_{t+2-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{r_1 - r_2} . \frac{1}{F} \left\{ \frac{r_1 - r_2}{F} + \frac{r_1^2 - r_2^2}{F^2} + \frac{r_1^3 - r_2^3}{F^3} + \ldots \right\} a_{t+2-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{r_1 - r_2} \left\{ B^2 (r_1 - r_2) + B^3 (r_1^2 - r_2^2) + B^4 (r_1^3 - r_2^3) + \ldots \right\} a_{t+2-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{r_1 - r_2} \sum_{j=0}^{\infty} \left(r_1^{j+1} - r_2^{j+1} \right) a_{t-k-j} \quad , \quad \theta_0 = 1 \quad , \qquad (2.4)$$

ARMA(3,q) process can be written in terms of it's white noise as follows,

$$X_t = \frac{\sum_{k=0}^{q} \theta_k a_{t+3-k}}{(F-r_1)(F-r_2)(F-r_3)} \qquad (2.5)$$

$$= \sum_{k=0}^{q} \theta_k \frac{\left(\frac{1}{F-r_2} - \frac{1}{F-r_3} \right)}{(F-r_1)(r_2-r_3)} \; a_{t+3-k}$$

$$= \sum_{k=0}^{q} \theta_k \frac{1}{r_2 - r_3} \left(\frac{1}{(F-r_1)(F-r_2)} - \frac{1}{(F-r_1)(F-r_3)} \right) a_{t+3-k}$$

$$= \sum_{k=0}^{q} \theta_k \frac{1}{r_2 - r_3} \left\{ \frac{1}{r_1 - r_2} \left(\frac{1}{F-r_1} - \frac{1}{F-r_2} \right) - \frac{1}{r_1 - r_3} \left(\frac{1}{F-r_1} - \frac{1}{F-r_3} \right) \right\} a_{t+3-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{(r_2 - r_3)F} \left\{ \frac{1}{r_1 - r_2} \left(\frac{r_1}{F} + \frac{r_1^2}{F^2} + \ldots - \frac{r_2}{F} - \frac{r_2^2}{F^2} - \ldots \right) - \frac{1}{r_1 - r_3} \left(\frac{r_1}{F} + \frac{r_1^2}{F^2} + \ldots - \frac{r_3}{F} - \frac{r_3^2}{F^2} - \ldots \right) \right\} a_{t+3-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{(r_2 - r_3)F} \left\{ \frac{1}{r_1 - r_2} \left(\frac{r_1 - r_2}{F} + \frac{r_1^2 - r_2^2}{F^2} + \ldots \right) - \frac{1}{r_1 - r_3} \left(\frac{r_1 - r_3}{F} + \frac{r_1^2 - r_3^2}{F^2} + \ldots \right) \right\} a_{t+3-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{r_2 - r_3} \left\{ \left(\frac{r_1^2 - r_2^2}{r_1 - r_2} - \frac{r_1^2 - r_3^2}{r_1 - r_3} \right) F^{-3} + \left(\frac{r_1^3 - r_2^3}{r_1 - r_2} - \frac{r_1^3 - r_3^3}{r_1 - r_3} \right) F^{-4} + \ldots \right\} a_{t+3-k}$$

$$= \sum_{k=0}^{q} \frac{\theta_k}{r_2 - r_3} \left\{ \sum_{j=0}^{\infty} \left(\frac{r_1^{j+2} - r_2^{j+2}}{r_1 - r_2} - \frac{r_1^{j+2} - r_3^{j+2}}{r_1 - r_3} \right) a_{t-j-k} \right\} \quad , \; \theta_0 = 1 \quad , \qquad (2.6)$$

Similarly, ARMA(4,q) process can be written as,

$$X_t = \frac{\sum_{k=0}^{q} \theta_k \, a_{t+4-k}}{(F-r_1)(F-r_2)(F-r_3)(F-r_4)} \qquad ----(2.7)$$

$$= \sum_{k=0}^{q} \theta_k \frac{\left(\frac{1}{F-r_3} - \frac{1}{F-r_4}\right) a_{t+4-k}}{(F-r_1)(F-r_2)(r_3-r_4)}$$

$$= \sum_{k=0}^{q} \theta_k \frac{1}{(F-r_1)(r_3-r_4)} \left(\frac{1}{(F-r_2)(F-r_3)} - \frac{1}{(F-r_2)(F-r_4)}\right) a_{t+4-k}$$

$$=\sum_{k=0}^{q}\frac{\theta_k}{(F-r_1)(r_3-r_4)}\left\{\frac{1}{r_2-r_3}\left(\frac{1}{F-r_2}-\frac{1}{F-r_3}\right)-\frac{1}{r_2-r_4}\left(\frac{1}{F-r_2}-\frac{1}{F-r_4}\right)\right\}a_{t+4-k}$$

$$=\sum_{k=0}^{q}\frac{\theta_k}{r_3-r_4}\left\{\frac{1}{r_2-r_3}\left[\frac{1}{r_1-r_2}\left(\frac{1}{F-r_1}-\frac{1}{F-r_2}\right)-\frac{1}{r_1-r_3}\left(\frac{1}{F-r_1}-\frac{1}{F-r_3}\right)\right]-\frac{1}{r_2-r_4}\left[\frac{1}{r_1-r_2}\left(\frac{1}{F-r_1}-\frac{1}{F-r_2}\right)-\frac{1}{r_1-r_4}\left(\frac{1}{F-r_1}-\frac{1}{F-r_4}\right)\right]\right\}a_{t+4-k}$$

$$=\sum_{k=0}^{q}\frac{\theta_k}{(r_3-r_4)F}\left\{\frac{1}{r_2-r_3}\left[\frac{1}{r_1-r_2}\left(\frac{r_1-r_2}{F}+\frac{r_1^2-r_2^2}{F^2}+\ldots\right)-\frac{1}{r_1-r_3}\left(\frac{r_1-r_3}{F}+\frac{r_1^2-r_3^2}{F^2}+\ldots\right)\right]-\frac{1}{r_2-r_4}\left[\frac{1}{r_1-r_2}\left(\frac{r_1-r_2}{F}+\frac{r_1^2-r_2^2}{F^2}+\ldots\right)-\frac{1}{r_1-r_4}\left(\frac{r_1-r_4}{F}+\frac{r_1^2-{}_4^2}{F^2}+\ldots\right)\right]\right\}a_{t+4-k}$$

$$=\sum_{k=0}^{q}\frac{\theta_k}{(r_3-r_4)F}\left\{\frac{1}{r_2-r_3}\left[\left(\frac{r_1^2-r_2^2}{r_1-r_2}-\frac{r_1^2-r_3^2}{r_1-r_3}\right)\frac{1}{F^2}+\left(\frac{r_1^3-r_2^3}{r_1-r_2}-\frac{r_1^3-r_3^3}{r_1-r_3}\right)\frac{1}{F^3}+\ldots\right]-\frac{1}{r_2-r_4}\left[\left(\frac{r_1^2-r_2^2}{r_1-r_2}-\frac{r_1^2-r_4^2}{r_1-r_4}\right)\frac{1}{F^2}+\left(\frac{r_1^3-r_2^3}{r_1-r_2}-\frac{r_1^3-r_4^3}{r_1-r_4}\right)\frac{1}{F^3}+\ldots\right]\right\}a_{t+4-k}$$

Since $\frac{d^2-b^2}{d-b}-\frac{d^2-c^2}{d-c}=b-c$ for every constants $\boldsymbol{b,c}$ and $\boldsymbol{d}$, then

$$X_t=\sum_{k=0}^{q}\frac{\theta_k}{r_3-r_4}\left\{\left[\frac{1}{r_2-r_3}\left(\frac{r_1^3-r_2^3}{r_1-r_2}-\frac{r_1^3-r_3^3}{r_1-r_3}\right)-\frac{1}{r_2-r_4}\left(\frac{r_1^3-r_2^3}{r_1-r_2}-\frac{r_1^3-r_4^3}{r_1-r_4}\right)\right]\frac{1}{F^4}+\left[\frac{1}{r_2-r_3}\left(\frac{r_1^4-r_2^4}{r_1-r_2}-\frac{r_1^4-r_3^4}{r_1-r_3}\right)-\frac{1}{r_2-r_4}\left(\frac{r_1^4-r_2^4}{r_1-r_2}-\frac{r_1^4-r_4^4}{r_1-r_4}\right)\right]\frac{1}{F^5}+\ldots\right\}a_{t+4-k}$$

$$=\sum_{k=0}^{q}\frac{\theta_k}{r_3-r_4}\sum_{j=0}^{\infty}\left[\frac{1}{r_2-r_3}\left(\frac{r_1^{3+j}-r_2^{3+j}}{r_1-r_2}-\frac{r_1^{3+j}-r_3^{3+j}}{r_1-r_3}\right)-\frac{1}{r_2-r_4}\left(\frac{r_1^{3+j}-r_2^{3+j}}{r_1-r_2}-\frac{r_1^{3+j}-r_4^{3+j}}{r_1-r_4}\right)\right]a_{t-j-k}\quad,\ \theta_0=1\quad,\qquad ----(2.8)$$

By using the same argument, we can generalize the above results to the following general formula for representation of ARMA(p,q) process in terms of it's white noise ,

$$X_t = \begin{cases} \sum_{k=0}^{q} \theta_k \sum_{j=0}^{\infty} \eta_p(j+p-1, r_1, r_2, \dots, r_p)\ a_{t-j-k} & , \quad p = 2,3,\dots \\ \\ \sum_{k=0}^{q} \theta_k \sum_{j=0}^{\infty} r_1^j\ a_{t-j-k} & , \quad p = 1 \end{cases} \qquad ----(2.9)$$

Where for $k = 3, 4, \dots$,

$$\eta_k(j+k-1, r_1, r_2, \dots, r_k) = \frac{1}{r_{k-1} - r_k} \{\eta_{k-1}(j+k-1, r_1, r_2, \dots, r_{k-1}) - \eta_{k-1}(j+k-1, r_1, r_2, \dots, r_{k-2}, r_k)\}$$

$$= \frac{1}{r_{k-1} - r_k} \left\{ \frac{1}{r_{k-2} - r_{k-1}} \left(\eta_{k-2}(j+k-1, r_1, \dots, r_{k-2}) - \eta_{k-2}(j+k-1, r_1, \dots, r_{k-3}, r_{k-1}) \right) - \frac{1}{r_{k-2} - r_k} \left(\eta_{k-2}(j+k-1, r_1, \dots, r_{k-2}) - \eta_{k-2}(j+k-1, r_1, \dots, r_{k-3}, r_k) \right) \right\} \qquad ----(2.10)$$

And so on until we reach to $\eta_2(l, a, b) = \frac{a^l - b^l}{a - b}$ where l is positive integer number .

THE CHARACTERISTIC FUNCTION OF ARMA (p,q) PROCESS

By using the general formula in (2.9), we can write $\psi_X(u)$ in terms of $\psi_a(u)$ for ARMA(p,q) process as follows,

$$\begin{aligned}\psi_X(u) &= E\ \exp\{iuX_t\}\\ &= E\ \exp\left\{iu\left(\sum_{k=0}^{q}\theta_k\sum_{j=0}^{\infty}\eta_p(j+p-1,r_1,r_2,\dots,r_p)\right)a_{t-j-k}\right\}\\ &= \prod_{k=0}^{q}\prod_{j=0}^{\infty}E\ \exp\{iu\theta_k\eta_p(j+p-1,r_1,r_2,\dots,r_p)\}a_{t-j-k}\\ &= \begin{cases}\prod_{k=0}^{q}\prod_{j=0}^{\infty}\psi_a(u\theta_k\eta_p(j+p-1,r_1,\dots,r_p)) & ,\quad p=2,3,\dots\\ \prod_{k=0}^{q}\prod_{j=0}^{\infty}\psi_a(u\theta_k r_1^j) & ,\quad p=1\end{cases}\end{aligned} \tag{3.1}$$

<u>Example (1)</u>

Suppose that $\{X_t\}$ is an ARMA(p,q) process and it's white noise follow normal distribution with characteristic function $\psi_a(u)=\exp\{iu\mu-(u^2\sigma^2/2)\}$, then the characteristic function of $\{X_t\}$ can be written as,

$$\psi_X(u)=\begin{cases}\prod_{k=0}^{q}\prod_{j=0}^{\infty}\exp\{iu\theta_k\eta_p(j+p-1,r_1,\dots,r_p)\mu-u^2\theta_k^2\sigma^2\eta_p^2(j+p-1,r_1,\dots,r_p)/2\} & ,\ p=2,3,\dots\\ \prod_{k=0}^{q}\prod_{j=0}^{\infty}\exp\{iu\theta_k r_1^j\mu-u^2\theta_k^2 r_1^{2j}\sigma^2/2\} & ,\ p=1\end{cases}$$

$$=\begin{cases}\exp\left\{iu\left(\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k\eta_p(j+p-1,r_1,\dots,r_p)\right)\mu-\frac{u^2\sigma^2}{2}\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k^2\eta_p^2(j+p-1,r_1,\dots,r_p)\right\} & ,\ p=2,3,\dots\\ \exp\left\{iu\mu\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k r_1^j-\frac{u^2\sigma^2}{2}\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k^2\ r_1^{2j}\right\} & ,\ p=1\end{cases}$$

So, the distribution of $\{X_t\}$ will be,

$$X_t\approx\begin{cases}N\left(\mu\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k\eta_p(j+p-1,r_1,\dots,r_p),\sigma^2\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k^2\eta_p^2(j+p-1,r_1,\dots,r_p)\right) & ,\ p=2,3,\dots\\ N\left(\mu\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k r_1^j,\sigma^2\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k^2\ r_1^{2j}\right) & ,\ p=1\end{cases}$$

As special case , if p=2 then, the distribution of ARMA(2,q) process will be normal with mean, $\mu\sum_{k=0}^{q}\sum_{j=0}^{\infty}\frac{r_1^{j+1}-r_2^{j+1}}{r_1-r_2}\theta_k$ and variance $\sigma^2\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left(\frac{r_1^{j+1}-r_2^{j+1}}{r_1-r_2}\right)^2\theta_k^2$.

ARMA(3,q) will distributed also as normal with mean,

$$\mu\sum_{k=0}^{q}\sum_{j=0}^{\infty}\frac{\theta_k}{r_2-r_3}\left(\frac{r_1^{j+2}-r_2^{j+2}}{r_1-r_2}-\frac{r_1^{j+2}-r_3^{j+2}}{r_1-r_3}\right)$$

, and variance

$$\sigma^2\sum_{k=0}^{q}\sum_{j=0}^{\infty}\frac{\theta_k^2}{(r_2-r_3)^2}\left(\frac{r_1^{j+2}-r_2^{j+2}}{r_1-r_2}-\frac{r_1^{j+2}-r_3^{j+2}}{r_1-r_3}\right)^2 .$$

Example (2)

Suppose that $\{X_t\}$ is an ARMA(p,q) process and it's white noise follow Cauchy distribution with characteristic function $\Phi_a(u)=\exp\{iud-m|u|\}$, then the characteristic function of $\{X_t\}$ can be written according to (3.1) as,

$$\psi_X(u)=\begin{cases}\prod_{k=0}^{q}\prod_{j=0}^{\infty}\exp\left(idu\theta_k\eta_p(j+p-1,r_1,\ldots,r_p)-m\left|u\theta_k\eta_p(j+p-1,r_1,\ldots,r_p)\right|\right) & ,\ p=2,3,\ldots\\ \prod_{k=0}^{q}\prod_{j=0}^{\infty}\exp\left(idu\theta_k r_1^j-m\left|u\theta_k r_1^j\right|\right) & ,\ p=1\end{cases}$$

$$=\begin{cases}\exp\left(iud\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k\eta_p(j+p-1,r_1,...,r_p)-m|u|\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left|\theta_k\eta_p(j+p-1,r_1,...,r_p)\right|\right) & ,\ p=2,3,...\\ \exp\left(iud\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k r_1^j-m|u|\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left|\theta_k r_1^j\right|\right) & ,\ p=1\end{cases}$$

So, the distribution of $\{X_t\}$ will be,

$$X_t\approx Cauchy\begin{cases}\left(d\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k\eta_p(j+p-1,r_1,...,r_p),\ m\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left|\theta_k\eta_p(j+p-1,r_1,...,r_p)\right|\right) & ,\ p=2,3,...\\ \left(d\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k r_1^j\ ,\ \sum_{k=0}^{q}\sum_{j=0}^{\infty}\left|\theta_k r_1^j\right|\right) & ,\ p=1\end{cases}$$

As special case , if p=2 then, the distribution of ARMA(2,q) process will be Cauchy with parameters , $d\sum_{k=0}^{q}\sum_{j=0}^{\infty}\theta_k\frac{r_1^{j+1}-r_2^{j+1}}{r_1-r_2}$ and $m\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left|\theta_k\frac{r_1^{j+1}-r_2^{j+1}}{r_1-r_2}\right|$.

ARMA(3,q) will distributed also as Cauchy with parameters ,

$$d\sum_{k=0}^{q}\sum_{j=0}^{\infty}\frac{\theta_k}{r_2-r_3}\left(\frac{r_1^{j+2}-r_2^{j+2}}{r_1-r_2}-\frac{r_1^{j+2}-r_3^{j+2}}{r_1-r_3}\right)$$ and

$$m\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left|\frac{\theta_k}{r_2-r_3}\left(\frac{r_1^{j+2}-r_2^{j+2}}{r_1-r_2}-\frac{r_1^{j+2}-r_3^{j+2}}{r_1-r_3}\right)\right|.$$

Example (3)

Let $\{X_t\}$ be an ARMA(p,q) process and it's white noise follow Inverse Gaussian (IG) distribution with characteristic function $\Phi_a(u)=\exp\{-(-2i\lambda u)^{1/2}\}$, then the characteristic function of $\{X_t\}$ can be written according to (3.1) as,

$$\psi_X(u)=\begin{cases}\prod_{k=0}^{q}\prod_{j=0}^{\infty}\exp\left\{-\left(-2i\lambda\, u\,\theta_k\eta_p(j+p-1,r_1,\ldots,r_p)\right)^{1/2}\right\} & , \quad p=2,3,\ldots\\ \prod_{k=0}^{q}\prod_{j=0}^{\infty}\exp\left\{-\left(-2i\lambda\, u\,\theta_k r_1^{j}\right)^{1/2}\right\} & , p=1\end{cases}$$

$$=\begin{cases}\exp\left\{-(-2i\lambda u)^{1/2}\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left(\theta_k\eta_p(j+p-1,r_1,\ldots,r_p)\right)^{1/2}\right\} & , p=2,3,\ldots\\ \exp\left\{-(-2i\lambda u)^{1/2}\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left(\theta_k r_1^{j}\right)^{1/2}\right\} & , p=1\end{cases}$$

So, the distribution of $\{X_t\}$ will be,

$$X_t\approx IG\begin{cases}\lambda\left(\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left(\theta_k\eta_p(j+p-1,r_1,\ldots,r_p)\right)^{1/2}\right)^2 & , \quad p=2,3,\ldots\\ \lambda\left(\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left(\theta_k r_1^{j}\right)^{1/2}\right)^2\end{cases}$$

As special case , if p=2 then, the distribution of ARMA(2,q) process will be Inverse Gaussian with parameter , $\lambda\left(\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left[\theta_k\,\frac{r_1^{j+1}-r_2^{j+1}}{r_1-r_2}\right]^{1/2}\right)^2$.

ARMA(3,q) will distributed also as Inverse Gaussian with parameter,

$$\lambda\left\{\sum_{k=0}^{q}\sum_{j=0}^{\infty}\left(\frac{\theta_k}{r_2-r_3}\left(\frac{r_1^{j+2}-r_2^{j+2}}{r_1-r_2}-\frac{r_1^{j+2}-r_3^{j+2}}{r_1-r_3}\right)\right)^{1/2}\right\}^2.$$

Example (4)

Let $\{X_t\}$ be an ARMA(p,q) process and it's white noise follow Gamma distribution with characteristic function $\Phi_a(u) = \left(\frac{m}{m-iu}\right)^d$, then the characteristic function of $\{X_t\}$ can be written according to (3.1) as,

$$\psi_X(u) = \begin{cases} \prod_{k=0}^{q}\prod_{j=0}^{\infty}\left(\frac{m}{m-iu\theta_k\eta_p(j+p-1,r_1,\ldots,r_p)}\right)^d & , p=2,3,\ldots \\ \prod_{k=0}^{q}\prod_{j=0}^{\infty}\left(\frac{m}{m-iu\theta_k r_1^j}\right)^d & , p=1 \end{cases}$$

The above formula does not assign any traditional probability distribution of $\{X_t\}$, so one can use the uniqueness relation between the probability distribution and it's characteristic function to find the probability distribution of $\{X_t\}$.

COEFFICIENTS OF THE CAUSAL ARMA(p,q) PROCESS

If we use the transformation $h = k + j$, equation (2.9) can be rewritten as ,

$$X_t = \begin{cases} \sum_{h=0}^{\infty}\sum_{j=h-q}^{h}\theta_{h-j}\eta_p(j+p-1,r_1,...,r_p)a_{t-h} & , p=2,3,... \\ \sum_{h=0}^{\infty}\sum_{j=h-q}^{h}\theta_{h-j}r_1^j a_{t-h} & , p=1 \end{cases} \quad (4.1)$$

Then from the above equation and equation (1.10) we can exactly determined the values of g_h $(h=0,1,2,...)$ to be ,

$$g_h = \begin{cases} \sum_{\substack{j=h-q \\ j\geq 0}}^{h}\theta_{h-j}\eta_p(j+p-1,r_1,...,r_p) & , p=2,3,... \\ \sum_{\substack{j=h-q \\ j\geq 0}}^{h}\theta_{h-j}r_1^j & , p=1 \end{cases} \quad (4.2)$$

Examples 5

Following special cases of ARMA(p,q) as examples to describe the above work .

A ARMA(2,1) process

From equation (1.2) we can write the ARMA(2,1) model as ,

$$X_t = (r_1 + r_2)\, X_{t-1} - r_1 r_2\, X_{t-2} + a_t + \theta_1 a_{t-1} \qquad (5.1)$$

If we substitute in X_{t-1} and X_{t-2} , (5.1) can be rewritten as ,

$$\begin{aligned} X_t &= (r_1 + r_2)\left\{(r_1 + r_2)X_{t-2} - r_1 r_2\, X_{t-3} + a_{t-1} + \theta_1 a_{t-2}\right\} - r_1 r_2 \left\{(r_1 + r_2)X_{t-3} - r_1 r_2\, X_{t-4} + a_{t-2} + \theta_1 a_{t-3}\right\} \\ &\qquad + a_t + \theta_1 a_{t-1} \\ &= a_t + (r_1 + r_2 + \theta_1)a_{t-1} + \left((r_1 + r_2)\theta_1 - r_1 r_2\right)a_{t-2} - r_1 r_2\, a_{t-3} + (r_1 + r_2)^2\, X_{t-2} - 2r_1 r_2 (r_1 + r_2) X_{t-3} \\ &= a_t + (r_1 + r_2 + \theta_1)a_{t-1} + \left((r_1 + r_2)\theta_1 - r_1 r_2\right)a_{t-2} - r_1 r_2\, a_{t-3} + (r_1 + r_2)^2 . \\ &\qquad \left\{(r_1 + r_2)X_{t-3} - r_1 r_2\, X_{t-4} + a_{t-2} + \theta_1 a_{t-3}\right\} - 2r_1 r_2 (r_1 + r_2) X_{t-3} \end{aligned}$$

By respectively substitutions as we did above , and then , compare the resulting equation with equation (1.10) , we can get ,

$$\left.\begin{aligned} g_0 &= 1 \\ g_1 &= \theta_1 + r_1 + r_2 \\ g_2 &= (r_1 + r_2)^2 + (r_1 + r_2)\theta_1 - r_1 r_2 \\ g_3 &= \left(r_1^2 + r_1 r_2 + r_2^2\right)\theta_1 + r_1^3 + r_1^2 r_2 + r_2^2 r_1 + r_2^3 \\ &: \\ &: \end{aligned}\right\} \qquad (5.2)$$

According to the traditional method .

The values of g_h $(h = 1,2,...)$ in (5.2) can be obtained simply by using our general formula , since from (4.2) , we can write ,

$$g_h = \sum_{\substack{j=h-1 \\ j \ge 0}}^{h} \theta_{h-j}\, \eta_2(j+1, r_1, r_2) \qquad (5.3)$$

If $h = 0$, then $g_0 = \theta_0\, \eta_2(1, r_1, r_2) = \theta_0 \dfrac{r_1 - r_2}{r_1 - r_2} = 1$, and so if $h = 1$, then ,

$g_1 = \theta_1\, \eta_2(1, r_1, r_2) + \theta_0\, \eta_2(2, r_1, r_2) = \theta_1 \dfrac{r_1 - r_2}{r_1 - r_2} + \theta_0 \dfrac{(r_1 - r_2)^2}{r_1 - r_2} = \theta_1 + r_1 + r_2$, and so on for the other values of $h = 2,3,4,...$.

B ARMA(2,2) process

From equation (1.2) we can write the ARMA(2,2) model as ,

$$X_t = (r_1 + r_2)\, X_{t-1} - r_1\, r_2\, X_{t-2} + a_t + \theta_1\, a_{t-1} + \theta_2\, a_{t-2} \qquad (5.4)$$

If we substitute in X_{t-1} and X_{t-2} , (5.4) can be rewritten as ,

$$\begin{aligned} X_t &= (r_1 + r_2)\{(r_1 + r_2)X_{t-2} - r_1 r_2\, X_{t-3} + a_{t-1} + \theta_1\, a_{t-2} + \theta_2\, a_{t-3}\} - r_1 r_2\, . \\ &\qquad \{(r_1 + r_2)X_{t-3} - r_1 r_2\, X_{t-4} + a_{t-2} + \theta_1\, a_{t-3} + \theta_2\, a_{t-4}\} + a_t + \theta_1\, a_{t-1} + \theta_2\, a_{t-2} \\ &= a_t + (r_1 + r_2 + \theta_1)a_{t-1} + (\theta_2 + (r_1 + r_2)\theta_1 - r_1 r_2)a_{t-2} + (\theta_2 (r_1 + r_2) - \theta_1 r_1 r_2)a_{t-3} - \theta_2\, r_1 r_2\, a_{t-4} \\ &\qquad + (r_1 + r_2)^2\, X_{t-2} - 2(r_1 + r_2) r_1 r_2\, X_{t-3} + (r_1 r_2)^2\, a_{t-4} \\ &= a_t + (r_1 + r_2 + \theta_1)a_{t-1} + (\theta_2 + (r_1 + r_2)\theta_1 - r_1 r_2)a_{t-2} + (\theta_2 (r_1 + r_2) - \theta_1 r_1 r_2)a_{t-3} - \theta_2\, r_1 r_2\, a_{t-4} \\ &\qquad + (r_1 + r_2)^2 \{(r_1 + r_2)X_{t-3} - r_1 r_2\, X_{t-4} + a_{t-2} + \theta_1\, a_{t-3} + \theta_2\, a_{t-4}\} - 2(r_1 + r_2) r_1 r_2 \\ &\qquad .\{(r_1 + r_2)X_{t-4} - r_1 r_2\, X_{t-5} + a_{t-3} + \theta_1\, a_{t-4} + \theta_2\, a_{t-5}\} + (r_1 r_2)^2\, a_{t-4} \end{aligned}$$

By respectively substitutions as we did above , and then , compare the resulting equation with equation (1.10) , we can get ,

$$\left.\begin{aligned} g_0 &= 1 \\ g_1 &= \theta_1 + r_1 + r_2 \\ g_2 &= (r_1 + r_2)^2 + (r_1 + r_2)\theta_1 - r_1\, r_2 + \theta_2 \\ g_3 &= (r_1 + r_2)^3 - (r_1\, r_2)\theta_1 - 2\, r_1\, r_2\, (r_1 + r_2) + \theta_1\, (r_1 + r_2)^2 + \theta_2\, (r_1 + r_2) \\ &: \end{aligned}\right\} \qquad (5.5)$$

According to the traditional method .

The values of $g_h \;\; (h = 1,2,...)$ in (5.5) can be obtained simply by using our general formula , since from (4.2) , we can write ,

$$g_h = \sum_{\substack{j=h-2 \\ j \geq 0}}^{h} \theta_{h-j}\, \eta_2(j + 1, r_1, r_2) \qquad (5.6)$$

If $h = 0$, then $g_0 = \theta_0\, \eta_2(1, r_1, r_2) = \theta_0 \dfrac{r_1 - r_2}{r_1 - r_2} = 1$, and so if $h = 1$, then ,

$g_1 = \theta_1\, \eta_2(1, r_1, r_2) + \theta_0\, \eta_2(2, r_1, r_2) = \theta_1 \dfrac{r_1 - r_2}{r_1 - r_2} + \theta_0 \dfrac{(r_1 - r_2)^2}{r_1 - r_2} = \theta_1 + r_1 + r_2$, and so on for the other values of $h = 2, 3, 4, ...$.

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